A SHARP BOUND FOR THE STEIN-WAINGER OSCILLATORY INTEGRAL

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ABSTRACT. Let $\mathcal{P}_d$ denote the space of all real polynomials of degree at most d . It is an old result of Stein and Wainger that

$$\sup_{P \in \mathcal{P}_d} \left| p.v. \int_{\mathbb{R}} e^{iP(t)} \frac{dt}{t} \right| \leq C_d$$

for some constant C_d depending only on d . On the other hand, Carbery, Wainger and Wright claim that the true order of magnitude of the above principal value integral is $\log d$. We prove that

$$\sup_{P \in \mathcal{P}_d} \left| p.v. \int_{\mathbb{R}} e^{iP(t)} \frac{dt}{t} \right| \sim \log d.$$

1. INTRODUCTION

Let $\mathcal{P}_d$ be the vector space of all real polynomials of degree at most d in $\mathbb{R}$. For $P \in \mathcal{P}_d$ we consider the principal value integral

$$I(P) = \left| p.v. \int_{\mathbb{R}} e^{iP(t)} \frac{dt}{t} \right|.$$

We wish to estimate the quantity $I(P)$ by a constant $C(d)$ depending only on the degree of the polynomial d . This amounts to estimating the integral

$$I_{(\epsilon, R)}(P) = \left| \int_{\epsilon \leq |t| \leq R} e^{iP(t)} \frac{dt}{t} \right|$$

by some constant $C(d)$ independent of ϵ , R and P .

This problem is quite old and in fact was answered some thirty years ago by Stein and Wainger in [4] and [6]. They showed that the quantity $I(P)$ is bounded by a constant C_d depending only on d . Their proof is very simple and uses a combination of induction and van der Corput's lemma. Let us recall the latter since we'll also be using it in what follows.

Proposition 1.1 (van der Corput). *Let $\phi : [a, b] \rightarrow \mathbb{R}$ be a C^k function and suppose that $|\phi^{(k)}(t)| \geq 1$ for some $k \geq 1$ and all $t \in [a, b]$. If $k = 1$ suppose in addition*

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that ϕ' is monotonic. Then, for every $\lambda \in \mathbb{R}$,

$$\left| \int_a^b e^{i\lambda\phi(x)} dx \right| \leq \frac{Ck}{|\lambda|^{\frac{1}{k}}},$$

where C is an absolute constant independent of a, b, k and ϕ .

For a proof of this very well-known result with Ck replaced by C_k , see, for example, [3]. A proof that the constant C_k can be taken to be linear in k can be found in [1].

On the other hand, Carbery, Wainger and Wright have conjectured in [2] that the true order of magnitude of the principal value integral is $\log d$. The main result of this paper is the proof of this conjecture. This is the content of:

Theorem 1.2. *There exist two absolute positive constants c_1 and c_2 such that*

$$c_1 \log d \leq \sup_{P \in \mathcal{P}_d} \left| p.v. \int_{\mathbb{R}} e^{iP(x)} \frac{dx}{x} \right| \leq c_2 \log d.$$

Remark 1.3. Suppose that K is a $-n$ homogeneous function on $\mathbb{R}^n$, odd and integrable on the unit sphere. Then, by the one-dimensional result, we trivially get that there is an absolute positive constant c , such that

$$\left| p.v. \int_{\mathbb{R}^n} e^{iP(x)} K(x) dx \right| \leq c \|K\|_{L^1(S^{n-1})} \log d,$$

for every polynomial P on $\mathbb{R}^n$, of degree at most d .

Notation. We will use the letter c to denote an absolute positive constant, which might change even in the same line of text. Also, the notation $A \sim B$ means that there exist absolute positive constants c_1 and c_2 such that $c_1 B \leq A \leq c_2 B$.

