SMALL VOLUME ON BIG n -SPHERES

CHRISTOPHER B. CROKE

(Communicated by Jon G. Wolfson)

ABSTRACT. We consider Riemannian metrics on the n -sphere for $n \geq 3$ such that the distance between any pair of antipodal points is bounded below by 1. We show that the volume can be arbitrarily small. This is in contrast to the 2-dimensional case where Berger has shown that $\text{Area} \geq 1/2$.

In 1977, Berger [B77] considered the set of Riemannian metrics g on the n -sphere such that $d(x, A(x)) \geq 1$ and defined the constants:

$$h(n) \equiv \inf\{\text{Vol}(g)\}.$$

We will also consider $\bar{h}(n) \equiv \inf\{\text{Vol}(g) | A^*g = g\}$, where you can think of A as the standard antipodal map or simply set A to be any fixed point free diffeomorphism of S^n with $A^2 = Id$.

One of Berger's reasons for his interest in these invariants was their use in getting lower bounds on the volume of balls $B(p, R)$ in a complete Riemannian manifold with R less than half the injectivity radius $\text{inj}(M)$. In this case all geodesics in the ball minimize distance. Hence for all $r \leq R$ the distance in $B(p, R)$ between antipodal points of $S(p, r) = \partial B(p, r)$ is $2r$, and thus the intrinsic distance in $S(p, r)$ is even larger. So we get $\text{Vol}(S(p, r)) \geq h(n-1)(2r)^{n-1}$ and

$$\begin{aligned} \text{Vol}(B(p, R)) &= \int_0^R \text{Vol}(S(p, r)) dr \\ &\geq \int_0^R h(n-1)(2r)^{n-1} dr = 2^{n-1} \frac{h(n-1)}{n} R^n. \end{aligned}$$

Berger noted that $h(1) = 2$ and proved $h(2) \geq \frac{1}{2}$ to get a (nonsharp) lower bound $\text{Vol}(B(p, R)) \geq 2R^2$ on the volume of 2-balls and $\text{Vol}(B(p, R)) \geq \frac{2}{3}R^3$ for 3-balls when $R \leq \text{inj}(M)/2$.

In this short note we show by relatively easy examples:

Theorem 0.1. $h(n) = 0$ for $n \geq 3$.

This question and some history surrounding it was also discussed in section 6 of [CK03]. In particular, Ivanov (see [I98]) has given examples of a sequence of metrics on S^3 that Gromov-Hausdorff converge to the standard metric but whose volumes go to zero.

Received by the editors November 9, 2006.

2000 *Mathematics Subject Classification.* Primary 53C20.

The author was supported by NSF grants DMS 02-02536 and 07-04145.

©2007 American Mathematical Society
 Reverts to public domain 28 years from publication

The main open question about volume of balls can be stated as a conjecture which is open in all dimensions (even $n = 2!$).

Conjecture 0.2. *If ω_n represents the volume of the unit n -sphere:*

- a) *For any $r \leq \frac{\text{inj}(M)}{2}$, $\text{Vol}(B(r)) \geq \frac{\omega_n}{2}(\frac{2}{\pi})^n r^n$ with equality holding only if the ball is isometric to a hemisphere.*
 b) *For any $r \leq \text{inj}(M)$, $\text{Vol}(B(r)) \geq \frac{\omega_n}{\pi^n} r^n$, where equality holds if and only if M is isometric to the round sphere of injectivity radius r (i.e. extrinsic radius $\frac{r}{\pi}$).*

As mentioned above, Berger [B77] gave nonsharp lower bounds for $n = 2$ and $n = 3$. The author gave nonsharp lower bounds in all dimensions in [Cr80] and showed that b) was true for the “average” ball (i.e. $\frac{1}{\text{Vol}(M)} \int_M \text{Vol}(B(x, r)) dx \geq \frac{\omega_n}{\pi^n} r^n$) for any compact manifold M in any dimension [Cr84]. This followed Berger’s isoeimboic inequality $\text{Vol}(M) \geq \frac{\omega_n}{\pi^n} \text{inj}(M)^n$, where equality holds if and only if M is isometric to the round sphere.

As to the question of the exact value of $h(2)$ we have the following interesting

Conjecture 0.3. $h(2) = \frac{4}{\pi}$, and the round sphere is the only space that achieves this.

The situation for $\bar{h}$ is quite different. By identifying antipodal points we get a metric on $\mathbb{R}P^n$ such that all noncontractible closed curves have length at least 1 (i.e. $\text{sys}(M) \geq 1$). Hence Pu’s Theorem [Pu52] shows $\bar{h}(2) = \frac{4}{\pi}$, while Gromov’s theorem [Gr83] gives us nonsharp positive lower bounds for $\bar{h}(n)$ for all n . The question as to the actual values of $\bar{h}(n)$ for $n \geq 3$ is a hard and very interesting one.

One other place where this notion came up is in [Cr02]. There it is shown that for a Riemannian metric on a 3-sphere with $d(x, A(x)) \geq D$ and $L = \text{length of the shortest closed geodesic}$, we have $\text{Vol}^{1/3} \geq \text{const} \min\{L, 2D\}$. In this note we show that there is no constant such that $\text{Vol}^{1/3} \geq \text{const} D$. It is still an open (and very interesting) question if there is a constant so that $\text{Vol}^{1/3} \geq \text{const} L$.

We now construct the examples on the 3-sphere.

