HYPERCYCLIC AND TOPOLOGICALLY MIXING COSINE FUNCTIONS ON BANACH SPACES

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ABSTRACT. Our first aim in this paper is to give sufficient conditions for the hypercyclicity and topological mixing of a strongly continuous cosine function. We apply these results to study the cosine function associated to translation groups. We also prove that every separable infinite dimensional complex Banach space admits a topologically mixing uniformly continuous cosine family.

INTRODUCTION

A bounded linear operator $T \in \mathcal{B}(X)$ on a separable complex Banach space X is said to be *hypercyclic* if there exists an $x \in X$ such that $\{T^n x\}_{n \in \mathbb{N}}$ is dense in X .

In 1969, Rolewicz [12] gave the first example of a hypercyclic operator on a Banach space. He showed that if B is the backward shift on $l^2(\mathbb{N})$, then λB is hypercyclic if and only if $|\lambda| > 1$. He also wondered if for every separable infinite dimensional Banach space there exists a hypercyclic operator. This question was independently answered in the affirmative by Ansari [1] and Bernal-González [5]. Bonet and Peris [7] generalized the result for Fréchet spaces.

It is well known that $T \in \mathcal{B}(X)$ is hypercyclic if and only if for any pair of nonvoid open sets $U, V \subset X$ there exists some positive integer n_0 such that

$$(*) \quad T^{n_0} U \cap V \neq \emptyset.$$

It is said that an operator is *topologically mixing* if the condition $(*)$ holds for every n large enough.

A one-parameter family $\{T(t)\}_{t \geq 0} \subset \mathcal{B}(X)$ of bounded linear operators is a one-parameter semigroup of operators in $\mathcal{B}(X)$ if it verifies the following two conditions:

- (i) $T(0) = I$,
- (ii) $T(t)T(s) = T(t + s)$ for all $t, s \geq 0$.

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The infinitesimal generator A of a semigroup $\{T(t)\}_{t \geq 0}$ can be calculated by

$$Ax := \lim_{t \rightarrow 0^+} \frac{T(t)x - x}{t}$$

for $x \in X$ where this limit exists. In the case where A and $-A$ generate semigroups, it is said A generates a *group*.

We say that a semigroup is *strongly continuous* or C_0 -semigroup provided that $\lim_{t \rightarrow s} T(t)x = T(s)x$ for all $s \geq 0$ and $x \in X$. The semigroup is *uniformly continuous* if this limit holds uniformly on the unit ball of X . If $\{T(t)\}_{t \geq 0}$ is a strongly continuous semigroup, then the infinitesimal generator is closed and densely defined. In the uniformly continuous case, the infinitesimal generator is everywhere defined and bounded. See definitions and properties, for example, in [2].

A semigroup $\{T(t)\}_{t \geq 0}$ is *hypercyclic* if there exists $x \in X$ such that $\{T(t)x\}_{t \geq 0}$ is dense in X . We say that a semigroup $\{T(t)\}_{t \geq 0}$ is *topologically mixing* if for every pair of nonvoid sets $U, V \subset X$ there exists t_0 such that

$$T(t)U \cap V \neq \emptyset \text{ for all } t \text{ such that } t \geq t_0.$$

In [4] ([3]) it is proved that every separable infinite dimensional complex Banach space admits a hypercyclic (topologically mixing) uniformly continuous semigroup.

A family of operators $C(t) : \mathbb{R} \rightarrow \mathcal{B}(X)$ is called a *cosine function* if $C(0) = I$ and

$$2C(t)C(s) = C(t+s) + C(t-s) \text{ for all } s, t \in \mathbb{R}.$$

To start, we assume that the family is strongly continuous, i.e., $t \rightarrow C(t)x$, for $t \in \mathbb{R}$, is a continuous X -valued function for each $x \in X$. The *infinitesimal generator* A of $C(t)$ is defined by

$$Ax = \lim_{t \rightarrow 0} \frac{2}{t^2} (C(t) - I)x, \quad x \in D(A),$$

where $D(A)$ consists of those vectors $x \in X$ such that the above limit exists. Then A is a closed, densely defined operator. It is known that there exist constants $M > 0$ and $w \geq 0$ such that $\|C(t)\| \leq Me^{w|t|}$ for all $t \in \mathbb{R}$. Moreover, if $z \in \mathbb{C}$ satisfies $\operatorname{Re} z > w$, then $z^2 \in \rho(A)$ and so $\sigma(A)$ is contained in a parabolic domain in $\mathbb{C}$ which is open to the left [2, Proposition 3.14.18].