2. THE LOWER BOUND IN THE THEOREM

In this section we will construct a real polynomial P of degree at most d such that the inequality

$$(2.1) \quad I(P) = \left| p.v. \int_{\mathbb{R}} e^{iP(t)} \frac{dt}{t} \right| \geq c \log d$$

holds. The general plan of the construction is as follows. We will first construct a function f (which will not be a polynomial) such that $I(f) \geq c \log n$. We will then construct a polynomial P of degree $d = 2n^2 - 1$ that approximates the function f in a way that $|I(f) - I(P)|$ is small (small means $o(\log n)$ here). Since $\log n \sim \log d$, this will yield our result.

Lemma 2.1. *For n a large positive integer, let $f(t)$ be the continuous function that is equal to 1 for $\frac{1}{n} \leq t \leq 1 - \frac{1}{n}$, equal to -1 for $-1 + \frac{1}{n} \leq t \leq -\frac{1}{n}$, equal to 0 for $|t| \geq 1$ and linear in each interval $[-1, -1 + \frac{1}{n}]$, $[-\frac{1}{n}, \frac{1}{n}]$ and $[1 - \frac{1}{n}, 1]$. Then,*

$$(2.2) \quad I(f) = \left| p.v. \int_{\mathbb{R}} e^{if(t)} \frac{dt}{t} \right| \geq c \log n.$$

Proof. The proof is more or less straightforward.

$$\begin{aligned}
 I(f) &= 2 \left| \int_0^1 \frac{\sin f(t)}{t} dt \right| \\
 &\geq 2 \left| \int_{\frac{1}{n}}^{1-\frac{1}{n}} \frac{\sin f(t)}{t} dt \right| - 2 \left| \int_0^{\frac{1}{n}} \frac{\sin f(t)}{t} dt \right| - 2 \left| \int_{1-\frac{1}{n}}^1 \frac{\sin f(t)}{t} dt \right| \\
 &\geq 2 \sin 1 \log(n-1) - 2 \int_0^{\frac{1}{n}} \frac{f(t)}{t} dt - 2 \int_{1-\frac{1}{n}}^1 \frac{f(t)}{t} dt \\
 &= 2 \sin 1 \log(n-1) - 2 - 2n \log \frac{n}{n-1} + 2 \\
 &\geq 2 \sin 1 \log(n-1) - 4 \geq c \log n. \quad \square
 \end{aligned}$$

We now want to construct a polynomial which approximates the function f . We will do so by convolving the function f with a “polynomial approximation to the identity”. To be more specific, for $k \in \mathbb{N}$ and $x \in \mathbb{R}$ define the function

$$(2.3) \quad \phi_k(x) = c_k \left(1 - \frac{x^2}{4} \right)^{k^2},$$

where the constant c_k is defined by means of the normalization

$$(2.4) \quad \int_{-2}^2 \phi_k(x) dx = 1.$$

Observe that

$$1 = c_k \int_{-2}^2 \left(1 - \frac{x^2}{4} \right)^{k^2} dx = 4c_k \int_0^1 (1-x^2)^{k^2} dx = 2c_k B\left(\frac{1}{2}, k^2 + 1\right),$$

where $B(\cdot, \cdot)$ is the beta function. Using standard estimates for the beta function we see that $c_k \sim k$.

Define, next, the functions P_k in $\mathbb{R}$ as

$$(2.5) \quad P_k(t) = \int_{-1}^1 f(x) \phi_k(t-x) dx,$$

where f is the function of Lemma 2.1. It is clear that the functions P_k are *polynomials* of degree at most $2k^2$. The following lemma deals with some technical issues concerning the polynomials P_k .

Lemma 2.2. *Let P_k be defined as in (2.5) above.*

(i) *P_k is an odd polynomial of degree $2k^2 - 1$ with leading coefficient*

$$a_k = (-1)^{k^2+1} \frac{2c_k k^2}{4k^2} \left(1 - \frac{1}{n} \right).$$

That is,

$$P_k(t) = a_k t^{2k^2-1} + \dots$$

(ii) *As a consequence of (i) we have for all t ,*

$$|P_k^{(2k^2-1)}(t)| \geq c(2k^2-1)! \frac{k^3}{4k^2}.$$

(iii) For $t \in [-1, 1]$ we have

$$P_k(t) = \int_0^2 (f(t+x) + f(t-x))\phi_k(x)dx.$$

Proof. (i) Using (2.5) we have

$$\begin{aligned} P_k(-t) &= \int_{-1}^1 f(x)\phi_k(-t-x)dx = \int_{-1}^1 f(x)\phi_k(t+x)dx \\ &= \int_{-1}^1 f(-x)\phi_k(t-x)dx = -P_k(t). \end{aligned}$$

