SEMICLASSICAL ANALYSIS FOR HIGHLY DEGENERATE POTENTIALS

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ABSTRACT. This paper characterizes the semi-classical limit of the fundamental energy,

$$E(h) := \sigma_1[-h^2\Delta + a(x); \Omega],$$

and ground state ψ_h of the Schrödinger operator $-h^2\Delta + a$ in a bounded domain Ω , in the highly degenerate case when $a \geq 0$ and $a^{-1}(0)$ consists of two components, say $\Omega_{0,1}$ and $\Omega_{0,2}$. The main result establishes that

$$\lim_{h \downarrow 0} \frac{E(h)}{h^2} = \min \{ \sigma_1[-\Delta; \Omega_{0,i}], i = 1, 2 \}$$

and that ψ_h approximates in $H_0^1(\Omega)$ the ground state of $-\Delta$ in $\Omega_{0,i}$ if

$$\sigma_1[-\Delta; \Omega_{0,i}] < \sigma_1[-\Delta; \Omega_{0,j}], \quad j \in \{1, 2\} \setminus \{i\}.$$

1. INTRODUCTION

This paper ascertains the limiting behavior of the lowest eigenvalue and associated ground state of the linear eigenvalue problem

$$(1.1) \quad \begin{cases} (-\Delta + \lambda a(x)) \varphi = \tau \varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

as $\lambda \uparrow \infty$, where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, with boundary $\partial\Omega$ of class $\mathcal{C}^{2,\alpha}$ for some $\alpha \in (0, 1)$, $\lambda \in \mathbb{R}$ is regarded as a parameter, Δ stands for the Laplacian operator in $\mathbb{R}^N$, and $a \in \mathcal{C}^\alpha(\bar{\Omega})$ is a non-negative function satisfying the following hypotheses:

1. The open set $\Omega_+ := \{x \in \Omega : a(x) > 0\}$ is a subdomain of Ω of class $\mathcal{C}^{2,\alpha}$ with $\bar{\Omega}_+ \subset \Omega$.
2. The open set $\Omega_0 := \Omega \setminus \bar{\Omega}_+$ consists of two components, $\Omega_{0,i}$, $i \in \{1, 2\}$, such that $\bar{\Omega}_{0,1} \cap \bar{\Omega}_{0,2} = \emptyset$, $\bar{\Omega}_{0,2} \subset \Omega$, and

$$(1.2) \quad \sigma_1[-\Delta; \Omega_{0,1}] < \sigma_1[-\Delta; \Omega_{0,2}].$$

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For a given subdomain $D \subset \Omega$ and $q \in \mathcal{C}^\alpha(\bar{D})$, we denote by $\sigma_1[-\Delta + q; D]$ the principal eigenvalue of $-\Delta + q$ in D under homogeneous Dirichlet boundary conditions, and consider the function

$$(1.3) \quad \sigma(\lambda) := \sigma_1[-\Delta + \lambda a; \Omega], \quad \lambda \in \mathbb{R}.$$

Also, for each $\lambda \in \mathbb{R}$, $\varphi_\lambda > 0$ stands for the unique eigenfunction associated to $\sigma(\lambda)$ normalized so that

$$(1.4) \quad \int_{\Omega} \varphi_\lambda^2 = 1.$$

For any $x > 0$, let B_x denote the ball of radius x centered at the origin. Then, a genuine situation case for which assumptions 1 and 2 are satisfied occurs if $\Omega = B_R$, $\Omega_{0,1} = B_R \setminus \bar{B}_\rho$, and $\Omega_{0,2} = B_r$, for some $0 < r < \rho < R$ with sufficiently small r . In this case, $\Omega_+ = B_\rho \setminus \bar{B}_r$.

The main goal of this paper is ascertaining the limiting behavior of the eigenpair $(\sigma(\lambda), \varphi_\lambda)$ as $\lambda \uparrow \infty$. Our results do substantially sharpen and considerably tidy up some previous findings going back to [12], [13], and Dancer and López-Gómez [6]. In particular, [6, Theorem 5.1] is substantially deepened and its proof greatly simplified by avoiding the use of some extremely sophisticated devices, coming from Kato [11, & IV.2.4], that were needed in [6] to prove Theorem 5.1 therein. Essentially, the main results of this paper establish that

$$(1.5) \quad \lim_{\lambda \uparrow \infty} \sigma(\lambda) = \sigma_1[-\Delta; \Omega_{0,1}], \quad \lim_{\lambda \uparrow \infty} [\lambda \sigma'(\lambda)] = 0,$$

and that

$$(1.6) \quad \lim_{\lambda \uparrow \infty} \|\varphi_\lambda - \Phi_\omega\|_{H_0^1(\Omega)} = 0,$$