An operator A generates a uniformly continuous cosine function $C(t)$; that is, $\lim_{t \rightarrow 0} \|C(t) - I\| = 0$ if and only if A is bounded. In this case the cosine function admits the following representation:

$$C(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{2n!} A^n.$$

Moreover, if B is the generator of a C_0 -group $T(t)$ on X , then $A := B^2$ generates a cosine function $C(t)$ on X given by

$$(0.1) \quad C(t) := \frac{1}{2} (T(t) + T(-t))$$

(see for example [2, Example 3.14.15]). Recall that a Banach space is said to have the UMD property or uniform martingale difference property if the Hilbert transform is bounded in $L^p(\mathbb{R}, X)$, $1 < p < \infty$; see for example [2]. Every bounded cosine function in a UMD-space X may be decomposed in this way ([2, Section 3.16]).

At first glance, C_0 -semigroups are connected with the first order Cauchy problem while cosine families are closer to the second order Cauchy problem; see more details in [2]. Now we introduce the following classes of cosine families.

Definition 0.1. A cosine function $\{C(t)\}_{t \geq 0}$ is *hypercyclic* if there exists $x \in X$ such that $\{C(t)x\}_{t \geq 0}$ is dense in X ; in this case the vector $x \in X$ is called *hypercyclic*. The cosine function $\{C(t)\}_{t \geq 0}$ is *topologically transitive* if for every pair of nonvoid sets U, V there exists t_0 such that

$$C(t_0)U \cap V \neq \emptyset.$$

Also, $\{C(t)\}_{t \geq 0}$ is *topologically mixing* if for every pair of nonvoid sets $U, V \subset X$ there exists t_0 such that

$$C(t)U \cap V \neq \emptyset \text{ for all } t \text{ such that } t \geq t_0.$$

In the first section we give sufficient conditions for the hypercyclicity and topological mixing of a strongly continuous cosine function. We apply these results to study cosine functions defined by translation groups in the second section. We also prove that every separable infinite dimensional complex Banach space admits a topologically mixing uniformly continuous cosine function in the last section.

1. TOPOLOGICALLY TRANSITIVE AND MIXING CRITERIA

The first result is a consequence of a general result of universality in the survey of Grosse-Erdmann [9, Theorem1]. The proof is given here for completeness.

Theorem 1.1. *Let $\{C(t)\}_{t \geq 0}$ be a strongly continuous cosine function on a separable Banach space X . Then the following assertions are equivalent:*

- (i) $\{C(t)\}_{t \geq 0}$ is topologically transitive.
- (ii) $\{C(t)\}_{t \geq 0}$ has a dense set of hypercyclic vectors.

Proof. We take the set of positive rational numbers and enumerate them as $\{t_1, t_2, \dots\}$. Consider now the sequence family $\{C(t_n)\}_{n \in \mathbb{N}}$. Take $p \in X$. Clearly the set $\{C(t)p\}_{t \geq 0}$ is dense if and only if the set $\{C(t_n)p\}_{n \in \mathbb{N}}$ is dense.

Consider now the covering of X by the enumerated sequence of balls B_m centered on a countable subset of X with rational radius. Also, define

$$G_m := \bigcup_{n \in \mathbb{N}} C(t_n)^{-1}(B_m).$$

Then the set of hypercyclic vectors for $\{C(t)\}_{t \geq 0}$ is equal to $\bigcap_{m \in \mathbb{N}} G_m$. This set is dense by the Baire category theorem if $\{C(t)\}_{t \geq 0}$ is topologically transitive and thus we show (i) $\Rightarrow$ (ii).

