DIRECTIONAL DERIVATIVE ESTIMATES FOR BEREZIN'S OPERATOR CALCULUS

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ABSTRACT. Directional derivative estimates for Berezin symbols of bounded operators on Bergman spaces of arbitrary bounded domains Ω in $\mathbb{C}^n$ are obtained. These estimates also hold in the setting of the Segal-Bargmann space on $\mathbb{C}^n$. It is also shown that our estimates are sharp at every point of Ω by exhibiting the optimizers explicitly.

1. INTRODUCTION

For Ω any bounded domain in $\mathbb{C}^n$ with Lebesgue measure dv , we consider the set $L^2(\Omega, dv)$ of square-integrable complex-valued functions defined on Ω , which forms a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{\Omega} f(z) \bar{g}(z) dv(z)$$

for $f, g \in L^2(\Omega, dv)$. It is well-known that the subset of holomorphic functions in $L^2(\Omega, dv)$ is a closed subspace, so it is also a Hilbert space with the inherited inner product. This is the Bergman space, $A^2(\Omega, dv)$. Moreover, $A^2(\Omega, dv)$ is a reproducing-kernel space with the Bergman kernel $K(z, w)$ defined on $\Omega \times \Omega$, which is holomorphic in z and conjugate-holomorphic in w and has the reproducing property

$$f(z) = \int_{\Omega} f(w) K(z, w) dv(w)$$

for $f \in A^2(\Omega, dv)$. Similarly for $\mathbb{C}^n$, we take the Gaussian measure

$$d\mu(z) = (2\pi)^{-n} e^{-\frac{|z|^2}{2}} dv(z),$$

and the corresponding Lebesgue space $L^2(\mathbb{C}^n, d\mu)$ and the Segal-Bargmann subspace $H^2(\mathbb{C}^n, d\mu)$ of holomorphic functions are defined in the same manner. Here the Bergman kernel is given by $K(z, w) = e^{\frac{\langle z, w \rangle}{2}}$ where $\langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j$.

It is known that the Bergman kernel function induces a (positive-definite) Riemannian metric on Ω or $\mathbb{C}^n$. The infinitesimal Bergman metric is defined by

$$g_{j,k}(z) = 2 \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log K(z, z)$$

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(the constant “2” conforms with [12]). In turn, this gives rise to the Bergman metric

$$s_{\Omega}(z, v) = \left\{ \sum_{j,k=1}^n g_{j,k}(z) v_j \bar{v}_k \right\}^{\frac{1}{2}}$$

for $v \in T_z(\Omega) \cong \mathbb{C}^n$ the tangent space at $z \in \Omega$, and the distance function $\beta(\cdot, \cdot)$ in the standard way [8]. In particular, the Bergman metric coincides with the usual Euclidean metric on $\mathbb{C}^n$.

We will deal with the Bergman spaces $A^2(\Omega, dv)$ and the Segal-Bargmann space $H^2(\mathbb{C}^n, d\mu)$ at the same time. Unless stated otherwise, $A^2(\Omega)$ stands for the above two types of spaces $A^2(\Omega, dv)$ and $H^2(\mathbb{C}^n, d\mu)$ in the sequel.

F.A. Berezin introduced a general symbol calculus for linear operators on reproducing kernel Hilbert space in his quantization program [1], [2]. More specifically, for $X \in Op(A^2(\Omega))$ the algebra of all bounded operators on $A^2(\Omega)$, the Berezin transform is defined as

$$\tilde{X}(z) = \langle Xk_z, k_z \rangle,$$

where $k_z(\cdot) = \frac{K(\cdot, z)}{\sqrt{K(z, z)}}$ is the normalized kernel function in the sense that $\|k_z\|_{A^2(\Omega)} = 1$ for any $z \in \Omega$. The function $\tilde{X}(\cdot)$ is called the covariant symbol of operator X , and is real analytic and is uniquely determined by X . It is easy to see that $|\tilde{X}(\cdot)| \leq \|X\|$ for $X \in Op(A^2(\Omega))$. In [3] and [4], the first author obtained Lipschitz estimates for the Berezin transform, namely,