where $\Phi_\omega = 0$ in $\Omega \setminus \bar{\Omega}_{0,1}$ and $\varphi_{0,1} := \Phi_\omega|_{\Omega_{0,1}} > 0$ provides us with the unique principal eigenfunction of $\sigma_1[-\Delta; \Omega_{0,1}]$ such that

$$\int_{\Omega_{0,1}} \varphi_{0,1}^2 = 1.$$

As these findings sharpen Dancer and López-Gómez [6], and the theory of [6] has been used in Donnelly [7] to improve some previous results of Bourgain [3], the analysis carried out in this paper might play a role in the theory of Riemannian manifolds. In quantum mechanics the auxiliary constant

$$h := \lambda^{-1/2}, \quad \lambda > 0,$$

is reminiscent of Planck's constant. In terms of h , we have that

$$(1.7) \quad \begin{cases} (-h^2\Delta + a(x)) \varphi_\lambda = \sigma_1[-h^2\Delta + a; \Omega] \varphi_\lambda & \text{in } \Omega, \\ \varphi_\lambda = 0 & \text{on } \partial\Omega, \end{cases}$$

and in semi-classical analysis all quantum effects are neglected by switching off to zero the Planck constant h . It was the intention of Simon in [16, pp. 296] to study the case when $a(x)$ vanishes on a manifold, though he only observed that if some zero of $a(x)$ does degenerate, then $\sigma_1[-h^2\Delta + a; \Omega]$ might go to zero faster than linearly. As a consequence from (1.5), the fundamental energy must decay quadratically. Precisely,

$$\sigma_1[-h^2\Delta + a; \Omega] = \sigma_1[-\Delta; \Omega_{0,1}]h^2 + o(h^4) \quad \text{as } h \downarrow 0,$$

whereas the fundamental states $\psi_h := \varphi_{1/h^2}$, $h \sim 0$, concentrate within $\Omega_{0,1}$, approximating the ground state of $-\Delta$ in $\Omega_{0,1}$, as $h \downarrow 0$, in the space $H_0^1(\Omega)$, which substantially improves [6, Theorem 5.1]. Though rather natural, the fact that the limiting ground state does concentrate within the component of $a^{-1}(0)$ for which $-\Delta$ has the lowest ground energy seems to be a new result, in the sense that it was never observed, nor conjectured, in the available literature. Similar results are satisfied if, instead of (1.2), the following condition holds:

$$(1.8) \quad \sigma_1[-\Delta; \Omega_{0,1}] > \sigma_1[-\Delta; \Omega_{0,2}];$$

however, in such a case,

$$\lim_{h \downarrow 0} \frac{\sigma_1[-h^2\Delta + a; \Omega]}{h^2} = \sigma_1[-\Delta; \Omega_{0,2}],$$

and the limiting ground state lives in $\Omega_{0,2}$, instead of in $\Omega_{0,1}$. Nevertheless, in the limiting case when

$$\sigma_1[-\Delta; \Omega_{0,1}] = \sigma_1[-\Delta; \Omega_{0,2}],$$

it remains an open problem to ascertain whether or not the limiting ground state does concentrate either in $\Omega_{0,1}$, or in $\Omega_{0,2}$, or in both components simultaneously.

This paper is organized as follows. Section 2 collects some known results that will be used later, as well as a self-contained proof of the fact that $\lambda \mapsto \sigma(\lambda)$ is analytic. Section 3 contains the proof of the second relation of (1.5). Section 4 contains the proof of the first relation of (1.5) as well as the proof of (1.6). Although the first part of (1.5) essentially goes back to López-Gómez [12], [13], where it was proven through an appropriate supersolution, the proof given here can be applied in the absence of a maximum principle, where the method of supersolutions fails.

Throughout this paper, given two Banach spaces U and V and a linear continuous operator $T \in \mathcal{L}(U, V)$, we shall denote by $N[T]$ and $R[T]$ the null space (kernel) and the rank (image) of T , respectively.

2. PRELIMINARIES. CHARACTERIZATION OF THE MAXIMUM PRINCIPLE

This section collects some well known results that are going to be used later. The next theorem is a very classical result, which establishes the existence of the principal eigenvalue and its dominance (see, e.g., Amann [1]).