Conversely, if $\{C(t)\}_{t \geq 0}$ has a dense set of hypercyclic vectors, each term of the intersection must be dense and thus $\{C(t)\}_{t \geq 0}$ is topologically transitive and we have proved (ii) $\Rightarrow$ (i). $\square$

In the next results, we give some topologically transitive criteria and one topologically mixing criterion for cosine functions. The following theorem can be shown from [6, Remark 2.6(3)] taking the particular case $T_n = C(t_n)$. However we also prefer to give the proof for completeness.

Theorem 1.2. *Let $\{C(t)\}_{t \geq 0}$ be a strongly continuous cosine function on a separable Banach space X . If there exists a sequence of positive reals $\{t_n\}$, $t_n \rightarrow \infty$, such that*

- (i) $X_0 := \{x \in X : \lim_{t_n \rightarrow \infty} C(t_n)x = 0\}$ and
- (ii) $X_\infty := \{y \in X : \exists u_n \rightarrow 0, \lim_{t_n \rightarrow \infty} C(t_n)u_n \rightarrow y\}$

are dense, then $\{C(t)\}_{t \geq 0}$ is a topologically transitive cosine function.

Proof. Given U and V a pair of nonempty open subsets in X , since X_0 and X_∞ are dense, there exist $x_0 \in X_0 \cap U$ and $y_0 \in X_\infty \cap V$. One has $C(t_n)x_0 \rightarrow 0$ and there exists $u_n \rightarrow 0$ with $\lim_{t_n \rightarrow \infty} C(t_n)u_n \rightarrow y_0$. Thus $C(t_n)(x_0 + u_n) \rightarrow y_0 \in V$. Since U and V are open, there exists $t' > 0$ such that $C(t')U \cap V \neq \emptyset$. $\square$

Corollary 1.3. *Let $\{C(t)\}_{t \geq 0}$ be a strongly continuous cosine function on a separable Banach space X . If there exists a sequence of positive reals $\{t_n\}$, $t_n \rightarrow \infty$, such that $X_1 := \{x \in X : \lim_{t_n \rightarrow \infty} C(t_n)x = \lim_{t_n \rightarrow \infty} C(2t_n)x = 0\}$ is dense, then $\{C(t)\}_{t \geq 0}$ is a topologically transitive cosine function.*

Proof. We consider the subspaces X_0 and X_∞ defined in Theorem 1.2. Then $X_1 \subset X_0$ and $X_1 \subset X_\infty$ because, given $y_0 \in X_1$, we define $u_n = 2C(t_n)y_0$. Then $u_n \rightarrow 0$ and $\lim_{t_n \rightarrow \infty} C(t_n)u_n = \lim_{t_n \rightarrow \infty} y_0 + C(2t_n)y_0 \rightarrow y_0$. Hence X_0 and X_∞ are dense sets and the proof is obtained as a consequence of Theorem 1.2. $\square$

Theorem 1.4. *Let $\{C(t)\}_{t \geq 0}$ be a strongly continuous cosine function on a separable Banach space X . If $X_0 := \{x \in X : \lim_{t \rightarrow \infty} C(t)x = 0\}$ is dense, then $\{C(t)\}_{t \geq 0}$ is a topologically mixing cosine function.*

Proof. Given U and V a pair of nonempty open subsets in X , since X_0 is dense, there exist $x_0 \in X_0 \cap U$ and $y_0 \in X_0 \cap V$. One has $C(t)x_0 \rightarrow 0$ as $t \rightarrow \infty$ and if we define $u_t := 2C(t)y_0$, then $u_t \rightarrow 0$ and $C(t)u_t = y_0 + C(2t)y_0$. Thus $\lim_{t \rightarrow \infty} C(t)u_t \rightarrow y_0$. Thus $C(t)(x_0 + u_t) \rightarrow y_0 \in V$. Since U and V are open, there exists t_0 such that $C(t)U \cap V \neq \emptyset$, for all $t > t_0$. $\square$

2. TOPOLOGICALLY TRANSITIVE AND MIXING TRANSLATIONS

By an *admissible weight function* on $\mathbb{R}$ we mean a measurable function $\rho : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the conditions

- (i) $\rho(x) > 0$ for all $x \in \mathbb{R}$;
- (ii) there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(x) \leq Me^{\omega t} \rho(t+x)$ for all $x \in \mathbb{R}$ and all $t > 0$.

