LIPSCOMB'S SPACE ω^A IS THE ATTRACTOR OF AN INFINITE IFS CONTAINING AFFINE TRANSFORMATIONS OF $l^2(A)$

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ABSTRACT. It is known that Lipscomb's space $L(A)$ can be imbedded in Hilbert's space $l^2(A)$. Let ω^A be the imbedded version of $L(A)$ endowed with the $l^2(A)$ -induced topology. We show how to construct ω^A as the attractor of an iterated function system containing an infinite number of affine transformations of $l^2(A)$. In this way we answer an open question of J.C. Perry.

1. INTRODUCTION

In [3] Lipscomb introduced $L(A)$ and in [2] Perry and Lipscomb proved that the Lipscomb's space $L(A)$ on an arbitrary index set A can be imbedded in Hilbert's space $l^2(A)$, showing that $L(A)$ can be endowed with the metric inherited from $l^2(A)$. In [4] Perry shows that $L(A)$ can be injected into the Tychonoff cube I^A and ω^A , the imbedded version of $L(A)$, is a subset of a standard $|A|$ -simplex Δ^A , which is both a subspace of $l^2(A)$ and a subset of the Tychonoff cube I^A . Let ω^A be the imbedded version of $L(A)$ endowed with the $l^2(A)$ -induced topology and let ω_c^A denote the space whose underlying set is that of ω^A but whose topology is induced from the Tychonoff cube I^A . Perry shows that ω_c^A is the attractor of an infinite iterated function system $W = \{w_a\}_{a \in A}$ containing affine transformations of I^A . He mentions that it is an open problem to construct ω^A as the attractor of an iterated function system containing an infinite number of affine transformations of $l^2(A)$.

In this paper we show how to construct ω^A as the attractor of an infinite iterated function system containing affine transformations of $l^2(A)$.

2. NOTATION

We follow the notation from [4]. Let A be an arbitrary set and single out a point z of A . Let us consider the set $A' = A - \{z\}$. The points of $l^2(A)$ are collections of real numbers indexed by points of A' . If E is the set of real numbers, then $x \in l^2(A)$ means $x = \{x_a\} \in E^{A'}$ such that $x_a = 0$ for all but countably many

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$a \in A'$ and $\sum x_a^2$ converges. The topology of $l^2(A)$ is induced from the metric $d(x, y) = \sqrt{\sum_a (x_a - y_a)^2}$, where we think of x_a as the a -th coordinate of x . Let us also consider, for the case when A is an arbitrary set with the discrete topology, the Baire space $N(A)$, which is the topological product of countably many copies A_n of A . Hence the points of $N(A)$ consist of all sequences $v = a_1 a_2 \cdots a_n \dots$, with $a_n \in A$. Moreover $N(A)$ is a metric space with the metric

$$d(v, v') = \begin{cases} \frac{1}{k}, & \text{if } k \text{ is the first index where } a_i \neq a'_i, \\ 0, & \text{if } v = v'. \end{cases}$$

Lipscomb's space $L(A)$ is a quotient space of $N(A)$ such that each equivalence class consists of either a single point or two points. Those classes with two points come from identifying $a_1 a_2 \dots a_{k-2} a_{k-1} a_k a_k \dots$ with the point $a_1 a_2 \dots a_{k-2} a_k a_{k-1} a_k \dots$, where $a_{k-1} \neq a_k$. Therefore Lipscomb's space $L(A)$ is obtained via a projection or identification map $p : N(A) \rightarrow L(A)$.

3. PRELIMINARY OBSERVATIONS

For the following Proposition 3.1 and Definition 3.2 one can consult [5], pages 39–43.

Let (X, d) be a complete metric space. If $\mathcal{M}_X = \{N \subseteq X : N \text{ is nonempty and bounded}\}$, for $x \in X$, $\varepsilon > 0$ and $A, B \in \mathcal{M}_X$, we consider $d(x, A) \stackrel{\text{def}}{=} \inf_{a \in A} d(x, a)$, $B(A, \varepsilon) \stackrel{\text{def}}{=} \{x \in X : d(x, A) < \varepsilon\}$, $B[A, \varepsilon] \stackrel{\text{def}}{=} \{x \in X : d(x, A) \leq \varepsilon\}$ and $d(A, B) \stackrel{\text{def}}{=} \sup_{x \in A} d(x, B)$.