Theorem 2.1. *There exists a lowest eigenvalue of (1.1), denoted as in (1.3) and called the principal eigenvalue of $-\Delta + \lambda a$ in Ω . Moreover, $\sigma(\lambda)$ is algebraically simple and it possesses a unique positive eigenfunction, normalized so that (1.4) holds; the so-called principal eigenfunction of $-\Delta + \lambda a$ in Ω ; subsequently denoted by φ_λ . Furthermore, φ_λ is strongly positive, $\sigma(\lambda)$ is the unique eigenvalue of (1.1) to a positive eigenfunction, and any other eigenvalue τ of (1.1) satisfies $\operatorname{Re} \tau > \sigma(\lambda)$.*

Next, we recall a pivotal characterization of the maximum principle through the positivity of the principal eigenvalue and the existence of a positive strict supersolution; it goes back to López-Gómez and Molina-Meyer [15] in the context of weakly cooperative systems.

Theorem 2.2. *The following assertions are equivalent:*

$$(a) \quad \sigma_1[-\Delta + \lambda a, \Omega] > 0.$$

(b) *There exists a function $h \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ such that $h > 0$ in Ω , and*

$$((-\Delta + \lambda a)h, h|_{\partial\Omega}) > (0, 0) \quad \text{in } \Omega \times \partial\Omega;$$

h is said to be a positive strict supersolution of $-\Delta + \lambda a$ in Ω .

(c) *The operator $-\Delta + \lambda a$ satisfies the strong maximum principle in Ω .*

From Theorem 2.2, one can easily obtain the properties (a), (b), and (c) of the next proposition.

Proposition 2.3. *Subsequently, $q, q_1, q_2, \dots$ stand for functions of the space $\mathcal{C}^\alpha(\bar{\Omega})$, $\alpha \in (0, 1)$, and $\Omega_1, \Omega_2, \dots$ are subdomains of Ω of class $\mathcal{C}^{2,\alpha}$. Then,*

(a) *$\sigma_1[-\Delta + q_1, \Omega] < \sigma_1[-\Delta + q_2, \Omega]$ if $q_1 < q_2$ (monotonicity with respect to the potential).*

(b) *The mapping $q \mapsto \sigma_1[-\Delta + q, \Omega]$, from $\mathcal{C}^\alpha(\bar{\Omega})$ to $\mathbb{R}$, is continuous (continuity with respect to the potential).*

(c) *$\sigma_1[-\Delta + q, \Omega_1] < \sigma_1[-\Delta + q, \Omega_2]$ whenever Ω_2 is a proper subdomain of Ω_1 (monotonicity with respect to the domain).*

(d) *The map $\sigma(\lambda)$ is concave in λ .*

(e) *There is continuity of $\sigma(\lambda)$ with respect to the domain, in the sense that if Ω_k , $k \geq 1$, is a sequence of subdomains of Ω of class $\mathcal{C}^{2,\alpha}$ such that*

$$\lim_{k \rightarrow \infty} \Omega_k = \Omega_0,$$

e.g., as discussed in [13], then,

$$\lim_{k \rightarrow \infty} \sigma_1[-\Delta + q; \Omega_k] = \sigma_1[-\Delta + q; \Omega_0].$$

(f) *The function $\sigma(\lambda)$ is real analytic in $\mathbb{R}$.*

Except for property (f), all these properties are well known, and detailed proofs of them can be found in the monograph of Hess [9] and in López-Gómez [13]. For the sake of completeness we will give a complete self-contained proof of part (f), for it might be difficult to give a reference including one. Our proof will be based upon [14, Lemma 2.1], which is a perturbation result going back to Crandall and Rabinowitz [4]. Subsequently, we fix $\lambda \in \mathbb{R}$ and consider the operator family

$$\mathfrak{L}(\mu) := -\Delta + \mu a - \sigma(\lambda) : \mathcal{C}^{2,\alpha}(\bar{\Omega}) \rightarrow \mathcal{C}^\alpha(\bar{\Omega}), \quad \mu \in \mathbb{R}.$$

As $\mathfrak{L}(\mu)$ is polynomial in μ , it provides us with a real analytic family in $\mu \in \mathbb{R}$. Moreover,

$$N[\mathfrak{L}(\lambda)] = \text{span}[\varphi_\lambda] \quad \text{and} \quad \varphi_\lambda \notin R[\mathfrak{L}(\lambda)].$$