([8, Definition 4.1]). We consider the function spaces $(L_\rho^p(\mathbb{R}), \|\cdot\|_p)$,

$$L_\rho^p(\mathbb{R}) = \{u : \mathbb{R} \rightarrow \mathbb{C} : u \text{ measurable, } \int_{\mathbb{R}} |u(\tau)|^p \rho(\tau) d\tau < \infty\},$$

with

$$\|u\|_p = \left\{ \int_{\mathbb{R}} |u(\tau)|^p \rho(\tau) d\tau \right\}^{\frac{1}{p}}, \quad p \geq 1$$

and $(C_{0,\rho}(\mathbb{R}), \|\cdot\|_\infty)$ where

$$C_{0,\rho}(\mathbb{R}) = \{u : \mathbb{R} \rightarrow \mathbb{C} : u \text{ continuous, } \lim_{|\tau| \rightarrow \infty} \rho(\tau)u(\tau) = 0\}$$

with

$$\|u\|_\infty = \sup_{\tau \in \mathbb{R}} |u(\tau)| \rho(\tau).$$

Lemma 2.1 ([8, Lemma 4.2]). *Let ρ be an admissible weight function on $\mathbb{R}$. For each $l > 0$ there are constants $0 < m \leq M$ (depending on ρ and l only) such that for each $\sigma \in \mathbb{R}$ and each $\tau \in [\sigma, \sigma + l]$*

$$m\rho(\sigma) \leq \rho(\tau) \leq M\rho(\sigma + l).$$

The translation semigroup $\{T(t)\}_{t \geq 0}$ with parameter $t \in [0, \infty)$ is defined as

$$[T(t)u](\tau) := u(\tau + t) \text{ for } u \in C_{0,\rho}(\mathbb{R}) \text{ or } L_\rho^p(\mathbb{R}).$$

The C_0 -semigroup $\{T(t)\}_{t \geq 0}$ was considered in [8, Section 4] and [3, Section 4].

Theorem 2.2. *Let X be one of the spaces $L_\rho^p(\mathbb{R})$ or $C_{0,\rho}(\mathbb{R})$ with an admissible weight function ρ . Moreover, let $\rho(-\tau)$ be an admissible weight function. Then the translation semigroup $\{T(t)\}_{t \geq 0}$ can be extended to a group on X and*

- (i) $C(t) = \frac{1}{2}(T(t) + T(-t))$ is topologically mixing if and only if

$$\lim_{|t| \rightarrow \infty} \rho(t) = 0,$$

- (ii) the cosine function $C(t) := \frac{1}{2}(T(t) + T(-t))$ is topologically transitive if there exists a sequence $\{t_j\}_{j \geq 0}$ of positive real numbers such that

$$\lim_{j \rightarrow \infty} \rho(t_j) = \lim_{j \rightarrow \infty} \rho(2t_j) = \lim_{j \rightarrow \infty} \rho(-t_j) = \lim_{j \rightarrow \infty} \rho(-2t_j) = 0.$$

Proof. We will prove the case $X = L_\rho^p(\mathbb{R})$. First, we look at (i).

Suppose that $\lim_{|t| \rightarrow \infty} \rho(t) = 0$. It is enough to prove that $\{C(t)\}_{t \geq 0}$ satisfies the condition of Theorem 1.4. Consider the set X_0 of all C^∞ -functions defined on $\mathbb{R}$ with compact support, which is a dense subspace of X . Let us see that $C(t)u$ converges to zero as $t \rightarrow \infty$ for all $u \in X_0$.