Next, from (2.5) we have that

$$\begin{aligned} P_k(t) &= c_k \int_{-1}^1 f(x) \sum_{m=0}^{k^2} \binom{k^2}{m} \left(-\frac{(t-x)^2}{4} \right)^m dx \\ &= c_k \sum_{m=0}^{k^2} \binom{k^2}{m} \frac{(-1)^m}{4^m} \int_{-1}^1 f(x)(t-x)^{2m} dx \\ &= c_k \frac{(-1)^{k^2}}{4^{k^2}} \int_{-1}^1 f(x)(x-t)^{2k^2} dx \\ &\quad + c_k \sum_{m=0}^{k^2-1} \binom{k^2}{m} \frac{(-1)^m}{4^m} \int_{-1}^1 f(x)(t-x)^{2m} dx. \end{aligned}$$

It is now easy to see that the two highest-order terms come from the first summand in the above formula. Therefore,

$$\begin{aligned} P_k(t) &= c_k \frac{(-1)^{3k^2}}{4^{k^2}} \int_{-1}^1 f(x)dx t^{2k^2} - c_k \frac{(-1)^{k^2} 2k^2}{4^{k^2}} \int_{-1}^1 f(x)x dx t^{2k^2-1} + \dots \\ &= (-1)^{k^2+1} \frac{2c_k k^2}{4^{k^2}} \left(1 - \frac{1}{n} \right) t^{2k^2-1} + \dots \end{aligned}$$

(ii) We just use the result of (i) and the fact that $c_k \sim k$.

(iii) Fix a $t \in [-1, 1]$. Then,

$$\begin{aligned} \int_{-2}^2 f(t-x)\phi_k(x)dx &= \int_{\mathbb{R}} f(t-x)\phi_k(x)\chi_{[-2,2]}(x)dx \\ &= \int_{-1}^1 f(x)\phi_k(t-x)\chi_{[-2,2]}(t-x)dx \\ &= \int_{-1}^1 f(x)\phi_k(t-x)dx \\ &= P_k(t). \end{aligned}$$

However, since ϕ_k is even,

$$P_k(t) = \int_{-2}^2 f(t-x)\phi_k(x)dx = \int_0^2 (f(t+x) + f(t-x))\phi_k(x)dx.$$

□

We are now ready to prove the lower bound for $I(P)$.

Proposition 2.3. *Let P_n be the polynomial defined in (2.5) where n is the large positive integer used to define the function f in Lemma 2.1. Then P_n is a polynomial of degree $d = 2n^2 - 1$ and*

$$I(P_n) = \left| p.v. \int_{\mathbb{R}} e^{iP_n(t)} \frac{dt}{t} \right| \geq c \log d.$$

Proof. Since P_n is odd,

$$I(P_n) = 2 \left| \int_0^{+\infty} \frac{\sin P_n(t)}{t} dt \right|,$$

and it suffices to show that for all $R \geq 1$,

$$(2.6) \quad \left| \int_0^R \frac{\sin P_n(t)}{t} dt \right| \geq c \log d \sim c \log n.$$

By part (ii) of Lemma 2.2 and a standard application of Proposition 1.1 (van der Corput) we see that

$$\left| \int_1^R \frac{\sin P_n(t)}{t} dt \right| \leq c$$

for all $R \geq 1$. As a result, the proof will be complete if we show that

$$(2.7) \quad I_1(P_n) = \left| \int_0^1 \frac{\sin P_n(t)}{t} dt \right| \geq c \log n.$$