The map $h^* : \mathcal{M}_X \times \mathcal{M}_X \rightarrow \mathbb{R}_+$, given by

$$h^*(A, B) = \max\{d(A, B), d(B, A)\},$$

for all $A, B \in \mathcal{M}_X$, satisfies the following properties:

- i) $h^*(A, A) = 0$, for all $A \in \mathcal{M}_X$;
- ii) $h^*(A, B) = h^*(B, A)$, for all $A, B \in \mathcal{M}_X$;
- iii) $h^*(A, B) \leq h^*(A, C) + h^*(C, B)$, for all $A, B, C \in \mathcal{M}_X$.

Therefore h^* is a pseudometric in $\mathcal{M}_X$ which is called the *Hausdorff-Pompeiu pseudometric*.

Proposition 3.1. *For all $A_1, A_2 \in \mathcal{M}_X$, the following properties are valid:*

- i) *for all $\varepsilon > 0$, $h^*(A_1, A_2) \leq \varepsilon$ if and only if $A_1 \subseteq B[A_2, \varepsilon]$ and $A_2 \subseteq B[A_1, \varepsilon]$;*
- ii) $h^*(A_1, A_2) = \inf\{r > 0 : A_1 \subseteq B(A_2, r) \text{ and } A_2 \subseteq B(A_1, r)\}$;
- iii) $h^*(A_1, A_2) = h^*(\overline{A_1}, \overline{A_2})$;
- iv) $h^*(A_1, A_2) = 0$ if and only if $\overline{A_1} = \overline{A_2}$;
- v) *if $f : X \rightarrow X$ is a Lipschitz function, then*

$$h^*(f(A_1), f(A_2)) \leq \text{Lip}(f) \cdot h^*(A_1, A_2);$$

vi) *if $\{A_1^i\}_{i \in I}$ and $\{A_2^i\}_{i \in I}$ are two families of elements of $\mathcal{M}_X$ such that $\overline{\bigcup_{i \in I} A_1^i} \in \mathcal{M}_X$ and $\overline{\bigcup_{i \in I} A_2^i} \in \mathcal{M}_X$, then*

$$h^*\left(\overline{\bigcup_{i \in I} A_1^i}, \overline{\bigcup_{i \in I} A_2^i}\right) = h^*\left(\bigcup_{i \in I} A_1^i, \bigcup_{i \in I} A_2^i\right) \leq \sup_{i \in I} h^*(A_1^i, A_2^i).$$

Definition 3.2. The restriction of h^* to $\mathcal{B}_X \times \mathcal{B}_X$, where $\mathcal{B}_X = \{N \subseteq X : N \text{ is nonempty, closed and bounded}\}$, is a metric. We call it the *Hausdorff-Pompeiu metric* and denote it by h .

Lemma 3.3. *In the above-mentioned frame, the metric space $(\mathcal{B}_X, h)$ is complete.*

Proof. Let $(N_n)_n$ be a Cauchy sequence of elements from $\mathcal{B}_X$. Then, for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq n_\varepsilon$ and all $p \in \mathbb{N}$, we have $h(N_{n+p}, N_n) < \varepsilon$. Let us note that $N_{n+p} \subseteq B[N_n, \varepsilon]$, for all $n \in \mathbb{N}$, with $n \geq n_\varepsilon$ and all $p \in \mathbb{N}$ (see Proposition 3.1 (i)). In particular, we obtain $\bigcup_{n \in \mathbb{N}} N_n \subseteq B(N_1, \varepsilon) \cup \dots \cup B(N_{n_\varepsilon-1}, \varepsilon) \cup B[N_{n_\varepsilon}, \varepsilon]$, so $\bigcup_{n \in \mathbb{N}} N_n$ is bounded.

Let $N = \{x \in X : x = \lim_{k \rightarrow \infty} x_{n_k} \text{ where each } n_k < n_{k+1} \text{ and each } x_{n_k} \in N_{n_k}\}$.

Since $N \subseteq \overline{\bigcup_{n \in \mathbb{N}} N_n}$, we infer that N is bounded.

We assert that N is closed.