Let u be a C^∞ -function whose support is contained in $[-l, l]$, $l > 0$, and let $t > 2l$. Following ideas similar to those in the proof of [8, Theorem 4.8], we have

$$\begin{aligned} \|C(t)u\|_p^p &= \int_{\mathbb{R}} \left| \frac{u(\tau + t) + u(\tau - t)}{2} \right|^p \rho(\tau) d\tau \\ &\leq M \max(\rho(-t), \rho(t)) \int_{-l}^l |u(\tau)|^p d\tau \end{aligned}$$

and

$$\begin{aligned} \|u\|_p^p &= \int_{-l}^l |u(\tau)|^p \rho(\tau) d\tau = \int_{-l}^0 |u(\tau)|^p \rho(\tau) d\tau + \int_0^l |u(\tau)|^p \rho(\tau) d\tau \\ &\geq m_1 \rho(0) \int_0^l |u(-\tau)|^p d\tau + m_2 \rho(0) \int_0^l |u(\tau)|^p d\tau \\ &\geq \min(m_1, m_2) \rho(0) \int_{-l}^l |u(\tau)|^p d\tau. \end{aligned}$$

Hence

$$\|C(t)u\|_p^p \leq M \frac{\max(\rho(-t), \rho(t))}{\min(m_1, m_2) \rho(0)} \|u\|_p^p.$$

Since $\lim_{|t| \rightarrow \infty} \rho(t) = 0$, we have that $\|C(t)u\|_p \rightarrow 0$ as $t \rightarrow \infty$.

Suppose now that $C(t)$ is topologically mixing. Then given $l > 0$ and $f \in X$ with $\text{supp } f \subset [-l, l]$, $f \geq 0$, and $\|f\|_p = 1$, we have $\forall \varepsilon > 0$, $\exists t_\varepsilon > 2l$ such that $\forall t > t_\varepsilon \exists v_t \in X$,

$$\|v_t - f\|_p < \varepsilon \quad \text{and} \quad \|C(t)v_t\|_p < \varepsilon.$$

We define

$$\begin{aligned} v_t^+ &:= \max(0, v_t) \geq 0, \\ w_t &:= v_t^+ \chi_{[-l, l]}. \end{aligned}$$

We obtain that $\|C(t)w_t\|_p < \varepsilon$ and $\|w_t\|_p > 1 - \varepsilon$. Moreover

$$\begin{aligned} \|C(t)w_t\|_p^p &= \left(\frac{1}{2}\right)^p \left(\int_{-l-t}^{l-t} (w_t(s+t))^p \rho(s) ds + \int_{-(l+t)}^{l-t} (w_t(-s-t))^p \rho(-s) ds \right) \\ &\geq (m_2 \rho(-l-t) + m_1 \rho(-(l+t))) \int_{-l}^l |w_t(s)|^p ds \\ &\geq (m_2 m_1 \rho(-t) + m_1 m_2 \rho(t)) \int_{-l}^l |w_t(s)|^p ds \end{aligned}$$

and

$$\begin{aligned} \|w_t\|_p^p &= \int_0^l (w_t(-s))^p \rho(-s) ds + \int_0^l (w_t(s))^p \rho(s) ds \\ &\leq (M_1 \rho(-l) + M_2 \rho(l)) \int_{-l}^l |w_t(s)|^p ds; \end{aligned}$$

see similar ideas in [8, Theorem 4.8]. Hence

$$\varepsilon^p > \|C(t)w_t\|_p^p \geq \frac{m_1^2 \rho(-t) + m_2^2 \rho(t)}{M_1 \rho(-l) + M_2 \rho(l)} \|w_t\|_p^p > \frac{m_1 m_2 (\rho(-t) + \rho(t))}{M_1 \rho(-l) + M_2 \rho(l)} (1 - \varepsilon)^p.$$

Thus, we have

$$\lim_{|t| \rightarrow \infty} \rho(t) = 0.$$

In order to prove (ii), we consider X_1 as the set of all C^∞ -functions defined on $\mathbb{R}$ with compact support, which is a dense subspace of X . By Corollary 1.3, we obtain (ii). $\square$