Indeed, if $\varphi_\lambda \in R[\mathfrak{L}(\lambda)]$, then there exists $u \in \mathcal{C}^{2,\alpha}(\bar{\Omega})$ such that

$$[-\Delta + \lambda a - \sigma(\lambda)]u = \varphi_\lambda.$$

Thus, multiplying this identity by φ_λ , and integrating by parts in Ω yields

$$\int_{\Omega} \varphi_\lambda^2 = \int_{\Omega} \varphi_\lambda [-\Delta + \lambda a - \sigma(\lambda)]u = \int_{\Omega} u [-\Delta + \lambda a - \sigma(\lambda)]\varphi_\lambda = 0,$$

which contradicts (1.4). Therefore, thanks to [14, Lemma 2.1], there exist $\varepsilon > 0$ and an analytic map $(a, \varphi) : (\lambda - \varepsilon, \lambda + \varepsilon) \rightarrow \mathbb{R} \times \mathcal{C}^{2,\alpha}(\bar{\Omega})$, such that

$$(2.1) \quad (a(\lambda), \varphi(\lambda)) = (0, \varphi_\lambda)$$

and, for every $\mu \in (\lambda - \varepsilon, \lambda + \varepsilon)$,

$$(2.2) \quad [-\Delta + \mu a - \sigma(\lambda)]\varphi(\mu) = a(\mu)\varphi(\mu).$$

As $\varphi(\mu) > 0$ for $\mu \sim \lambda$, because φ_λ is strongly positive, we find from the uniqueness of the principal eigenvalue that

$$(2.3) \quad a(\mu) = \sigma_1[-\Delta + \mu a - \sigma(\lambda); \Omega] = \sigma(\mu) - \sigma(\lambda).$$

Consequently, $\sigma(\mu)$ is real analytic as well. This shows Proposition 2.3(f).

3. ASYMPTOTIC BEHAVIOR OF $\sigma(\lambda)$ AS $\lambda \uparrow \infty$

As the map $\lambda \mapsto \sigma(\lambda)$ is increasing and for any λ we have that $\sigma(\lambda) < \sigma_1[-\Delta; \Omega_{0,1}]$, because $a = 0$ in $\Omega_{0,1}$, the following limit is well defined:

$$\ell := \lim_{\lambda \uparrow \infty} \sigma(\lambda) \leq \sigma_1[-\Delta; \Omega_{0,1}].$$

In particular, $\lim_{\lambda \uparrow \infty} \sigma'(\lambda) = 0$, where we are denoting $\sigma'(\lambda) = \frac{d\sigma}{d\lambda}(\lambda)$. The following result is much sharper.

Theorem 3.1. *The principal eigenvalue $\sigma(\lambda)$ satisfies*

$$(3.1) \quad \lim_{\lambda \uparrow \infty} [\lambda \sigma'(\lambda)] = 0.$$

In other words, $\sigma'(\lambda) = o(\lambda^{-1})$ as $\lambda \uparrow \infty$.

Proof. Pick a $\lambda \in \mathbb{R}$ and let $\varphi(\mu)$, $\mu \sim \lambda$, denote the unique map satisfying (2.2). Then, by (2.2) and (2.3), we have that

$$(3.2) \quad -\Delta \varphi(\mu) + \mu a \varphi(\mu) = \sigma(\mu) \varphi(\mu), \quad \mu \sim \lambda,$$

and, hence, differentiating with respect to μ shows that

$$(3.3) \quad (-\Delta + \mu a - \sigma(\mu)) \varphi'(\mu) = (\sigma'(\mu) - a) \varphi(\mu),$$

where $' = \frac{d}{d\mu}$. Thus, multiplying (3.3) by $\varphi(\mu)$ and integrating in Ω gives

$$\begin{aligned} \int_{\Omega} (\sigma'(\mu) - a) \varphi^2(\mu) &= \int_{\Omega} \varphi(\mu) (-\Delta + \mu a - \sigma(\mu)) \varphi'(\mu) \\ &= \int_{\Omega} \varphi'(\mu) (-\Delta + \mu a - \sigma(\mu)) \varphi(\mu) = 0 \end{aligned}$$

and, therefore,

$$\int_{\Omega} (\sigma'(\mu) - a) \varphi^2(\mu) = 0, \quad \mu \sim \lambda.$$

In particular, by (1.4) and (2.1), we find that

$$(3.4) \quad \sigma'(\lambda) = \frac{\int_{\Omega} a \varphi^2(\lambda)}{\int_{\Omega} \varphi^2(\lambda)} = \frac{\int_{\Omega} a \varphi_{\lambda}^2}{\int_{\Omega} \varphi_{\lambda}^2} = \int_{\Omega} a \varphi_{\lambda}^2.$$

Note that, in general, we cannot guarantee that $\int_{\Omega} \varphi^2(\mu) = 1$, unless $\mu = \lambda$.