$$(1) \quad |\tilde{X}(a) - \tilde{X}(b)| \leq \sqrt{2} \|X\| \beta(a, b),$$

for any $a, b \in \Omega$ and $X \in Op(A^2(\Omega))$. Furthermore, the above estimates are shown to be sharp in the following sense:

$$(2) \quad \sup_{\substack{a \neq b \in \Omega \\ X \neq 0 \in Op(A^2(\Omega))}} \frac{|\tilde{X}(a) - \tilde{X}(b)|}{\|X\| \beta(a, b)} = \sqrt{2}$$

by varying X, a .

J. Xia pointed out that the proof of Theorem 4 of [3] can be used, along with a direct calculation, to provide a strengthened version of the above Lipschitz estimate (1) for $\Omega = \mathbb{C}^n$: for any $X \in Op(H^2(\mathbb{C}^n, d\mu))$, $\tilde{X}$ and its partial derivatives of all orders are bounded. Subsequently, Engliš and Zhang [5] improved upon this result and generalized it for invariant differential operators applied to $\tilde{X}$ on bounded symmetric domains Ω . The proof of their results depends strongly on the homogeneity of the domain, that is, the group of holomorphic automorphisms acts transitively on Ω , and also on the invariance of the differential operators considered with respect to this transitive group. The second author also investigated the same problem for some particular domains such as the unit ball of $\mathbb{C}^n$ and the *Laplace-Beltrami* operator in [11].

In this paper, directional derivative estimates for $\tilde{X}$ with $X \in Op(A^2(\Omega))$ are obtained without the help of the automorphism group, for any bounded domain $\Omega \subset \mathbb{C}^n$. We also show these estimates are sharp by finding appropriate bounded operators realizing the equality. As by-products, we recover some well-known results relating to our estimates.

2. DIRECTIONAL DERIVATIVE ESTIMATES FOR THE BEREZIN TRANSFORM

For $P_a = k_a \otimes k_a$ the projection onto the span of k_a , it is easy to see that for $X \in Op(A^2(\Omega))$

$$\tilde{X}(a) = tr(XP_a),$$

and additivity of the trace operation implies that

$$\tilde{X}(a) - \tilde{X}(b) = tr[X(P_a - P_b)].$$

For X bounded and T in trace-class, it is standard [6] that XT is in trace-class with

$$|tr(XT)| \leq \|X\| \|T\|_{tr}.$$

Thus,

$$|\tilde{X}(a) - \tilde{X}(b)| \leq \|X\| \|P_a - P_b\|_{tr}.$$

[3] and [4] show that

$$\|P_a - P_b\|_{tr} = 2\{1 - |\langle k_a, k_b \rangle|^2\}^{\frac{1}{2}}.$$

This quantity is related to the biholomorphically invariant Skwarczynski distance function [12]

$$\rho(a, b) = \{1 - |\langle k_a, k_b \rangle|\}^{\frac{1}{2}}$$

by

$$(3) \quad \|P_a - P_b\|_{tr} \leq 2\sqrt{2}\rho(a, b).$$

In turn, this gives the desired Lipschitz estimate since $\rho(a, b) \leq \frac{1}{2}\beta(a, b)$.

For $\gamma : [0, 1] \rightarrow \Omega$ a C^1 curve with $\gamma(0) = b$ and $\gamma'(0) = v$, a fixed nonzero vector in $\mathbb{C}^n$, [12] shows that

$$(4) \quad \lim_{t \rightarrow 0^+} \frac{\rho(\gamma(t), b)}{t} = \frac{1}{2}s_\Omega(b, v).$$

For $v = (v_1, \dots, v_n) \neq 0 \in \mathbb{C}^n$, D_v denotes the differentiation along the direction v , that is,

$$(D_v f)(b) = \lim_{t \rightarrow 0} \frac{f(b + tv) - f(b)}{t}$$

for C^1 functions f on Ω . Then it is standard that

$$D_v = \sum_{j=1}^n \left(v_j \frac{\partial}{\partial z_j} + \bar{v}_j \frac{\partial}{\partial \bar{z}_j} \right).$$

If we denote the complex gradient $\nabla = (\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n})$, then formally we can write

$$D_v = \langle \nabla, \bar{v} \rangle + \langle \bar{\nabla}, v \rangle.$$

Now we are ready to state our main result.