3. ON THE EXISTENCE OF TOPOLOGICALLY MIXING COSINE FUNCTIONS

We consider the weighted ℓ^1 -space with a weight sequence $\beta = (\beta_i)_{i=1}^\infty$ of positive numbers defined by

$$\ell^1(\beta) := \{(x_i)_{i=1}^\infty : x_i \in \mathbb{C}, \sum_{i=1}^\infty \beta_i |x_i| < \infty\}$$

equipped with the norm

$$\|(x_i)\| := \sum_{i=1}^\infty \beta_i |x_i|.$$

If $\sup_{i \in \mathbb{N}} \frac{\beta_i}{\beta_{i+1}} \leq M$ for some constant M , then the backward shift B defined by

$$B(x_1, x_2, \dots) := (x_2, x_3, \dots)$$

is a bounded linear operator on $\ell^1(\beta)$.

Following some ideas of [3, Lemma 2.1], where it was proved that $\{e^{tB^2}\}_{t \geq 0}$ is topologically mixing in $\ell^1(\beta)$, we will prove that $\{C(t) = \frac{1}{2}(e^{tB} + e^{-tB})\}_{t \geq 0}$ is also a topologically mixing cosine function.

Lemma 3.1. *Let $\beta = (\beta_i)_{i=1}^\infty$ be a sequence of positive numbers and let B be the backward shift operator. If $\sup_{i \in \mathbb{N}} \frac{\beta_i}{\beta_{i+1}} \leq M$ for some constant M , then the uniformly continuous cosine function $\{C(t) = \frac{1}{2}(e^{tB} + e^{-tB})\}_{t \geq 0}$ is topologically mixing in $\ell^1(\beta)$.*

Proof. Consider

$$\varphi := \{(x_i)_{i=1}^\infty \subset \mathbb{C} : \exists m, x_i = 0 \ \forall i > m\},$$

which is dense in $\ell^1(\beta)$. Given any pair $y = (y_i)_{i=1}^\infty$ and $z = (z_i)_{i=1}^\infty$ of elements in φ , and $\varepsilon > 0$, our purpose is to construct vectors $v(t) = (v_i(t))_{i=1}^\infty \in \ell^1(\beta)$, $t \geq 0$, such that there exists $t_0 \geq 0$ with

$$\|v(t) - y\| < \varepsilon \text{ and } \|C(t)v(t) - z\| < \varepsilon, \text{ for all } t > t_0.$$

To do this, we first select and fix $k \in \mathbb{N}$ with

$$y_i = z_i = 0 \text{ for } i \geq 2k.$$

We now define $v(t)$ as follows:

$$v_i(t) = \begin{cases} y_i & \text{for } i = 1, \dots, 2k, \\ 0 & \text{for } i = 2k+1, \dots, 4k \text{ and for } i > 6k. \end{cases}$$

In order to define $v_i(t)$ for $i = 4k+1, \dots, 6k$, we use that

$$(C(t)v(t))_i = \sum_{j=0}^{3k} \frac{t^{2j}}{(2j)!} v_{2j+i}(t) \text{ for } i = 1, \dots, 2k.$$

In fact, we choose $v_i(t)$ for $i = 4k+1, \dots, 6k$ as the solutions of the following system:

$$(3.1) \quad A \begin{pmatrix} y_1 \\ \vdots \\ y_{2k} \end{pmatrix} + D \begin{pmatrix} v_{4k+1}(t) \\ \vdots \\ v_{6k}(t) \end{pmatrix} = \begin{pmatrix} z_1 \\ \vdots \\ z_{2k} \end{pmatrix}$$

where

$$A := \begin{pmatrix} 1 & 0 & \mu_1 & 0 & \mu_2 & 0 & \cdots & 0 & \mu_{k-1} & 0 \\ 0 & 1 & 0 & \mu_1 & 0 & \mu_2 & \cdots & \mu_{k-2} & 0 & \mu_{k-1} \\ \cdot & 0 & 1 & 0 & \mu_1 & 0 & \cdots & 0 & \mu_{k-2} & 0 \\ \cdot & \cdot & 0 & 1 & 0 & \mu_1 & \cdots & \cdot & 0 & \mu_{k-2} \\ \cdot & \cdot & \cdot & 0 & 1 & 0 & \cdots & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & 1 & \cdots & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & \mu_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