On the other hand, multiplying

$$-\Delta \varphi_{\lambda} + \lambda a \varphi_{\lambda} = \sigma(\lambda) \varphi_{\lambda},$$

by φ_{λ} and integrating by parts in Ω , we find that

$$\int_{\Omega} |\nabla \varphi_{\lambda}|^2 + \int_{\Omega} \lambda a \varphi_{\lambda}^2 = \sigma(\lambda) \int_{\Omega} \varphi_{\lambda}^2 = \sigma(\lambda)$$

and, consequently, (3.4) implies

$$(3.5) \quad \int_{\Omega} |\nabla \varphi_{\lambda}|^2 + \lambda \sigma'(\lambda) = \sigma(\lambda).$$

As $\sigma(\lambda)$ is bounded above by $\sigma_1[-\Delta; \Omega_{0,1}]$ and $\sigma' \geq 0$, it follows from (3.5) that there exists a constant $K > 0$ such that

$$(3.6) \quad \int_{\Omega} |\nabla \varphi_{\lambda}|^2 \leq K, \quad \lambda \sigma'(\lambda) \leq K, \quad \text{for all } \lambda > 0.$$

Now, we claim that

$$(3.7) \quad \liminf_{\lambda \rightarrow \infty} [\lambda \sigma'(\lambda)] = 0.$$

Our proof of (3.7) proceeds by contradiction. Suppose (3.7) fails. Then, there exist $\lambda_0 > 0$ and $\alpha > 0$ such that

$$\lambda \sigma'(\lambda) > \alpha > 0 \quad \text{if } \lambda > \lambda_0,$$

and, hence, integrating in (λ_0, λ) we find that

$$\sigma(\lambda) \geq \sigma(\lambda_0) + \alpha \log \frac{\lambda}{\lambda_0}$$

and, consequently,

$$\lim_{\lambda \uparrow \infty} \sigma(\lambda) = \infty$$

which contradicts the fact that $\sigma(\lambda)$ is bounded above by $\sigma_1[-\Delta; \Omega_{0,1}]$, and concludes the proof of (3.7).

Note that, as a by-product from the identity (3.5), we also find that the map $\lambda \mapsto \int_{\Omega} |\nabla \varphi_{\lambda}|^2$ is real analytic, and, hence, by a further differentiation with respect to λ , we are driven to the identity

$$(3.8) \quad \frac{d}{d\lambda} \int_{\Omega} |\nabla \varphi_{\lambda}|^2 + \lambda \sigma''(\lambda) = 0.$$

According to Proposition 2.3(d), we already know that $\sigma(\lambda)$ is concave (a celebrated result of Kato [10]) and, hence, $\sigma''(\lambda) \leq 0$ for every $\lambda \in \mathbb{R}$. Thus, it follows from (3.8) that, for every $\lambda \geq 0$,

$$(3.9) \quad \frac{d}{d\lambda} \int_{\Omega} |\nabla \varphi_{\lambda}|^2 = -\lambda \sigma''(\lambda) \geq 0.$$

Hence, the map

$$\lambda \mapsto \int_{\Omega} |\nabla \varphi_{\lambda}|^2, \quad \lambda \geq 0,$$

is non-decreasing and, due to (3.6), bounded above. Therefore, the limit

$$d := \lim_{\lambda \uparrow \infty} \int_{\Omega} |\nabla \varphi_{\lambda}|^2$$

is well defined. Consequently, we also obtain from (3.5) that the following limit is well defined:

$$\lim_{\lambda \uparrow \infty} [\lambda \sigma'(\lambda)] = \ell - d.$$

Necessarily, due to (3.7), $\lim_{\lambda \uparrow \infty} [\lambda \sigma'(\lambda)] = 0$, and, hence, $\lim_{\lambda \uparrow \infty} \int_{\Omega} |\nabla \varphi_{\lambda}|^2 = \ell$. The proof is completed. $\square$

4. LIMITING BEHAVIOR OF THE GROUND STATE

The following theorem is the main result of this paper. Remember that, for every $\lambda \in \mathbb{R}$, $\varphi_\lambda > 0$ stands for the principal eigenfunction of $\sigma(\lambda)$ normalized by (1.4).

Theorem 4.1. *Let $\varphi_{0,1} \in H_0^1(\Omega_{0,1})$ be the unique principal eigenfunction associated to $\sigma[-\Delta; \Omega_{0,1}]$ such that*

$$\int_{\Omega_{0,1}} \varphi_{0,1}^2 = 1,$$

and set

$$(4.1) \quad \Phi_\omega := \begin{cases} \varphi_{0,1} & \text{in } \Omega_{0,1}, \\ 0 & \text{in } \Omega \setminus \Omega_{0,1}. \end{cases}$$

Then,

$$(4.2) \quad \lim_{\lambda \uparrow \infty} \sigma_1[-\Delta + \lambda a; \Omega] = \sigma_1[-\Delta; \Omega_{0,1}]$$

and

$$(4.3) \quad \lim_{\lambda \uparrow \infty} \|\varphi_\lambda - \Phi_\omega\|_{H_0^1(\Omega)} = 0.$$

