ON FIXED POINT THEOREMS OF LERAY–SCHAUDER TYPE

MARCIN BORKOWSKI AND DARIUSZ BUGAJEWSKI

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ABSTRACT. In this paper we prove a few fixed points theorems of Leray–Schauder type in hyperconvex metric spaces.

1. INTRODUCTION

The class of hyperconvex metric spaces, although introduced quite a long time ago (see [1]), is still very interesting, in particular for mathematicians working in fixed point theory. This fact seems to be closely connected with nice properties of these spaces. For example, it is commonly known that a bounded hyperconvex metric space has the fixed point property; that is, each nonexpansive mapping of such a space into itself has a fixed point (see [3], [18] and [19]). Note that if we have a nonexpansive mapping f which maps a bounded, closed and convex subset D of a Banach space into itself, then f has approximate fixed points in D ; more precisely,

$$\inf\{\|x - f(x)\| : x \in D\} = 0$$

(see [12]).

Recall also that the famous Schauder fixed point principle and its generalizations involving measures of noncompactness hold for hyperconvex metric spaces (see [9], [15], [4] and [5] for a more general case).

On the other hand, the famous nonlinear Leray–Schauder alternatives have a lot of applications, especially in the theory of nonlinear differential as well as integral equations (see, e.g., [13], [11], [16], [17] and the references therein). In this paper we formulate and prove a metric analogue of the nonlinear alternative ([8], Th. 5.1, p. 61). For this purpose we prove a hyperconvex version of the fixed point theorem from [20].

The paper is organized as follows: in Section 2 we collect some definitions and facts which will be needed in the sequel. Section 3 contains fixed point theorems for maps acting in hyperconvex metric spaces. Finally, in Section 4 (the Appendix) we discuss the extension property for nonexpansive mappings considered by Aronszajn and Panitchpakdi in their celebrated paper [1]—we complete details of the proof given by these authors.

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2. PRELIMINARIES

At the beginning of this section we recall the classical notions of hyperconvexity and $\mathfrak{m}$ -hyperconvexity introduced by Aronszajn and Panitchpakdi.

Definition 1. Let $\mathfrak{m}$ be a cardinal number. A metric space (X, d) is called $\mathfrak{m}$ -hyperconvex, if for any family $\{\overline{B}(x_i, r_i)\}_{i \in I}$ of closed balls in X such that $\text{card } I < \mathfrak{m}$ and $d(x_i, x_j) \leq r_i + r_j$ for any $i, j \in I$, the intersection $\bigcap_{i \in I} \overline{B}(x_i, r_i)$ is nonempty.

Definition 2. A metric space is called *hyperconvex*, if it is $\mathfrak{m}$ -hyperconvex for any cardinal $\mathfrak{m}$.

Remark 1 ([1]). An $\mathfrak{m}$ -hyperconvex space is complete if $\mathfrak{m}$ is uncountable.

Classical examples of hyperconvex spaces are the well-known l_∞ , L_∞ ones. It is also known that a Banach space is hyperconvex if and only if its unit ball is hyperconvex (see [7], Th. 3.5).

We will also need the fact, proved by J.-B. Baillon in [3, Theorem 7], that any chain of hyperconvex subsets of a bounded hyperconvex metric space has a nonempty intersection. Using this fact and the method from the proof of [10, Theorem 5.1], the following can be shown.

Proposition 1. *The intersection of any chain of hyperconvex subsets of a (not necessarily bounded) metric space is hyperconvex provided it is nonempty.*

In what follows we shall also use the concept of a hyperconvex hull, introduced by Isbell in his celebrated paper [14].

Definition 3. Let X be a metric space. The pair (E, e) , where E is a hyperconvex metric space and e is an isometric embedding of X into E , is called a *hyperconvex hull* of the metric space X if no hyperconvex proper subset of E includes $e(X)$.