Indeed, let us consider $x \in \overline{N}$. Then, there exists a sequence $(x_m)_m \subseteq N$ such that $\lim_{m \rightarrow \infty} x_m = x$. Since $x_1 \in N$, there exists a strictly increasing sequence $(n_k^1)_k \subseteq \mathbb{N}$, such that, for each $k \in \mathbb{N}$, there exists $t_{n_k^1} \in N_{n_k^1}$, having the property $\lim_{k \rightarrow \infty} t_{n_k^1} = x_1$. Therefore, we can choose k_1 such that $d(t_{n_{k_1}^1}, x_1) < 1$. Let us denote $n_{k_1}^1$ by n_1 . Hence $d(t_{n_1}, x_1) < 1$ and $t_{n_1} \in N_{n_1}$. Since $x_2 \in N$, there exists a strictly increasing sequence $(n_k^2)_k \subseteq \mathbb{N}$, such that, for each $k \in \mathbb{N}$, there exists $t_{n_k^2} \in N_{n_k^2}$, having the property $\lim_{k \rightarrow \infty} t_{n_k^2} = x_2$. Therefore, we can choose k_2 such that $d(t_{n_{k_2}^2}, x_2) < \frac{1}{2}$ and $n_{k_2}^2 > n_1$. Let us denote $n_{k_2}^2$ by n_2 . Then $d(t_{n_2}, x_2) < \frac{1}{2}$ and $t_{n_2} \in N_{n_2}$. Inductively we can construct a strictly increasing sequence $(n_k)_k \subseteq \mathbb{N}$, for which there exists $t_{n_k} \in N_{n_k}$ with the property $d(t_{n_k}, x_k) < \frac{1}{k}$, for each $k \in \mathbb{N}$. Then, since $d(t_{n_k}, x) \leq d(t_{n_k}, x_k) + d(x_k, x) < \frac{1}{k} + d(x_k, x)$ and $\lim_{k \rightarrow \infty} (\frac{1}{k} + d(x_k, x)) = 0$, we obtain that $x = \lim_{k \rightarrow \infty} t_{n_k}$, so $x \in N$.

Hence $N \in \mathcal{B}_X$.

Moreover $\lim_{n \rightarrow \infty} h(N_n, N) = 0$.

Indeed, for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$, such that for all $n \in \mathbb{N}$ with $n \geq n_\varepsilon$ and all $p \in \mathbb{N}$, we have $h(N_{n+p}, N_n) < \varepsilon$. Since $\bigcup_{p \in \mathbb{N}} N_{n+p} \subseteq B[N_n, \varepsilon]$, for all $n \in \mathbb{N}$, $n \geq n_\varepsilon$ and all $p \in \mathbb{N}$, we have $N \subseteq \overline{\bigcup_{p \in \mathbb{N}} N_{n+p}} \subseteq B[N_n, \varepsilon]$; hence $N \subseteq B[N_n, \varepsilon]$, for all $n \in \mathbb{N}$, $n \geq n_\varepsilon$. Therefore, for all $x \in N$, $x \in B[N_n, \varepsilon]$, so $d(x, N_{n_\varepsilon}) \leq \varepsilon$. Consequently, since $\sup_{x \in N} d(x, N_{n_\varepsilon}) = d(N, N_{n_\varepsilon})$, we get