Theorem 2.1. For $X \in Op(A^2(\Omega))$ and $v \in \mathbb{C}^n$ with $v \neq 0$,

$$(5) \quad s_\Omega(b, v)^{-1} |(D_v \tilde{X})(b)| \leq \sqrt{2} \|X\|$$

for all $b \in \Omega$.

Proof. For $X \in Op(A^2(\Omega))$ and given $v \in \mathbb{C}^n$ and $v \neq 0$, take $\gamma(t) = b + tv$ for $0 \leq t \leq t_0(b)$ for some $t_0(b) > 0$ so that the Euclidean ball $B(b, t_0(b)\|v\|) \subset \Omega$. Now

$$\begin{aligned} \left| \frac{\tilde{X}(\gamma(t)) - \tilde{X}(b)}{t} \right| &= \left| tr \left\{ \frac{X(P_{\gamma(t)} - P_b)}{t} \right\} \right| \\ &\leq \|X\| \frac{\|P_{\gamma(t)} - P_b\|_{tr}}{t} \\ &\leq 2\sqrt{2}\|X\| \frac{\rho(\gamma(t), b)}{t}, \end{aligned}$$

where (3) is applied. For arbitrary $\epsilon > 0$, according to (4), there is a $0 < t_1(b) < t_0(b)$ so that

$$\frac{\rho(\gamma(t), b)}{t} \leq \frac{1}{2}s_\Omega(b, v)(1 + \epsilon)$$

for $0 < t < t_1(b)$. Thus, for $t \in (0, t_1(b))$, we have

$$s_\Omega(b, v)^{-1} \left| \frac{\tilde{X}(\gamma(t)) - \tilde{X}(b)}{t} \right| \leq \sqrt{2}(1 + \epsilon)\|X\|.$$

Since $\tilde{X}$ is real analytic,

$$\lim_{t \rightarrow 0^+} \frac{\tilde{X}(\gamma(t)) - \tilde{X}(b)}{t} = (D_v \tilde{X})(b)$$

and so

$$s_\Omega(b, v)^{-1} |(D_v \tilde{X})(b)| \leq \sqrt{2}(1 + \epsilon)\|X\|.$$

Since $\epsilon > 0$ is arbitrary, we see that

$$s_\Omega(b, v)^{-1} |(D_v \tilde{X})(b)| \leq \sqrt{2}\|X\|.$$

□

Let

$$(6) \quad Q_f(b, v) = \frac{|(D_v f)(b)|}{s_\Omega(b, v)}$$

for a C^1 function f defined on Ω and v in $\mathbb{C}^n$ with $v \neq 0$. It is easy to see that $Q_f(b, tv) = Q_f(b, v)$ for any $t > 0$. For any smooth vector field Y on Ω , we know $(D_Y f)(b) = \{D_{Y(b)} f\}(b)$. Therefore Theorem 2.1 still holds with the vector v replaced by a smooth vector field Y with $Y(b) \neq 0$ for any $b \in \Omega$.