We will let MB_ϵ be the ϵ neighborhood in the flat plane $\mathbb{R}^2$ of a tripod (three unit length line segments from the origin making angles $\frac{2\pi}{3}$ with each other). As ϵ goes to 0 the area goes to 0 while the length of the boundary goes to 6. This has been known for a while as an example of a metric on a 2-disc whose area goes to 0 but such that on the boundary antipodal points satisfy $d_{MB_\epsilon}(p, A_\epsilon(p)) \geq d_{\mathbb{R}^2}(p, A_\epsilon(p)) \geq 1$. Here $A_\epsilon(p)$ is the point half way around the boundary. (Note that the nonconvexity of MB_ϵ is not the point here since one could use equilateral triangles in a more and more negatively curved simply connected space form.)

Our examples are just $M_\epsilon^3 = \partial(MB_\epsilon \times MB_\epsilon)$, which as the boundary of a 4-ball is a 3-sphere. Since $M_\epsilon = (\partial(MB_\epsilon) \times MB_\epsilon) \cup (MB_\epsilon \times \partial(MB_\epsilon))$, $\text{Vol}(M_\epsilon) = 2 \text{Area}(MB_\epsilon)L(\partial(MB_\epsilon))$. Hence $\lim_{\epsilon \rightarrow 0} \text{Vol}(M_\epsilon) = 0$.

We need to define the antipodal map $\bar{A}_\epsilon$ on M_ϵ . First extend A_ϵ to a continuous homeomorphism $A_\epsilon : MB_\epsilon \rightarrow MB_\epsilon$ with a single fixed point (the origin in $\mathbb{R}^2$ say). One can do this simply by mapping the line segment between the origin and $p \in \partial(MB_\epsilon)$ to the line segment between the origin and $A_\epsilon(p)$. Then let

$$\bar{A}_\epsilon = (A_\epsilon \times A_\epsilon)|_{M_\epsilon}.$$

For a point $(p, q) \in \partial(MB_\epsilon) \times MB_\epsilon \subset M_\epsilon$ we have

$$\begin{aligned} d_{M_\epsilon}((p, q), \bar{A}_\epsilon(p, q)) &= d_{M_\epsilon}((p, q), (A_\epsilon(p), A_\epsilon(q))) \\ &\geq d_{\mathbb{R}^4}((p, q), (A_\epsilon(p), A_\epsilon(q))) \geq d_{\mathbb{R}^2}(p, A_\epsilon(p)) \geq 1. \end{aligned}$$

Similarly if $(p, q) \in MB_\epsilon \times \partial(MB_\epsilon) \subset M_\epsilon$, then $d_{M_\epsilon}((p, q), \bar{A}_\epsilon(p, q)) \geq 1$.

This completes the proof of the theorem in the 3-dimensional case after smoothing the metric and $\bar{A}_\epsilon$. The higher-dimensional cases are handled by spherical suspension. Specifically given a metric g on the n -sphere S^n we construct a (singular) warped product metric $g' = \cos^2(t)g + dt^2$ on the $(n+1)$ -sphere $S^{n+1} = (S^n \times [-\frac{\pi}{2}, \frac{\pi}{2}]) / \sim$, where $(x, \frac{\pi}{2}) \sim (y, \frac{\pi}{2})$ and $(x, -\frac{\pi}{2}) \sim (y, -\frac{\pi}{2})$ for all $x, y \in S^n$. The antipodal map $A_{n+1} : S^{n+1} \rightarrow S^{n+1}$ is just $A_{n+1}(x, t) = (A_n(x), -t)$. It is easy to check that if g has $d(x, A_n(x)) \geq \pi$ and $\text{Vol}(g) = V$, then $D(x, A_{n+1}(x)) \geq \pi$ and $\text{Vol}(g') = \text{Vol}(g) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n(t) dt = \frac{\omega_{n+1}}{\omega_n} \text{Vol}(g)$. So spherically suspending (and smoothing) a sequence of examples that shows $h(n) = 0$ gives us a sequence showing that $h(n+1) = 0$.

REFERENCES

- [B77] M. Berger, *Volume et rayon d'injectivité dans les variétés riemanniennes de dimension 3*, Osaka J. Math., **14** (1977), 191-200. MR0467595 (57:7451)
- [Cr80] C. Croke, *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Scient. Ec. Norm. Sup., 4e série, t.**13** (1980), 419-435. MR608287 (83d:58068)
- [Cr84] C. Croke, *Curvature Free Volume Estimates*, Inventiones Mathematicae **76** (1984), 515-521. MR746540 (85f:53044)
- [Cr02] C. Croke, *The volume and lengths on a three sphere*, Comm. Anal. Geom., **10** (2002) no. 3, 467-474. MR1912255 (2003c:53041)
- [CK03] Croke, C.; Katz, M., Universal volume bounds in Riemannian manifolds, *Surveys in Differential Geometry VIII*, Lectures on Geometry and Topology held in honor of Calabi, Lawson, Siu, and Uhlenbeck at Harvard University, May 3-5, 2002, edited by S.T. Yau (Somerville, MA: International Press, 2003.) pp. 109-137. See [arXiv:math.DG/0302248](#) MR2039987 (2005d:53061)
- [Gr83] Gromov, M., Filling Riemannian manifolds, *J. Diff. Geom.* **18** (1983), 1-147. MR697984 (85h:53029)
- [I98] S. Ivanov, *Gromov-Hausdorff Convergence and volumes of manifolds*, St. Petersburg Math. J., **9**(1998) No.5, 945-959 MR1604389 (98k:53052)
- [Pu52] Pu, P.M., Some inequalities in certain nonorientable Riemannian manifolds, *Pacific J. Math.* **2** (1952), 55-71. MR0048886 (14:87e)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PENNSYLVANIA 19104-6395

E-mail address: ccroke@math.upenn.edu