with $\mu_i = \frac{t^{2i}}{(2i)!}$ for $i = 1, \dots, k-1$ and

$$D := \begin{pmatrix} \lambda_0 & 0 & \lambda_1 & 0 & \cdots & \lambda_{k-1} & 0 \\ 0 & \lambda_0 & 0 & \lambda_1 & \cdots & 0 & \lambda_{k-1} \\ \lambda_{-1} & 0 & \lambda_0 & 0 & \cdots & \cdot & 0 \\ 0 & \lambda_{-1} & 0 & \lambda_0 & \cdots & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \lambda_{-k+1} & 0 & \cdot & \cdot & \cdots & \lambda_0 & 0 \\ 0 & \lambda_{-k+1} & \cdot & \cdot & \cdots & 0 & \lambda_0 \end{pmatrix}$$

where $\lambda_i = \frac{t^{4k+2i}}{(4k+2i)!}$ for $i = -k+1, \dots, k-1$.

Observe that $D = UWS$ where

$$U = \begin{pmatrix} t^{4k} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & t^{4k} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & t^{4k-2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & t^{4k-2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & t^{2k+2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & t^{2k+2} \end{pmatrix},$$

$$W := \begin{pmatrix} \eta_0 & 0 & \eta_1 & 0 & \cdots & \eta_{k-1} & 0 \\ 0 & \eta_0 & 0 & \eta_1 & \cdots & 0 & \eta_{k-1} \\ \eta_{-1} & 0 & \eta_0 & 0 & \cdots & \cdot & 0 \\ 0 & \eta_{-1} & 0 & \eta_0 & \cdots & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \eta_{-k+1} & 0 & \cdot & \cdot & \cdots & \eta_0 & 0 \\ 0 & \eta_{-k+1} & \cdot & \cdot & \cdots & 0 & \eta_0 \end{pmatrix}$$

with $\eta_i = \frac{1}{(4k+2i)!}$ for $i = -k+1, \dots, k-1$ and

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & t^2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & t^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & t^{2k-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & t^{2k-2} \end{pmatrix}$$

are regular matrices with $2k$ rows and $2k$ columns for $t \neq 0$. Thus the solution of system (3.1) is given by

$$\begin{pmatrix} v_{4k+1}(t) \\ \vdots \\ v_{6k}(t) \end{pmatrix} = S^{-1}W^{-1}U^{-1} \left[\begin{pmatrix} z_1 \\ \vdots \\ z_{2k} \end{pmatrix} - A \begin{pmatrix} y_1 \\ \vdots \\ y_{2k} \end{pmatrix} \right].$$

Let $(w_i)_{i=1}^{2k}$ be defined as

$$\begin{pmatrix} w_1 \\ \vdots \\ w_{2k} \end{pmatrix} = U^{-1} \left[\begin{pmatrix} z_1 \\ \vdots \\ z_{2k} \end{pmatrix} - A \begin{pmatrix} y_1 \\ \vdots \\ y_{2k} \end{pmatrix} \right].$$

Then, if we consider t with $|t| > 1$, there exists C_1 independent of t such that

$$|w_j| \leq C_1 |t|^{-2k-2} \text{ for } j = 1, \dots, 2k.$$

Thus there exists C_2 independent of t satisfying

$$|v_i(t)| \leq C_2 |t|^{2k-i} \text{ for } i = 4k+1, \dots, 6k.$$

Since $v_i(t) = y_i$ for $i \leq 4k$ and $y_i = 0$ for $i > 2k$, we have

$$\|v(t) - y\| = \sum_{i=4k+1}^{6k} \beta_i |v_i(t)| \leq C_2 \sum_{i=4k+1}^{6k} \beta_i |t|^{-2k}.$$