The quantity (6) was introduced and studied in several complex variables a few decades ago. From the differential geometrical point of view, it is related to the norm of $d\tilde{X}(b)$, the cotangent vector (1-form) at b on Ω . In particular, K. T. Hahn [7] obtained a comparison theorem between the Carathéodory differential metric $\alpha_\Omega(b, v)$ and the Bergman metric $s_\Omega(b, v)$ on bounded domains in $\mathbb{C}^n$, which is the special case of our Theorem 2.1 for the multiplication operators $X = M_f$ with the symbol f a bounded holomorphic function. More specifically, we recover Hahn's Theorem (with a "less sharp" multiplying constant than his) as follows:

Corollary 2.2. *For f a bounded holomorphic function on Ω a bounded domain in $\mathbb{C}^n$, $v \neq 0 \in \mathbb{C}^n$,*

$$(7) \quad \frac{|\langle \nabla f(b), \bar{v} \rangle|}{s_\Omega(b, v)} \leq \sqrt{2}\|f\|_\infty.$$

Moreover

$$\alpha_{\Omega}(b, v) \leq \sqrt{2}s_{\Omega}(b, v).$$

Proof. It is easy to check that $\widetilde{M}_f = f$, $\|M_f\| = \|f\|_{\infty}$ and $D_v f = \langle \nabla f, \bar{v} \rangle$. Recall that the Carathéodory differential metric is defined as

$$\alpha_{\Omega}(b, v) = \sup_{f \in H(\Omega, U)} |\langle \nabla f(b), \bar{v} \rangle|,$$

where $H(\Omega, U)$ denotes the class of holomorphic functions f on Ω with values in the unit disk U . The result follows directly from Theorem 2.1. $\square$

For Ω any bounded homogeneous domain, Timoney [14] applied the quantity (6) to define Bloch functions in higher dimensions. Recall that a holomorphic function f on Ω is said to be a Bloch function if the Bloch norm

$$\sup_{b \in \Omega} Q_f(b) < \infty,$$

where

$$Q_f(b) = \sup_{v \neq 0 \in \mathbb{C}^n} Q_f(b, v).$$

In order to show that the Banach algebra $H^{\infty}(\Omega)$ of bounded holomorphic functions is contained in the space $Bloch(\Omega)$ of Bloch functions, the concept of *schlicht disk* is used in [14]. Based on our estimates or those of [7], it is easy to see that

Corollary 2.3. $H^{\infty}(\Omega) \subset Bloch(\Omega)$.

Proof. From (7), we have

$$\sup_{b \in \Omega} Q_f(b) \leq \sqrt{2}\|f\|_{\infty}. \quad \square$$

3. THE SHARPNESS OF DIRECTIONAL DERIVATIVE ESTIMATES

Similar to [4], we consider the problem of sharpness of our directional derivative estimates (5). First let us recall some properties of the rank-two selfadjoint operator $P_a - P_b$ for $a \neq b \in \Omega$. In [4], the following identities are established:

$$\begin{aligned} \|P_a - P_b\| &= \{1 - |\langle k_a, k_b \rangle|^2\}^{\frac{1}{2}}, \\ \|P_a - P_b\|_{tr} &= 2\{1 - |\langle k_a, k_b \rangle|^2\}^{\frac{1}{2}}, \\ tr(P_a - P_b)^2 &= \|(P_a - P_b)^2\|_{tr} = 2\{1 - |\langle k_a, k_b \rangle|^2\}. \end{aligned}$$

Next we would like to consider the vector-valued function $\alpha(z) = k_z$ from Ω to $A^2(\Omega)$, and trace-class operator-valued function $A(z) = P_z = k_z \otimes k_z$. It is shown in [5] that $\alpha(z)$ and $A(z)$ have strong derivatives of all orders at $z \in \Omega$, respectively. Therefore we can define

$$(D_v A)(z) = \sum_{j=1}^n \left(v_j \frac{\partial A}{\partial z_j}(z) + \bar{v}_j \frac{\partial A}{\partial \bar{z}_j}(z) \right) = D_v k_z \otimes k_z + k_z \otimes D_v k_z$$

for $v \neq 0 \in \mathbb{C}^n$, where the second equality follows from the Leibnitz rule.