Using Lemma 2.1 and the triangle inequality we get

$$(2.8) \quad I_1(P_n) \geq c \log n - |I_1(P_n) - I(f)|$$

and, in order to show (2.7), it suffices to show that

$$(2.9) \quad |I_1(P_n) - I(f)| = o(\log n).$$

We have that

$$\begin{aligned} |I_1(P_n) - I(f)| &= \left| \int_0^1 \frac{\sin P_n(t) - \sin f(t)}{t} dt \right| \\ &\leq \int_0^1 \frac{|P_n(t) - f(t)|}{t} dt. \end{aligned}$$

Using part (iii) of Lemma 2.2 and (2.4), we get

$$|P_n(t) - f(t)| \leq \int_0^2 |f(t+x) + f(t-x) - 2f(t)| \phi_n(x) dx$$

for $0 \leq t \leq 1$. Hence

$$|I_1(P_n) - I(f)| \leq \int_0^2 \int_0^1 \frac{|f(t+x) + f(t-x) - 2f(t)|}{t} dt \phi_n(x) dx.$$

Now, the desired result, condition (2.9), is the content of the following lemma. $\square$

Lemma 2.4. *Let $A(x, t) = |f(t+x) + f(t-x) - 2f(t)|$. Then,*

$$\int_0^2 \int_0^1 \frac{A(x, t)}{t} dt \phi_n(x) dx = o(\log n).$$

Proof. Firstly, it is not difficult to establish that

$$(2.10) \quad A(x, t) \leq 4 \min(nx, nt, 1),$$

$$(2.11) \quad A(x, t) = 0, \quad \text{when } \frac{1}{n} \leq t - x \leq t + x \leq 1 - \frac{1}{n}.$$

Indeed,

$$\begin{aligned} A(x, t) &\leq |f(t+x) - f(t)| + |f(t-x) - f(t)| \\ &\leq nx + nx \leq 2nx. \end{aligned}$$

On the other hand,

$$\begin{aligned} A(x, t) &= |f(t+x) - f(x) + f(t-x) - f(-x) - 2f(t)| \\ &\leq |f(t+x) - f(x)| + |f(t-x) - f(-x)| + 2|f(t)| \\ &\leq nt + nt + 2nt = 4nt. \end{aligned}$$

Inequality (2.10) now follows by the fact that $|f|$ is bounded by 1 and (2.11) is trivial to prove.

We split the integral $\int_0^2 \int_0^1 \cdots dt dx$ into seven integrals:

$$\begin{aligned} &\int_0^2 \int_{\frac{1}{2}}^1 \cdots dt dx + \int_0^{\frac{1}{n}} \int_0^x \cdots dt dx + \int_{\frac{1}{n}}^2 \int_0^{\frac{1}{n}} \cdots dt dx + \int_0^{\frac{1}{n}} \int_x^{x+\frac{1}{n}} \cdots dt dx \\ &+ \int_0^{\frac{1}{2}-\frac{1}{n}} \int_{x+\frac{1}{n}}^{\frac{1}{2}} \cdots dt dx + \int_{\frac{1}{n}}^{\frac{1}{2}-\frac{1}{n}} \int_{\frac{1}{n}}^{x+\frac{1}{n}} \cdots dt dx + \int_{\frac{1}{2}-\frac{1}{n}}^2 \int_{\frac{1}{n}}^{\frac{1}{2}} \cdots dt dx. \end{aligned}$$

We estimate each of the seven integrals separately:

$$\begin{aligned} \int_0^2 \int_{\frac{1}{2}}^1 \frac{A(x, t)}{t} dt \phi_n(x) dx &\leq 4 \log 2 \int_0^2 \phi_n(x) dx = 2 \log 2, \\ \int_0^{\frac{1}{n}} \int_0^x \frac{A(x, t)}{t} dt \phi_n(x) dx &\leq \int_0^{\frac{1}{n}} \int_0^x \frac{4nt}{t} dt \phi_n(x) dx \\ &= \int_0^{\frac{1}{n}} 4nx \phi_n(x) dx \leq 2, \\ \int_{\frac{1}{n}}^2 \int_0^{\frac{1}{n}} \frac{A(x, t)}{t} dt \phi_n(x) dx &\leq \int_{\frac{1}{n}}^2 \int_0^{\frac{1}{n}} \frac{4nt}{t} dt \phi_n(x) dx \\ &= \int_0^{\frac{1}{n}} 4\phi_n(x) dx \leq 2. \int_0^{\frac{1}{n}} \int_x^{x+\frac{1}{n}} \frac{A(x, t)}{t} dt \phi_n(x) dx \\ &\leq \int_0^{\frac{1}{n}} \int_x^{x+\frac{1}{n}} \frac{4nx}{t} dt \phi_n(x) dx \\ &= \int_0^{\frac{1}{n}} 4nx \log \left(1 + \frac{1}{nx} \right) \phi_n(x) dx \leq 2. \end{aligned}$$

For $\int_0^{\frac{1}{2}-\frac{1}{n}} \int_{x+\frac{1}{n}}^{\frac{1}{2}} \cdots$ we have $\frac{1}{n} \leq t - x \leq t + x \leq 1 - \frac{1}{n}$ and, by (2.11), $A(x, t) = 0$. Hence

$$\int_0^{\frac{1}{2}-\frac{1}{n}} \int_{x+\frac{1}{n}}^{\frac{1}{2}} \frac{A(x, t)}{t} dt \phi_n(x) dx = 0.$$

Next

$$\begin{aligned} \int_{\frac{1}{n}}^{\frac{1}{2}-\frac{1}{n}} \int_{\frac{1}{n}}^{x+\frac{1}{n}} \frac{A(x,t)}{t} dt \phi_n(x) dx &\leq \int_{\frac{1}{n}}^{\frac{1}{2}-\frac{1}{n}} \int_{\frac{1}{n}}^{x+\frac{1}{n}} \frac{4}{t} dt \phi_n(x) dx \\ &\leq 4 \int_{\frac{1}{n}}^1 \log(nx+1) \phi_n(x) dx. \end{aligned}$$

Now, fix some $\alpha \in (0, 1)$. Write

$$\begin{aligned} \int_{\frac{1}{n}}^1 \log(nx+1) \phi_n(x) dx &= \int_{\frac{1}{n}}^{\frac{1}{n^\alpha}} \cdots dx + \int_{\frac{1}{n^\alpha}}^1 \cdots dx \\ &\leq \frac{\log(n^{1-\alpha}+1)}{2} + c_n \log(n+1) \int_{\frac{1}{n^\alpha}}^1 \left(1 - \frac{x^2}{4}\right)^{n^2} dx \\ &\leq \frac{\log(n^{1-\alpha}+1)}{2} + cn \log(n+1) e^{-\frac{1}{4}n^{2(1-\alpha)}}. \end{aligned}$$

Therefore,

$$\limsup_{n \rightarrow \infty} \frac{\int_{\frac{1}{n}}^1 \log(nx+1) \phi_n(x) dx}{\log n} \leq \frac{1-\alpha}{2}$$

and, since α is arbitrary in $(0, 1)$,

$$\int_{\frac{1}{n}}^{\frac{1}{2}-\frac{1}{n}} \int_{\frac{1}{n}}^{x+\frac{1}{n}} \frac{A(x,t)}{t} dt \phi_n(x) dx = o(\log n).$$

Finally,

$$\begin{aligned} \int_{\frac{1}{2}-\frac{1}{n}}^2 \int_{\frac{1}{n}}^{\frac{1}{2}} \frac{A(x,t)}{t} dt \phi_n(x) dx &\leq \int_{\frac{1}{2}-\frac{1}{n}}^2 \int_{\frac{1}{n}}^{\frac{1}{2}} \frac{4}{t} dt \phi_n(x) dx \\ &\leq 4 \log \frac{n}{2} c_n \int_{\frac{1}{2}-\frac{1}{n}}^2 \left(1 - \frac{x^2}{4}\right)^{n^2} dx \\ &\leq cn \log n e^{-\frac{1}{16}n^2} = o(1). \quad \square \end{aligned}$$