Hence, for sufficiently large $|t|$, we get

$$\|v(t) - y\| < \varepsilon.$$

Moreover

$$\begin{aligned} \|C(t)v(t) - z\| &= \sum_{i=2k+1}^{6k} \beta_i |(C(t)v(t))_i| = \sum_{i=2k+1}^{6k} \beta_i \left| \sum_{j=0}^{2k-1} \frac{t^{2j}}{(2j)!} v_{2j+i}(t) \right| \\ &\leq \sum_{i=2k+1}^{6k} \beta_i \left(\sum_{j=0}^{2k-1} \frac{|t|^{2j}}{(2j)!} |t|^{2k-2j-i} \right) \leq \sum_{i=2k+1}^{6k} \beta_i \left(\sum_{j=0}^{2k-1} \frac{|t|^{-1}}{(2j)!} \right). \end{aligned}$$

Therefore we conclude

$$\|C(t)v(t) - z\| < \varepsilon,$$

if $|t|$ is sufficiently large. $\square$

The proof of the following lemma is similar to [10, Lemma 2.1].

Lemma 3.2. *Let X_1, X_2 be separable Banach spaces, let $\Phi : X_1 \rightarrow X_2$ be a continuous mapping with dense range, and let $\{C(t)\}_{t \geq 0}$ and $\{\tilde{C}(t)\}_{t \geq 0}$ be strongly continuous cosine functions on X_1 and X_2 , respectively, such that $\tilde{C}(t)\Phi = \Phi C(t)$ for all $t \geq 0$. If $\{C(t)\}_{t \geq 0}$ is topologically mixing (topologically transitive, hypercyclic), then $\{\tilde{C}(t)\}_{t \geq 0}$ is also topologically mixing (topologically transitive, hypercyclic).*

The following result due to Ovsepian and Pełczyński [11] is needed to show the existence result.

Theorem 3.3 ([11]). *Let X be a separable infinite dimensional Banach space. Then there exist $(x_n)_{n=1}^\infty \subset X$ and $(f_m)_{m=1}^\infty \subset X^*$, the dual space of X , satisfying the following conditions:*

- (1) $f_m(x_n) = \delta_{m,n}$, $m, n \in \mathbb{N}$.
- (2) $\text{span}\{x_n : n \in \mathbb{N}\} = X$.
- (3) $f_m(x) = 0$, for all $m \in \mathbb{N} \Rightarrow x = 0$.
- (4) $\|x_n\| = 1$, for all $n \in \mathbb{N}$ and $\sup_{m \in \mathbb{N}} \|f_m\| = c < \infty$.

Theorem 3.4. *Every separable infinite dimensional Banach space X admits a topologically mixing uniformly continuous cosine function.*

Proof. Let $(x_n)_{n=1}^\infty \subset X$ and $(f_m)_{m=1}^\infty \subset X^*$ as in Theorem 3.3. Consider the bounded linear operator $S : X \rightarrow X$ defined by

$$(3.2) \quad Sx := \sum_{n=1}^{\infty} \frac{1}{2^n} f_{n+1}(x) x_n.$$

Our purpose is to show that $\{\frac{1}{2}(e^{tS} + e^{-tS})\}_{t \geq 0}$ is a topologically mixing uniformly continuous cosine function.

Define $\Phi : \ell^1 \rightarrow X$, by $\Phi((\alpha_j)_j) := \sum_{j=1}^{\infty} \alpha_j x_j$, which is a bounded linear operator with dense range.

Claim. The cosine function $\frac{1}{2}(e^{t\tilde{S}} + e^{-t\tilde{S}}) : \ell^1 \rightarrow \ell^1$ is topologically mixing, where

$$\tilde{S}((\alpha_j)_j) := \left(\frac{\alpha_2}{2}, \dots, \frac{\alpha_{n+1}}{2^n}, \dots \right).$$

From the definition it follows that $S\Phi = \Phi\tilde{S}$ on ℓ^1 and therefore $\frac{1}{2}(e^{tS} + e^{-tS})\Phi = \Phi\frac{1}{2}(e^{t\tilde{S}} + e^{-t\tilde{S}})$ for all $t \geq 0$. Applying Lemma 3.2, we have that $\{\frac{1}{2}(e^{tS} + e^{-tS})\}_{t \geq 0}$ is topologically mixing in X .

The proof of the claim is similar to the proof of [3, Theorem 2.4] where we use Lemma 3.1 and Lemma 3.2. $\square$

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