Lemma 3.1. For $b \in \Omega$ and $v \neq 0 \in \mathbb{C}^n$,

$$\begin{aligned} \lim_{t \rightarrow 0} \left\| \frac{A(b+tv) - A(b)}{t} - (D_v A)(b) \right\| &= 0, \\ \lim_{t \rightarrow 0} \left\| \frac{A(b+tv) - A(b)}{t} - (D_v A)(b) \right\|_{tr} &= 0, \\ \lim_{t \rightarrow 0} \left\| \left(\frac{A(b+tv) - A(b)}{t} \right)^2 - [(D_v A)(b)]^2 \right\|_{tr} &= 0. \end{aligned}$$

Proof. It suffices to prove the second limit since $\|X\| \leq \|X\|_{tr}$ and

$$\|X^2 - Y^2\|_{tr} \leq \|X\|_{tr} \|X - Y\|_{tr} + \|X - Y\|_{tr} \|Y\|_{tr}$$

for any trace-class operators X and Y . It is routine to check that

$$\begin{aligned} & \left\| \frac{A(b+tv) - A(b)}{t} - D_v A(b) \right\|_{tr} \\ & \leq \left\| (k_{b+tv} - k_b) \otimes \frac{k_{b+tv} - k_b}{t} \right\|_{tr} + \left\| k_b \otimes \left(\frac{k_{b+tv} - k_b}{t} - D_v k_b \right) \right\|_{tr} \\ & \quad + \left\| \left(\frac{k_{b+tv} - k_b}{t} - D_v k_b \right) \otimes k_b \right\|_{tr}. \end{aligned}$$

For rank-one operator $x \otimes y$ we know that $\|x \otimes y\| = \|x \otimes y\|_{tr} = \|x\| \|y\|$. It follows that

$$\begin{aligned} & \left\| \frac{A(b+tv) - A(b)}{t} - D_v A(b) \right\|_{tr} \\ & \leq \|k_{b+tv} - k_b\| \left\| \frac{k_{b+tv} - k_b}{t} \right\| + 2 \left\| \frac{k_{b+tv} - k_b}{t} - D_v k_b \right\| \\ & \longrightarrow 0 \quad \text{as } t \rightarrow 0, \end{aligned}$$

since the function $\alpha(z) = k_z$ is continuous as a vector-valued function and $D_v k_b$ exists at $b \in \Omega$. $\square$

We can now give an infinitesimal version of the above three identities about $P_a - P_b$ as follows:

Proposition 3.2. For $b \in \Omega$ and $v \neq 0 \in \mathbb{C}^n$,

$$\begin{aligned} \|(D_v A)(b)\| &= \frac{\sqrt{2}}{2} s_\Omega(b, v), \\ \|(D_v A)(b)\|_{tr} &= \sqrt{2} s_\Omega(b, v), \\ \|[(D_v A)(b)]^2\|_{tr} &= s_\Omega(b, v)^2. \end{aligned}$$

Proof. It is easy to see that

$$\left\| \frac{A(b+tv) - A(b)}{t} \right\| = \{1 + |\langle k_{b+tv}, k_b \rangle|\}^{\frac{1}{2}} \frac{\rho(b+tv, b)}{t}.$$

Thus, by (4),

$$\lim_{t \rightarrow 0^+} \left\| \frac{A(b+tv) - A(b)}{t} \right\| = \frac{\sqrt{2}}{2} s_\Omega(b, v)$$

since it is easy to check that

$$\lim_{t \rightarrow 0^+} \{1 + |\langle k_{b+tv}, k_b \rangle|\}^{\frac{1}{2}} = \sqrt{2}.$$