Proof. Let $\{\lambda_n\}_{n \geq 1}$ be an increasing unbounded sequence, i.e., such that $0 < \lambda_n < \lambda_m$ if $n < m$ and $\lim_{n \rightarrow \infty} \lambda_n = \infty$. Then, for every $n \geq 1$, $\varphi_{\lambda_n} > 0$ is the unique principal eigenfunction associated to $\sigma(\lambda_n)$ such that

$$(4.4) \quad \int_{\Omega} \varphi_{\lambda_n}^2 = 1,$$

by definition. It follows from

$$\int_{\Omega} |\nabla \varphi_{\lambda_n}|^2 + \lambda_n \int_{\Omega} a \varphi_{\lambda_n}^2 = \sigma(\lambda_n), \quad n \geq 1,$$

that $\{\varphi_{\lambda_n}\}_{n \geq 1}$ is bounded in $H_0^1(\Omega)$. As the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, there exist a subsequence of $\{\lambda_n\}_{n \geq 1}$, again labeled by n , and a $\varphi_\omega \in L^2(\Omega)$ such that

$$\lim_{n \rightarrow \infty} \|\varphi_{\lambda_n} - \varphi_\omega\|_{L^2(\Omega)} = 0.$$

Next, we will prove that $\{\varphi_{\lambda_n}\}_{n \geq 1}$ is a Cauchy sequence in $H_0^1(\Omega)$. This implies $\varphi_\omega \in H_0^1(\Omega)$ and

$$(4.5) \quad \lim_{n \rightarrow \infty} \|\varphi_{\lambda_n} - \varphi_\omega\|_{H_0^1(\Omega)} = 0.$$

Indeed, for every $n < m$, we have that $0 < \lambda_n < \lambda_m$ and

$$\begin{aligned} \int_{\Omega} |\nabla(\varphi_{\lambda_n} - \varphi_{\lambda_m})|^2 &= \int_{\Omega} |\nabla \varphi_{\lambda_n}|^2 + \int_{\Omega} |\nabla \varphi_{\lambda_m}|^2 - 2 \int_{\Omega} \langle \nabla \varphi_{\lambda_n}, \nabla \varphi_{\lambda_m} \rangle \\ &= \int_{\Omega} (\sigma_1[-\Delta + \lambda_n a; \Omega] \varphi_{\lambda_n} - \lambda_n a \varphi_{\lambda_n}) \varphi_{\lambda_n} \\ &\quad + \int_{\Omega} (\sigma_1[-\Delta + \lambda_m a; \Omega] \varphi_{\lambda_m} - \lambda_m a \varphi_{\lambda_m}) \varphi_{\lambda_m} \\ &\quad - 2 \int_{\Omega} (\sigma_1[-\Delta + \lambda_n a; \Omega] \varphi_{\lambda_n} - \lambda_n a \varphi_{\lambda_n}) \varphi_{\lambda_m}. \end{aligned}$$

Hence, rearranging terms, we find that

$$\begin{aligned} \int_{\Omega} |\nabla(\varphi_{\lambda_n} - \varphi_{\lambda_m})|^2 &= \sigma_1[-\Delta + \lambda_n a; \Omega] \int_{\Omega} \varphi_{\lambda_n} (\varphi_{\lambda_n} - \varphi_{\lambda_m}) \\ &\quad + \sigma_1[-\Delta + \lambda_m a; \Omega] \int_{\Omega} \varphi_{\lambda_m} (\varphi_{\lambda_m} - \varphi_{\lambda_n}) \\ &\quad + (\sigma_1[-\Delta + \lambda_m a; \Omega] - \sigma_1[-\Delta + \lambda_n a; \Omega]) \int_{\Omega} \varphi_{\lambda_n} \varphi_{\lambda_m} \\ &\quad - \lambda_n \int_{\Omega} a \varphi_{\lambda_n}^2 - \lambda_m \int_{\Omega} a \varphi_{\lambda_m}^2 + 2\lambda_n \int_{\Omega} a \varphi_{\lambda_n} \varphi_{\lambda_m}. \end{aligned}$$