3. THE UPPER BOUND IN THE THEOREM

We set

$$(3.1) \quad K_d = \sup_{P \in \mathcal{P}_d, \epsilon, R} \left| \int_{\epsilon \leq |t| \leq R} e^{iP(t)} \frac{dt}{t} \right|.$$

We take any polynomial P , of degree at most d , which we can assume has no constant term, that is, $P(0) = 0$. We set $k = \lfloor \frac{d}{2} \rfloor$ and we write

$$\begin{aligned} P(t) &= a_1 t + a_2 t^2 + \cdots + a_k t^k + a_{k+1} t^{k+1} + \cdots + a_d t^d \\ &= Q(t) + R(t), \end{aligned}$$

where $Q(t) = a_1 t + a_2 t^2 + \cdots + a_k t^k$ and $R(t) = a_{k+1} t^{k+1} + \cdots + a_d t^d$. Let $|a_l| = \max_{k+1 \leq j \leq d} |a_j|$ for some $k+1 \leq l \leq d$. By a change of variables in the

integral in (3.1) we can assume that $|a_l| = 1$ and thus that $|a_j| \leq 1$ for every $k+1 \leq j \leq d$. Now split the integral in (3.1) into two parts as follows:

$$(3.2) \quad \left| \int_{\epsilon \leq |t| \leq R} e^{iP(t)} \frac{dt}{t} \right| \leq \left| \int_{\epsilon \leq |t| \leq 1} e^{iP(t)} \frac{dt}{t} \right| + \left| \int_{1 \leq |t| \leq R} e^{iP(t)} \frac{dt}{t} \right| \\ = I_1 + I_2.$$

For I_1 we have that

$$\begin{aligned} I_1 &\leq \left| \int_{\epsilon \leq |t| \leq 1} [e^{iP(t)} - e^{iQ(t)}] \frac{dt}{t} \right| + \left| \int_{\epsilon \leq |t| \leq 1} e^{iQ(t)} \frac{dt}{t} \right| \\ &\leq \int_{\epsilon \leq |t| \leq 1} |e^{iP(t)} - e^{iQ(t)}| \frac{dt}{t} + K_{[\frac{d}{2}]} \\ &\leq \int_{0 \leq |t| \leq 1} \frac{|R(t)|}{t} dt + K_{[\frac{d}{2}]} \\ &\leq 2 \sum_{j=k+1}^d \frac{|a_j|}{j} + K_{[\frac{d}{2}]} \leq \sum_{j=k+1}^d \frac{1}{j} + K_{[\frac{d}{2}]} \leq c + K_{[\frac{d}{2}]}. \end{aligned}$$

For the second integral in (3.2) we have that

$$I_2 \leq \left| \int_{1 \leq t \leq R} e^{iP(t)} \frac{dt}{t} \right| + \left| \int_{-R \leq t \leq -1} e^{iP(t)} \frac{dt}{t} \right| = I_2^+ + I_2^-.$$

For some $\alpha > 0$ to be defined later split I_2^+ into two parts as follows:

$$I_2^+ \leq \int_{\{t \in [1, +\infty) : |P'(t)| \leq \alpha\}} \frac{dt}{t} + \left| \int_{\{t \in [1, R] : |P'(t)| > \alpha\}} e^{iP(t)} \frac{dt}{t} \right|.$$

Since $\{t \in [1, R] : |P'(t)| > \alpha\}$ consists of at most $O(d)$ intervals where P' is monotonic, using Proposition 1.1 we get the bound

$$\left| \int_{\{t \in [1, R] : |P'(t)| > \alpha\}} e^{iP(t)} \frac{dt}{t} \right| \leq c \frac{d}{\alpha}.$$