$$(3.1) \quad d(N, N_{n_\varepsilon}) \leq \varepsilon.$$

Let us consider $y \in N_{n_\varepsilon}$. Let us choose $n_0 = n_\varepsilon$. For each $k \in \mathbb{N}$, there exists n_k , such that $h(N_{n+p}, N_n) < \frac{\varepsilon}{2^k}$, for all $n \in \mathbb{N}$, $n \geq n_k$ and all $p \in \mathbb{N}$. Let us note that we can assume that $n_k < n_{k+1}$, for all $k \in \mathbb{N}$. Let us consider $y_{n_0} = y$. Then $d(y_{n_0}, N_{n_1}) \leq d(N_{n_0}, N_{n_1}) \leq h(N_{n_0}, N_{n_1}) < \varepsilon$, so there exists $y_{n_1} \in N_{n_1}$ such that $d(y_{n_0}, y_{n_1}) < \varepsilon$. In this manner we obtain a sequence $(y_{n_k})_k$ such that $y_{n_k} \in N_{n_k}$ and $d(y_{n_k}, y_{n_{k+1}}) < \frac{\varepsilon}{2^k}$, for all $k \in \mathbb{N}$. Let us note that, since for $m, p \in \mathbb{N}$, $m \geq p$, we have $d(y_{n_p}, y_{n_m}) \leq d(y_{n_p}, y_{n_{p+1}}) + d(y_{n_{p+1}}, y_{n_{p+2}}) + \dots + d(y_{n_{m-1}}, y_{n_m}) \leq \frac{\varepsilon}{2^p} + \frac{\varepsilon}{2^{p+1}} + \dots + \frac{\varepsilon}{2^{m-1}} < \frac{\varepsilon}{2^{p-1}}$, $(y_{n_k})_k$ is a Cauchy sequence. Therefore, taking into account the fact that (X, d) is a complete metric space, there exists $x \in X$ such that $\lim_{k \rightarrow \infty} d(y_{n_k}, x) = 0$, so $x \in \overline{N} = N$. But $d(y_{n_k}, y_{n_0}) < 2\varepsilon$, so $\lim_{k \rightarrow \infty} d(y_{n_k}, y_{n_0}) = d(x, y) \leq 2\varepsilon$. Hence $d(x, y) \leq 2\varepsilon$, for all $y \in N_{n_\varepsilon}$, so $d(y, N) = \inf_{u \in N} d(u, y) \leq d(x, y) \leq 2\varepsilon$, for all $y \in N_{n_\varepsilon}$, and therefore

$$\sup_{y \in N_{n_\varepsilon}} d(y, N) = d(N_{n_\varepsilon}, N) \leq 2\varepsilon, \text{ i.e.,}$$

$$(3.2) \quad d(N_{n_\varepsilon}, N) \leq 2\varepsilon.$$

From 3.1 and 3.2 we obtain that

$$h(N_{n_\varepsilon}, N) \leq 2\varepsilon.$$

Then, for $n \in \mathbb{N}$, $n \geq n_\varepsilon$, we have $h(N_n, N) \leq h(N, N_{n_\varepsilon}) + h(N_{n_\varepsilon}, N_n) \leq \varepsilon + 2\varepsilon = 3\varepsilon$, i.e.,

$$\lim_{n \rightarrow \infty} h(N_n, N) = 0. \quad \square$$

Lemma 3.4. *Let (X, d) be a complete metric space, let $(f_i)_{i \in I}$ be a family of c -contractions, where $f_i : X \rightarrow X$ and $c \in (0, 1)$, and let $\mathcal{F} : \mathcal{B}_X \rightarrow \mathcal{B}_X$ be given by*

$$\mathcal{F}(N) = \overline{\bigcup_{i \in I} f_i(N)},$$

for all $N \in \mathcal{B}_X$. Then $\mathcal{F}$ is a c -contraction.

Proof. Indeed, using v) and vi) from Proposition 3.1, we have

$$\begin{aligned} h(\mathcal{F}(N_1), \mathcal{F}(N_2)) &= h\left(\overline{\bigcup_{i \in I} f_i(N_1)}, \overline{\bigcup_{i \in I} f_i(N_2)}\right) = h\left(\bigcup_{i \in I} f_i(N_1), \bigcup_{i \in I} f_i(N_2)\right) \\ &\leq \sup_{i \in I} h(f_i(N_1), f_i(N_2)) \leq \sup_{i \in I} c \cdot h(N_1, N_2) = c \cdot h(N_1, N_2), \end{aligned}$$

for all $N_1, N_2 \in \mathcal{B}_X$. $\square$

Theorem 3.5. *Let us consider (X, d) to be a complete metric space, $c \in (0, 1)$ and $(f_i)_{i \in I}$ a family of c -contractions, where $f_i : X \rightarrow X$. If the family $(f_i)_{i \in I}$ is bounded (i.e., $\bigcup_{i \in I} f_i(A)$ is bounded for each bounded subset A of X), then there exists a unique closed and bounded subset M of X such that $M = \overline{\bigcup_{i \in I} f_i(M)}$.*

Proof. According to the Banach contraction principle, since $(\mathcal{B}_X, h)$ is a complete metric space (see Lemma 3.3) and $\mathcal{F} : \mathcal{B}_X \rightarrow \mathcal{B}_X$ is a c -contraction (see Lemma 3.4), there exists a unique $M \in \mathcal{B}_X$ such that $\mathcal{F}(M) = M$, i.e., $M = \overline{\bigcup_{i \in I} f_i(M)}$. $\square$