Moreover,

$$\begin{aligned} -\lambda_n \int_{\Omega} a \varphi_{\lambda_n}^2 - \lambda_m \int_{\Omega} a \varphi_{\lambda_m}^2 + 2\lambda_n \int_{\Omega} a \varphi_{\lambda_n} \varphi_{\lambda_m} \\ = -\lambda_n \int_{\Omega} a (\varphi_{\lambda_n} - \varphi_{\lambda_m})^2 + (\lambda_n - \lambda_m) \int_{\Omega} a \varphi_{\lambda_m}^2 < 0, \end{aligned}$$

since $0 < \lambda_n < \lambda_m$, and, consequently, we find that

$$\begin{aligned} \int_{\Omega} |\nabla(\varphi_{\lambda_n} - \varphi_{\lambda_m})|^2 &\leq \sigma(\lambda_n) \int_{\Omega} \varphi_{\lambda_n} (\varphi_{\lambda_n} - \varphi_{\lambda_m}) \\ &\quad + \sigma(\lambda_m) \int_{\Omega} \varphi_{\lambda_m} (\varphi_{\lambda_m} - \varphi_{\lambda_n}) \\ &\quad + (\sigma(\lambda_m) - \sigma(\lambda_n)) \int_{\Omega} \varphi_{\lambda_n} \varphi_{\lambda_m}. \end{aligned} \tag{4.6}$$

Now, according to the Hölder inequality and the monotonicity of the principal eigenvalue with respect to the domain, it follows from (1.4) that

$$\begin{aligned} \sigma(\lambda_n) \int_{\Omega} \varphi_{\lambda_n} (\varphi_{\lambda_n} - \varphi_{\lambda_m}) &\leq \sigma_1[-\Delta; \Omega_{0,1}] \|\varphi_{\lambda_n} - \varphi_{\lambda_m}\|_{L^2(\Omega)}, \\ \sigma(\lambda_m) \int_{\Omega} \varphi_{\lambda_m} (\varphi_{\lambda_m} - \varphi_{\lambda_n}) &\leq \sigma_1[-\Delta; \Omega_{0,1}] \|\varphi_{\lambda_m} - \varphi_{\lambda_n}\|_{L^2(\Omega)}, \end{aligned}$$

and

$$(\sigma(\lambda_m) - \sigma(\lambda_n)) \int_{\Omega} \varphi_{\lambda_n} \varphi_{\lambda_m} \leq \sigma(\lambda_m) - \sigma(\lambda_n),$$

since $0 < \sigma(\lambda_n) < \sigma(\lambda_m)$. We already know that $\{\varphi_{\lambda_n}\}_{n \geq 1}$ is a Cauchy sequence in $L^2(\Omega)$. Moreover, the numerical sequence $\{\sigma(\lambda_n)\}_{n \geq 1}$ is convergent, because it is monotone and bounded above by $\sigma_1[-\Delta; \Omega_{0,1}]$. Thus, owing to these estimates, we find from (4.6) that $\{\varphi_{\lambda_n}\}_{n \geq 1}$ is a Cauchy sequence in $H_0^1(\Omega)$ and, therefore, $\varphi_{\omega} \in H_0^1(\Omega)$ and (4.5) holds. Note that, moreover,

$$\varphi_{\omega} \geq 0 \quad \text{and} \quad \int_{\Omega} \varphi_{\omega}^2 = 1. \tag{4.7}$$

Next, we show that

$$\varphi_{\omega} = 0 \quad \text{in} \quad \Omega_+ = \{x \in \Omega : a(x) > 0\}. \tag{4.8}$$

Indeed, by (3.4),

$$\sigma'(\lambda_n) = \int_{\Omega} a \varphi_{\lambda_n}^2, \quad n \geq 1,$$

and, hence, (3.1) implies

$$\lim_{n \rightarrow \infty} \int_{\Omega} a \varphi_{\lambda_n}^2 = 0.$$

On the other hand, by Hölder inequality, we find from (1.4) and (4.7) that

$$\begin{aligned} \left| \int_{\Omega} a \varphi_{\lambda_n}^2 - \int_{\Omega} a \varphi_{\omega}^2 \right| &\leq \int_{\Omega} a(\varphi_{\lambda_n} + \varphi_{\omega}) |\varphi_{\lambda_n} - \varphi_{\omega}| \\ &\leq \|a\|_{\infty} \|\varphi_{\lambda_n} - \varphi_{\omega}\|_{L^2(\Omega)} \left(\int_{\Omega} \varphi_{\lambda_n}^2 + \int_{\Omega} \varphi_{\omega}^2 + 2 \int_{\Omega} \varphi_{\lambda_n} \varphi_{\omega} \right)^{\frac{1}{2}} \\ &\leq 2 \|a\|_{\infty} \|\varphi_{\lambda_n} - \varphi_{\omega}\|_{L^2(\Omega)} \end{aligned}$$

and, hence, we conclude that

$$0 = \int_{\Omega} a \varphi_{\omega} = \int_{\Omega_+} a \varphi_{\omega},$$

which implies (4.8). Subsequently, we will check that (4.8) implies

$$(4.9) \quad \varphi_{\omega}|_{\Omega_{0,i}} \in H_0^1(\Omega_{0,i}), \quad i \in \{1, 2\}.$$