In particular, Isbell proved that a hyperconvex hull exists for any metric space, and although it need not be uniquely determined, any two hyperconvex hulls are isometric. It also turns out (see [3]) that if A is a nonempty subset of a hyperconvex space X , then we can choose its hyperconvex hull (H, i) so that $A \subset H \subset X$ and i is an inclusion map. The set of all such hyperconvex hulls of A will be denoted by $\mathcal{H}(A)$ (or $\mathcal{H}_X(A)$, when it will be important that we consider A as a subset of the space X).

Remark 2. Notice that $\mathcal{H}(A) = \mathcal{H}(\overline{A})$ for a nonempty subset A of a hyperconvex metric space X . (Indeed, hyperconvex hulls are complete and hence closed.)

By $\varrho(\cdot, \cdot)$ we will denote the Euclidean metric in $\mathbb{R}^2$. Let us also recall the definitions of another two metrics in $\mathbb{R}^2$.

Definition 4. The following metric:

$$d(v_1, v_2) = \begin{cases} |y_1 - y_2|, & \text{if } x_1 = x_2, \\ |y_1| + |y_2| + |x_1 - x_2|, & \text{if } x_1 \neq x_2, \end{cases}$$

where $v_i = (x_i, y_i) \in \mathbb{R}^2$ for $i = 1, 2$, is called the “*river*” metric in $\mathbb{R}^2$.

Next, the *radial metric* in $\mathbb{R}^2$ is defined as follows:

$$d(v_1, v_2) = \begin{cases} \varrho(v_1, v_2), & \text{if } 0 = (0, 0), v_1, v_2 \text{ are collinear,} \\ \varrho(v_1, 0) + \varrho(0, v_2), & \text{otherwise.} \end{cases}$$

It was proved in [6] that these two metric spaces are also hyperconvex. We will apply them in Section 3 to illustrate our results.

For completeness, we will also give the definition of the Kuratowski measure of noncompactness.

Definition 5. Let A be a bounded subset of a metric space X . The *Kuratowski measure of noncompactness* of the set A is the infimum of the set of all positive numbers ε such that A may be covered by finitely many sets with diameters less than ε .

3. A FIXED POINT THEOREM FOR HYPERCONVEX SPACES

Let us begin with the main result of this section. (Throughout this section, by I we will denote the real unit interval $[0, 1]$.)

Theorem 2. Let X be a hyperconvex metric space and Ω its nonempty, open subset. Let us assume that there exists a continuous homotopy $H: I \times \overline{\Omega} \rightarrow X$ such that:

- (1) $H(0, \cdot)$ has a subadditive modulus of continuity (in particular, it is uniformly continuous) and $H(\{0\} \times \overline{\Omega}) \subset V$, where V is a compact and hyperconvex subset of $\overline{\Omega}$;
- (2) none of the mappings $H(\lambda, \cdot)$, where $\lambda \in [0, 1]$, has a fixed point in $\partial\Omega := \overline{\Omega} \setminus \Omega$;
- (3) each subset $C \subset \Omega$ such that $C = \Omega \cap P$ for some $P \in \mathcal{H}_X(H(I \times C) \cup V)$ is relatively compact.

Then $H(1, \cdot)$ has a fixed point in $\overline{\Omega}$.

Proof. First we will show that there exists a compact subset $C \subset X$ which is a hyperconvex hull of $H(I \times (C \cap \Omega)) \cup V$ in X . Let $\Sigma := \{U \subset X : V \subset U, U \text{ hyperconvex}, H(I \times (U \cap \Omega)) \subset U\}$. The set Σ is nonempty, because $X \in \Sigma$, and Proposition 1 shows that the intersection of any chain in Σ belongs to Σ . Let U_0 be minimal in Σ . Let $C \in \mathcal{H}_{U_0}(H(I \times (U_0 \cap \Omega)) \cup V)$; it is easy to see that $C \in \Sigma$. As $C \subset U_0$ and U_0 is minimal in Σ we must have $C = U_0$. In particular, $C \in \mathcal{H}_X(H(I \times (C \cap \Omega)) \cup V)$. Let $C' := C \cap \Omega$. From (3) we know that C' is relatively compact. Now we have:

$$H(I \times \overline{C'}) \cup V = H(\overline{I \times C'}) \cup V = \overline{H(I \times C')} \cup \overline{V} = \overline{H(I \times C') \cup V},$$

where the second equality follows from the compactness of $\overline{I \times C'}$ and V . Using Remark 2, we obtain:

$$\begin{aligned} C &\in \mathcal{H}_{U_0}(H(I \times ((C \cap \Omega) \cup V))) \\ &= \mathcal{H}_{U_0}(\overline{H(I \times ((C \cap \Omega) \cup V))}) \\ &= \mathcal{H}_{U_0}(H(I \times (\overline{(C \cap \Omega)} \cup V))). \end{aligned}$$

As $\overline{C \cap \Omega}$ is compact, so is C .

In the rest of the proof we use the method from [20]; for convenience of the reader, we will repeat it here. Let S be the set of all fixed points of the mappings $H(\lambda, \cdot)$ ($\lambda \in I$) in the set $C \cap \overline{\Omega}$, i.e., $S := \bigcup_{\lambda \in I} \text{Fix}_{C \cap \overline{\Omega}} H(\lambda, \cdot)$. If $H(1, \cdot)$ has a fixed point in $\partial\Omega$, the proof is finished; in the other case, from (2) we know that $S \cap \partial\Omega = \emptyset$. Consider the function $(\lambda, x) \mapsto (x, H(\lambda, x))$ that maps the compact set $I \times (C \cap \overline{\Omega})$ into $X \times X$. The preimage of the diagonal $\{(y, y) : y \in X\}$ with respect to this mapping is closed and hence compact and so is S , being its image

under the projection onto the second coordinate. By definition we have $S \subset \overline{\Omega}$; as $S \cap \partial\Omega = \emptyset$, so $S \subset \Omega$. This means that the closed sets $X \setminus \Omega$ and S are disjoint. Let $\lambda: X \rightarrow I$ be a continuous function such that $\lambda|_S \equiv 1$ and $\lambda|_{X \setminus \Omega} \equiv 0$.

Denote $H_0 := H(0, \cdot)|_{C \cap \overline{\Omega}}$. It maps the set $C \cap \overline{\Omega}$ uniformly into the hyperconvex space V and has a subadditive modulus of continuity, so in view of [1] we can extend it to a continuous map $\tilde{H}_0: C \rightarrow V$. Let us define the function $F: C \rightarrow X$ by the formula:

$$F(x) := \begin{cases} H(\lambda(x), x) & \text{for } x \in C \cap \overline{\Omega}, \\ \tilde{H}_0(x) & \text{for } x \in C \setminus \overline{\Omega}. \end{cases}$$

For $x \in C \cap \Omega$ we have $F(x) \in C$ and for $x \in C \setminus \Omega$ we have $\lambda(x) = 0$, and hence $F(x) = \tilde{H}_0(x) \in V \subset C$. To see that F is continuous, it is enough to check that the two expressions from its definition coincide on the intersection of the closed sets $C \cap \overline{\Omega}$ and $C \setminus \Omega$, whose union is C . Let $x \in (C \cap \overline{\Omega}) \cap (C \setminus \Omega)$. Then $\lambda(x) = 0$, so $H(\lambda(x), x) = H(0, x) = H_0(x) = \tilde{H}_0(x)$. The function F is therefore a continuous mapping of the nonempty, compact and hyperconvex set C into itself, so by [9] it has a fixed point $x_0 \in C$.