Lemma 3.6. *Let (X, d) be a complete metric space and let $\mathcal{F}$ be defined as in Lemma 3.4. If $T \in \mathcal{B}_X$ has the property that $\mathcal{F}(T) \subseteq T$, then $M \subseteq T$.*

Proof. Indeed, let us consider the sequence $(T_n)_{n \in \mathbb{N}}$, defined by $T_0 = T$ and $T_{n+1} = \mathcal{F}(T_n)$, for each $n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} h(T_n, M) = 0$ and $T_{n+1} \subseteq T_n$, for each $n \in \mathbb{N}$. Let us consider $a \in M$. Then there exists $n_1 \in \mathbb{N}$, such that $h(T_{n_1}, M) < 1$. Hence (see Proposition 3.1 (ii)) $M \subseteq B(T_{n_1}, 1)$, so $a \in B(T_{n_1}, 1)$, i.e. $d(a, T_{n_1}) < 1$, which implies the existence of $x_{n_1} \in T_{n_1}$ such that $d(a, x_{n_1}) < 1$. There exists $n_2 \in \mathbb{N}$, $n_2 > n_1$, such that $h(T_{n_2}, M) < \frac{1}{2}$. Hence (see Proposition 3.1 (ii)) $M \subseteq B(T_{n_2}, \frac{1}{2})$, so $a \in B(T_{n_2}, \frac{1}{2})$, i.e. $d(a, T_{n_2}) < \frac{1}{2}$, which implies the existence of $x_{n_2} \in T_{n_2}$ such that $d(a, x_{n_2}) < \frac{1}{2}$. In this way we obtain a strictly increasing sequence of natural numbers $(n_k)_{k \in \mathbb{N}}$ and a sequence $(x_{n_k})_{k \in \mathbb{N}}$ such that $x_{n_k} \in T_{n_k}$ and $d(a, x_{n_k}) < \frac{1}{k}$, for each $k \in \mathbb{N}$. Therefore $a = \lim_{k \rightarrow \infty} x_{n_k}$. Since $x_{n_k} \in T_{n_k} \subseteq T_0 = T$, for all $k \in \mathbb{N}$, we infer that $a \in \overline{T} = T$. Hence $M \subseteq T$. $\square$

Remark 3.7. To the best of the authors' knowledge the results from this section are new. By replacing, in these results, $\mathcal{B}_X$ with $\mathcal{H}(X) = \{N \subseteq X : N \text{ is nonempty and compact}\}$ and taking I finite, one obtains some classical results (see [1]). Similar results concerning infinite iterated function systems have also been obtained for the case when the attractor is compact (for example, see [6], where the case of a countable iterated function system on a compact metric space is considered).

4. AN ITERATED FUNCTION SYSTEM

Let us consider the complete metric space $(l^2(A), d)$. For each $a \in A \setminus \{z\}$, let $f_a : l^2(A) \rightarrow l^2(A)$ be the function given by $f_a(x) = \frac{1}{2}(x + u_a)$, for all $x \in l^2(A)$, and let $f_z : l^2(A) \rightarrow l^2(A)$ be the function given by $f_z(x) = \frac{1}{2}x$, for all $x \in l^2(A)$, where $u_a = (\alpha_j)_{j \in A'} \in l^2(A)$ is described by $\alpha_j = 0$, for $j \neq a$ and $\alpha_a = 1$. Let us note that, for each $a \in A$, f_a is a $\frac{1}{2}$ -contraction. Moreover, the family $(f_a)_{a \in A}$ is bounded. Indeed, if K is a bounded subset of $l^2(A)$, then there exists $r > 0$ such that $K \subseteq B_r(0)$. Then $f_a(B_r(0)) \subseteq B_{\frac{r+1}{2}}(0)$, for each $a \in A$, so $\bigcup_{a \in A} f_a(K) \subseteq B_{\frac{r+1}{2}}(0)$, i.e., $\bigcup_{a \in A} f_a(K)$ is bounded. According to Theorem 3.5, there exists a unique bounded closed subset M of $l^2(A)$ such that

$$M = \overline{\bigcup_{a \in A} f_a(M)}.$$

5. ω^A IS THE ATTRACTOR OF $(l^2(A), \{f_a\}_{a \in A})$

Theorem 5.1. *The imbedded version of $L(A)$ endowed with the $l^2(A)$ -induced topology is the attractor for the infinite iterated function system $(l^2(A), \{f_a\}_{a \in A})$.*