For the logarithmic measure of the set $\{t \in [1, +\infty) : |P'(t)| \leq \alpha\}$, observe that

$$\begin{aligned} \int_{\{t \in [1, +\infty) : |P'(t)| \leq \alpha\}} \frac{dt}{t} &\leq \sum_{m=0}^{\infty} \int_{\{t \in [2^m, 2^{m+1}] : |P'(t)| \leq \alpha\}} \frac{dt}{t} \\ &= \sum_{m=0}^{\infty} \int_{\{2^m t \in [2^m, 2^{m+1}] : |P'(2^m t)| \leq \alpha\}} \frac{dt}{t} \\ &= \sum_{m=0}^{\infty} \int_{2^m \{t \in [1, 2] : |P'(2^m t)| \leq \alpha\}} \frac{dt}{t} \\ &= \sum_{m=0}^{\infty} \int_{\{t \in [1, 2] : |P'(2^m t)| \leq \alpha\}} \frac{dt}{t}. \end{aligned}$$

We have thus shown that

$$(3.3) \quad \int_{\{t \in [1, +\infty) : |P'(t)| \leq \alpha\}} \frac{dt}{t} \leq \sum_{m=0}^{\infty} |\{t \in [1, 2] : |P'(2^m t)| \leq \alpha\}|.$$

In order to finish the proof we need a suitable estimate for the sublevel set of a polynomial. This is the content of the following lemma.

Lemma 3.1 (Vinogradov). *Let $h(t) = b_0 + b_1 t + \cdots + b_n t^n$ be a real polynomial of degree n . Then,*

$$|\{t \in [1, 2] : |h(t)| \leq \alpha\}| \leq c \left(\frac{\alpha}{\max_{0 \leq k \leq n} |b_k|} \right)^{\frac{1}{n}}.$$

This lemma is due to Vinogradov [5]. We postpone the proof of Lemma 3.1 until after the end of the proof of the upper bound.

Consider the polynomial $P'(2^m t)$ with coefficients $j a_j 2^{m(j-1)}$, $1 \leq j \leq d$. Clearly, $\max_{1 \leq j \leq d} |j a_j 2^{m(j-1)}| \geq |l a_l 2^{m(l-1)}| \geq (\lfloor \frac{d}{2} \rfloor + 1) 2^{m \lfloor \frac{d}{2} \rfloor}$. Using Lemma 3.1 and (3.3), we get

$$\int_{\{t \in [1, +\infty) : |P'(t)| \leq \alpha\}} \frac{dt}{t} \leq c \alpha^{\frac{1}{d-1}} \sum_{m=0}^{\infty} \left(\frac{1}{(\lfloor \frac{d}{2} \rfloor + 1) 2^{m \lfloor \frac{d}{2} \rfloor}} \right)^{\frac{1}{d-1}} \leq c \alpha^{\frac{1}{d-1}}.$$

Obviously, a similar estimate holds for I_2^- . Summing the estimates we get

$$\left| \int_{\epsilon \leq |t| \leq R} e^{iP(t)} \frac{dt}{t} \right| \leq c + c \frac{d}{\alpha} + c \alpha^{\frac{1}{d-1}} + K_{\lfloor \frac{d}{2} \rfloor}.$$

Optimizing in α we get that

$$(3.4) \quad \left| \int_{\epsilon \leq |t| \leq R} e^{iP(t)} \frac{dt}{t} \right| \leq c + K_{\lfloor \frac{d}{2} \rfloor}$$

and hence

$$K_d \leq c + K_{\lfloor \frac{d}{2} \rfloor}.$$

In particular we have

$$K_{2^n} \leq c + K_{2^{n-1}}.$$

Using induction on n we get that $K_{2^n} \leq cn$. It is now trivial to show the inequality for general d . Indeed, if $2^{n-1} < d \leq 2^n$, then $K_d \leq K_{2^n} \leq cn \leq c \log d$.

For the sake of completeness we give the proof of Lemma 3.1.