If $x_0 \in C \setminus \overline{\Omega}$, we would have $x_0 = F(x_0) = \tilde{H}_0(x_0) \in V \subset \overline{\Omega}$, which is a contradiction. Hence $x_0 \in \overline{\Omega}$. This means that x_0 is a fixed point of the mapping $H(\lambda(x_0), \cdot)$. From the definition of S we have $x_0 \in S$ and so $\lambda(x_0) = 1$. It turns out that x_0 is a fixed point of the mapping $H(1, \cdot)$. $\square$

We will show three examples of constructing a homotopy joining any given continuous mapping with a mapping to a compact and hyperconvex set V .

Example 1. Let $X := \langle \mathbb{R}^2, d \rangle$, where d is the radial metric. Let $F: X \rightarrow X$ be continuous and $H: I \times X \rightarrow X$ be defined by the formula $H(\lambda, x) := \lambda F(x)$. It is easy to check that H is continuous, $H(1, \cdot) = F$ and $H(0, \cdot) \equiv 0$.

Example 2. Let $X := \langle \mathbb{R}^2, d \rangle$, where d is the “river” metric. Again let $F = (F_1, F_2): X \rightarrow X$ be any continuous mapping; define $H: I \times X \rightarrow X$ by the formula $H(\lambda, x) := (F_1(x), \lambda F_2(x))$. Obviously $H(1, \cdot) = F$ and $H(\{0\} \times X) \subset \{(x, 0) | x \in \mathbb{R}\}$; it is also possible to check that H is continuous. Now we can use the fact that any mapping with a range in $\mathbb{R}$ is null homotopic and add another homotopy to H in order to obtain a homotopy joining F with a constant function $F'(x) = (0, 0)$.

Example 3. In this example we will use the notation and facts from [2, Section 3]. Let us recall the basic definitions. Let $(X, \|\cdot\|)$ be a real normed space, $C \subset X$ its compact Chebyshev subset, d_C a hyperconvex metric on C . By x^p we will denote the (unique) metric projection of x onto C . We define the metric d on X by the following formula:

$$d(x, y) := \begin{cases} \|x - y\| & \text{if } x^p = y^p \text{ and } x, x^p, y \text{ are collinear,} \\ \|x - x^p\| + d_C(x^p, y^p) + \|y^p - y\| & \text{otherwise.} \end{cases}$$

It is proved in [2] that a metric so defined is hyperconvex.

Let $F: X \rightarrow X$ be continuous and let $H: I \times X \rightarrow X$ be defined by the formula $H(\lambda, x) := (F(x))^p + \lambda(F(x) - (F(x))^p)$. Obviously $H(1, \cdot) = F$ and $H(\{0\} \times X) \subset C$. Moreover, note that if $F(x) \in C$, then $H(\lambda, x) = F(x)$ for any $\lambda \in I$. We will now show that H is continuous.

Note that if $y \in C$, then $d(x, y) = \|x - x^p\| + d_C(x^p, y^p) + \|y^p - y\| = \|x - x^p\| + d_C(x^p, y)$ irrespective of whether the condition “ $x^p = y^p$ and x, x^p, y are collinear”

is satisfied or not. Let $\lambda_0 \in I$, $x_0 \in X$, $\varepsilon > 0$. We will consider two cases: (i) $F(x_0) \in C$ and (ii) $F(x_0) \notin C$. We shall endow $I \times X$ with the “maximum” metric.

(i) If $F(x_0) \in C$, then $H(\lambda, x_0) = F(x_0)$ for each $\lambda \in I$. Let $\delta > 0$ be such that $d(F(x_0), F(x)) \leq \varepsilon$ if $d(x_0, x) \leq \delta$. We have for any $\lambda \in I$:

$$\begin{aligned} d(H(\lambda_0, x_0), H(\lambda, x)) &= d(F(x_0), (F(x))^p + \lambda(F(x) - (F(x))^p)) \\ &= \|\lambda(F(x) - (F(x))^p)\| + d_C((F(x))^p, F(x_0)) \\ &\leq \|F(x) - (F(x))^p\| + d_C((F(x))^p, F(x_0)) = d(F(x_0), F(x)) \leq \varepsilon. \end{aligned}$$