Proof. Using [2], let us consider the function $s : L(A) \rightarrow l^2(A)$ given by $s(\alpha) = (\alpha_b)_{b \in A'}$, where $\alpha = p(v)$, $v = a_1 a_2 \cdots \in N(A)$ and

$$\alpha_b = \begin{cases} \sum_{k \text{ with } a_k=b} \frac{1}{2^k}, & \text{if there exists } k \text{ such that } a_k = b, \\ 0, & \text{if there exists no } k \text{ such that } a_k = b. \end{cases}$$

Note that ω^A is $s(L(A))$.

According to lines 16, 17, 20 and 21, page 2482, from [4], we have $s(L(A)) \subseteq \Delta^A = \{x = (x_b)_{b \in A'} \in l^2(A) : \text{each } x_b \geq 0 \text{ and } \sum_{A'} x_b \leq 1\}$, so $s(L(A)) \subseteq B(0_{l^2(A)}, 1)$ and therefore $s(L(A))$ is a bounded subset of $l^2(A)$.

According to Corollary 15, from [4], $s(L(A))$ is a closed subset of $l^2(A)$.

Therefore $s(L(A)) \in \mathcal{B}_{l^2(A)}$.

For M as given in Theorem 3.5, we assert that $M = s(L(A))$.

On the one hand, we have $M \subseteq s(L(A))$. Indeed, if $x \in s(L(A))$, then there exists $\alpha = p(v)$, where $v = a_1 a_2 \cdots \in N(A)$, such that $x = s(\alpha) = (\alpha_b)_{b \in A'}$. Therefore $f_a(x) = \frac{1}{2}(x + u_a) = (\gamma_b)_{b \in A'}$, where $\gamma_b = \frac{1}{2}\alpha_b$, for $b \in A' - \{a\}$, and $\gamma_a = \frac{1}{2}(\alpha_a + 1)$, so $f_a(x) = (\gamma_b)_{b \in A'} = s(p(v_0))$, where $v_0 = a a_1 a_2 \cdots \in N(A)$. Hence $f_a(s(L(A))) \subseteq s(L(A))$, for all $a \in A$. Consequently,

$$\bigcup_{a \in A} f_a(s(L(A))) \subseteq s(L(A));$$

i.e., using the notation from our previous results and taking into account the fact that $s(L(A))$ is a closed subset of $l^2(A)$, we obtain $\mathcal{F}(s(L(A))) \subseteq s(L(A))$. Then, according to Lemma 3.6, $M \subseteq s(L(A))$. On the other hand, $s(L(A)) \subseteq M$.

Indeed, if $\alpha = p(v)$, where $v = a_1 a_2 \cdots \in N(A)$, then, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(f_{a_1} \circ f_{a_2} \circ \cdots \circ f_{a_n}(0_{l^2(A)}), s(\alpha)) \\ = d\left(\frac{u_{a_n} + 2u_{a_{n-1}} + \cdots + 2^{n-1}u_{a_1}}{2^n}, (\alpha_b)_{b \in A'}\right) \\ \leq \sqrt{\sum_{k>n} \frac{1}{2^{k-1}}} \end{aligned}$$

so, since

$$\lim_{n \rightarrow \infty} \sqrt{\sum_{k>n} \frac{1}{2^{k-1}}} = 0,$$

we obtain that $s(\alpha) = \lim_{n \rightarrow \infty} f_{a_1} \circ f_{a_2} \circ \cdots \circ f_{a_n}(0_{l^2(A)})$. Let us note that $0_{l^2(A)} \in M$. Indeed, $0_{l^2(A)} \in M$ because $M \neq \emptyset$ yields $y \in M$, and the sequence $f_z(y), f_z \circ f_z(y), \dots$ in the closed set M converges to $0_{l^2(A)}$. Therefore, since $f_a(M) \subseteq M$, for all $a \in A$, $s(\alpha) \in \overline{M} = M$, and, consequently, $s(L(A)) \subseteq M$.

Hence $M = \omega^A$; i.e., ω^A is the attractor of $(l^2(A), \{f_a\}_{a \in A})$. $\square$

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