Proof of Lemma 3.1. The set $E_\alpha = \{t \in [1, 2] : |h(t)| \leq \alpha\}$ is a union of intervals. We slide them together to form a single interval I of length $|E_\alpha|$ and pick $n+1$ equally spaced points in I . If we slide the intervals back to their original position, we end up with $n+1$ points $x_0, x_1, x_2, \dots, x_n \in E_\alpha$ that satisfy

$$(3.5) \quad |x_j - x_k| \geq |E_\alpha| \frac{|j - k|}{n}.$$

The Lagrange polynomial that interpolates the values $h(x_0), h(x_1), \dots, h(x_n)$ coincides with $h(x)$:

$$h(x) = \sum_{j=0}^n h(x_j) \frac{(x - x_0)(x - x_1) \cdots (x - x_{j-1})(x - x_{j+1}) \cdots (x - x_n)}{(x_j - x_0)(x_j - x_1) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_n)}.$$

Therefore we get for the coefficients of h that

$$b_k = \sum_{j=0}^n h(x_j) \frac{(-1)^{n-k} \sigma_{n-k}(x_0, \dots, \hat{x}_j, \dots, x_n)}{(x_j - x_0)(x_j - x_1) \cdots (x_j - x_{j-1})(x_j - x_{j+1}) \cdots (x_j - x_n)}$$

for $k = 0, 1, \dots, n$. In the above formula, $\sigma_{n-k}(x_0, \dots, \hat{x}_j, \dots, x_n)$ is the $(n-k)$ -th elementary symmetric function of $x_0, \dots, \hat{x}_j, \dots, x_n$ where x_j is omitted. Using the estimate $\sigma_{n-k}(x_0, \dots, \hat{x}_j, \dots, x_n) \leq \binom{n}{n-k} 2^{n-k}$ together with (3.5) we get that, for every $k = 0, 1, \dots, n$,

$$\begin{aligned} |b_k| &\leq \binom{n}{n-k} 2^{n-k} n^n \frac{\alpha}{|E_\alpha|^n} \sum_{j=0}^n \frac{1}{j!(n-j)!} \\ &= \binom{n}{n-k} 2^{2n-k} \frac{n^n}{n!} \frac{\alpha}{|E_\alpha|^n} \leq c \frac{8^n}{\sqrt{n}} \frac{n^n}{n!} \frac{\alpha}{|E_\alpha|^n}, \end{aligned}$$

where we used the estimate $\binom{n}{n-k} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \leq c \frac{2^n}{\sqrt{n}}$. Hence

$$\max_{0 \leq k \leq n} |b_k| \leq c \frac{8^n}{\sqrt{n}} \frac{n^n}{n!} \frac{\alpha}{|E_\alpha|^n},$$

and solving with respect to $|E_\alpha|$ we get

$$|E_\alpha| \leq c \left(\frac{\alpha}{\max_{0 \leq k \leq n} |b_k|} \right)^{\frac{1}{n}}.$$

□

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REFERENCES

1. G. I. Arhipov, A. A. Karacuba, and V. N. Čubarikov, *Trigonometric integrals*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 5, 971–1003, 1197. MR552548 (81f:10050)
2. Anthony Carbery, Stephen Wainger, and James Wright, *Personal communication*, 2005.
3. E. M. Stein, *Oscillatory integrals in Fourier analysis*, Beijing lectures in harmonic analysis (Beijing, 1984), Ann. of Math. Stud., vol. 112, Princeton Univ. Press, Princeton, NJ, 1986, pp. 307–355. MR864375 (88g:42022)
4. Elias M. Stein and Stephen Wainger, *The estimation of an integral arising in multiplier transformations*, Studia Math. **35** (1970), 101–104. MR0265995 (42:904)
5. Ivan Matveevič Vinogradov, *Selected works*, Springer-Verlag, Berlin, 1985, With a biography by K. K. Mardzhanishvili, Translated from the Russian by Naidu Psv [P. S. V. Naidu], Translation edited by Yu. A. Bakhturin. MR807530 (87a:01042)
6. Stephen Wainger, *Averages and singular integrals over lower-dimensional sets*, Beijing lectures in harmonic analysis (Beijing, 1984), Ann. of Math. Stud., vol. 112, Princeton Univ. Press, Princeton, NJ, 1986, pp. 357–421. MR864376 (89a:42026)

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