For every $i \in \{1, 2\}$ and sufficiently small $\delta > 0$, consider the open set

$$\Omega_{\delta,i} := \{x \in \Omega : \text{dist}(x, \partial\Omega_{0,i}) < \delta\}.$$

According to (4.8), we have that $\varphi_{\omega}|_{\Omega_{\delta,i}} \in H_0^1(\Omega_{\delta,i})$, $i \in \{1, 2\}$, and, hence, there exists $\delta_0 > 0$ such that

$$\varphi_{\omega} \in \bigcap_{0 < \delta < \delta_0} H_0^1(\Omega_{\delta,i}), \quad i \in \{1, 2\}.$$

On the other hand, as we are assuming that $\Omega_{0,1}$ and $\Omega_{0,2}$ are smooth subdomains of Ω , they are stable in the sense of Babuska and V'ýborný [2] (cf. López-Gómez [13]), and, therefore,

$$H_0^1(\Omega_{0,i}) = \bigcap_{0 < \delta < \delta_0} H_0^1(\Omega_{\delta,i}), \quad i \in \{1, 2\},$$

which concludes the proof of (4.9).

Subsequently, we should recall that, by monotonicity, the limit

$$0 < \ell := \lim_{\lambda \uparrow \infty} \sigma(\lambda) = \lim_{n \rightarrow \infty} \sigma(\lambda_n) \leq \sigma[-\Delta; \Omega_{0,1}]$$

is well defined. Pick $i \in \{1, 2\}$ and a test function $\psi \in \mathcal{C}_0^{\infty}(\Omega_{0,i})$. Then, multiplying the corresponding differential equation by ψ , integrating in $\Omega_{0,i}$ and applying the formula of integration by parts, we find that

$$\int_{\Omega_{0,i}} \langle \nabla \psi, \nabla \varphi_{\lambda_n} \rangle = \sigma(\lambda_n) \int_{\Omega_{0,i}} \psi \varphi_{\lambda_n}, \quad n \geq 1,$$

since $a = 0$ in $\bar{\Omega}_{0,i}$. Thus, by (4.5), passing to the limit as $n \rightarrow \infty$ in these identities shows that

$$\int_{\Omega_{0,i}} \langle \nabla \psi, \nabla \varphi_{\omega} \rangle = \ell \int_{\Omega_{0,i}} \psi \varphi_{\omega}.$$

Consequently, for each $i \in \{1, 2\}$,

$$\varphi_{\omega}|_{\Omega_{0,i}} \in H_0^1(\Omega_{0,i}), \quad \varphi_{\omega}|_{\Omega_{0,i}} \geq 0,$$

provides us with a weak solution of the linear boundary value problem

$$(4.10) \quad \begin{cases} -\Delta u = \ell u & \text{in } \Omega_{0,i}, \\ u = 0 & \text{on } \partial\Omega_{0,i}. \end{cases}$$

By elliptic regularity, for each $i \in \{1, 2\}$, $\varphi_\omega|_{\Omega_{0,i}} \in C^{2,\alpha}(\bar{\Omega}_{0,i})$ provides us with a classical solution of (4.10). By (1.2), we have that

$$\ell < \sigma_1[-\Delta; \Omega_{0,2}]$$

and, hence, according to Theorem 2.1, we find that

$$(4.11) \quad \varphi_\omega = 0 \quad \text{in } \Omega_{0,2}.$$

Moreover, necessarily $\ell = \sigma_1[-\Delta; \Omega_{0,1}]$, and, hence, (4.2) holds. Indeed, if $\ell < \sigma_1[-\Delta; \Omega_{0,1}]$, then it follows from Theorem 2.1 that $\varphi_\omega = 0$ in $\Omega_{0,1}$ and, therefore, $\varphi_\omega = 0$ in Ω , which contradicts (4.7). Consequently, (4.2) is satisfied,

$$\varphi_\omega|_{\Omega_{0,1}} > 0, \quad \int_{\Omega_{0,1}} \varphi_\omega^2 = 1,$$

and, necessarily, $\varphi_\omega = \varphi_{0,1}$ in $\Omega_{0,1}$, where $\varphi_{0,1}$ is the principal eigenfunction associated to $\sigma_1[-\Delta; \Omega_{0,1}]$ that was introduced in the statement of the theorem. Consequently, we conclude from (4.8) and (4.11) that $\Phi_\omega = \varphi_\omega$. As the previous convergence holds along any increasing sequence $\{\lambda_n\}_{n \geq 1}$, the proof is completed. $\square$

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