By Lemma 3.1 and continuity of the operator norm, we have

$$\|(D_v A)(b)\| = \frac{\sqrt{2}}{2} s_\Omega(b, v).$$

The other two identities follow similarly using (4) and Lemma 3.1. $\square$

Theorem 3.3. *For fixed $b \in \Omega$ and $v \neq 0 \in \mathbb{C}^n$, let the operator $X_{b,v} = (D_v A)(b)$; then*

$$s_\Omega(b, v)^{-1} |(D_v \tilde{X}_{b,v})(b)| = \sqrt{2} \|X_{b,v}\|.$$

Proof. For $\tilde{X}(z) = \text{tr}(XA(z))$, [5] shows that for any $v \neq 0 \in \mathbb{C}^n$,

$$(D_v \tilde{X})(z) = \text{tr}[X(D_v A)(z)].$$

In particular, we take, for fixed $b \in \Omega$ and $v \neq 0 \in \mathbb{C}^n$, $X_{b,v} = (D_v A)(b)$ and then

$$(D_v \tilde{X}_{b,v})(z) = \text{tr}[X_{b,v}(D_v A)(z)].$$

Therefore, evaluating at $z = b$,

$$(D_v \tilde{X}_{b,v})(b) = \text{tr}[X_{b,v}(D_v A)(b)] = \text{tr}\{[(D_v A)(b)]^2\}.$$

The remainder of the proof is just the application of Proposition 3.2. $\square$

4. CONCLUDING REMARKS

For real analytic functions f on Ω , we can follow Timoney to define real analytic Bloch functions by requiring

$$\sup_{b \in \Omega} Q_f(b) < \infty$$

where $Q_f(b)$ is defined as above. So Theorem 2.1 implies that the range of the Berezin transform on $Op(A^2(\Omega))$ is a subspace of the space of real analytic Bloch functions. Our estimate (5) gives the greatest possible rate of growth of $D_v \tilde{X}$ for fixed X, v .

In particular, for the annulus $B = \{z \in \mathbb{C} : \frac{1}{R} < |z| < R\}$ with $R > 1$, which is bounded but not homogeneous, the Bergman metric is given by Herbort [9] (also see [10], p. 187) as

$$s_B(b, v) = \frac{\sqrt{2}|v|}{|b|\sqrt{\chi(\log |b|^2)}},$$

where $\chi : (-2 \log R, 2 \log R) \rightarrow \mathbb{R}$ is a positive real analytic function which is symmetric about 0, with a unique local maximum at 0, and such that

$$0 < \lim_{t \rightarrow 2 \log R^-} \frac{\chi(t)}{(t - 2 \log R)^2} < \infty.$$

For $|v| = 1$ and $X \in Op(A^2(B))$, we have

$$|(D_v \tilde{X})(b)| \leq 2R \|X\| \{\chi(\log |b|^2)\}^{-\frac{1}{2}},$$

and it is easy to check that the boundary behavior of $D_v \tilde{X}$ is asymptotically dominated by

$$\left\{ \log \frac{|b|}{R} \right\}^{-1} \quad \text{near } |b| = R$$

and

$$\{\log(R|b|)\}^{-1} \quad \text{near } |b| = \frac{1}{R}.$$

For fixed $X \in Op(A^2(\Omega))$ and $v \neq 0 \in \mathbb{C}^n$, can $|(D_v \tilde{X})(b)|$ “blow-up” near the boundary $\partial\Omega$? It is easy to see that this can occur for $\Omega = B$ by taking $X = M_\varphi$ with φ bounded and holomorphic on B with $\varphi'(z)$ unbounded near a point on ∂B .

Note that $s_\Omega(b, v)$ need not blow-up near every boundary point of Ω . Consider the punctured disk

$$U_0 = \{z \in \mathbb{C} : 0 < |z| < 1\}.$$

It is known that $A^2(U_0) = A^2(U)|_{U_0}$ so that

$$s_{U_0}(b, v) = s_U(b, v)$$

for $b \in U_0$ and

$$\lim_{b \rightarrow 0} s_{U_0}(b, v) = 2|v|.$$

For $b_0 \in \partial\Omega$, the general problem of finding X, v so that

$$\lim_{b \rightarrow b_0} Q_{\tilde{X}}(b, v) > 0,$$

remains open.

Finally, we note that our Theorem 2.1 provides an affirmative answer to the covariant differentiation conjecture of [5] (p. 2287) for the power $k = 1$. Their conjecture remains open for $k \geq 2$. For technical background on this conjecture, see [8], [13].

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