(ii) The case $F(x_0) \notin C$ is slightly more complicated. We can safely assume that $\varepsilon < 2\|F(x_0) - (F(x_0))^p\|$. Let $\delta > 0$ be such that $d(F(x_0), F(x)) \leq \frac{1}{2}\varepsilon < \|F(x_0) - (F(x_0))^p\|$ if $d(x_0, x) \leq \delta$; this means that $F(x) \notin C$ for such x 's and the condition “ $(F(x_0))^p = (F(x))^p$ and $F(x_0), (F(x_0))^p, F(x)$ are collinear” is satisfied and hence $d(F(x_0), F(x)) = \|F(x_0) - F(x)\|$. We can decrease δ if necessary so that $\delta \leq \frac{\varepsilon}{2\|F(x_0) - (F(x_0))^p\| + \varepsilon}$. Let $d((\lambda_0, x_0), (\lambda, x)) \leq \delta$, i.e., $|\lambda_0 - \lambda| \leq \delta$ and $d(x_0, x) \leq \delta$. Denote $z_0 := F(x_0) - (F(x_0))^p$, $z := F(x) - (F(x))^p$, $y_0 := (F(x_0))^p + \lambda_0 z_0$, $y := (F(x))^p + \lambda z$. Note that $y_0 \in [(F(x_0))^p, F(x_0)]$, so $y_0^p = (F(x_0))^p$ and by a similar argument $y^p = (F(x))^p$. Knowing this, we can estimate as follows:

$$\begin{aligned} d(H(\lambda_0, x_0), H(\lambda, x)) &= d(y_0, y) = \|y_0 - y\| = \|\lambda_0 z_0 - \lambda z\| \\ &\leq \|\lambda_0 z_0 - \lambda_0 z\| + \|\lambda_0 z - \lambda z\| \leq |\lambda_0| \|z_0 - z\| + |\lambda_0 - \lambda| \|z\| \\ &\leq \|z_0 - z\| + |\lambda_0 - \lambda| (\|z - z_0\| + \|z_0\|) \\ &\leq \|F(x_0) - F(x)\| + \delta (\|F(x_0) - F(x)\| + \|F(x_0) - (F(x_0))^p\|) \\ &\leq d(F(x_0), F(x)) + \delta (\|F(x_0) - (F(x_0))^p\| + \frac{1}{2}\varepsilon) \leq \varepsilon, \end{aligned}$$

which finishes the proof.

Corollary 3. *Let Ω be a nonempty, open subset of a hyperconvex metric space X . Let there exist a continuous homotopy $H: I \times \bar{\Omega} \rightarrow X$ satisfying conditions (1)–(2) from Theorem 2. In addition, let $\mu(H(I \times C)) < \mu(C)$ whenever $\mu(C) > 0$, where μ is either Kuratowski's or Hausdorff's measure of noncompactness. Then $H(1, \cdot)$ has a fixed point in $\bar{\Omega}$.*

Proof. It is enough to check that the condition (3) from Theorem 2 is satisfied. Assume that for some $C \subset \Omega$ we have $C = \Omega \cap P$, where $P \in \mathcal{H}_X(H(I \times C) \cup V)$. Then, by [10, Corollary 5.11], $\mu(C) = \mu(\Omega \cap P) \leq \mu(P) = \mu(H(I \times C) \cup V) = \mu(H(I \times C))$; but this is only possible when $\mu(C) = 0$, i.e., when C is relatively compact. $\square$

Corollary 4. *Let X be a metrizable topological vector space, $K \subset X$ be star-shaped with respect to 0 and hyperconvex, $\Omega \subset K$ be open in K and such that $0 \in \Omega$. Let $F: \bar{\Omega} \rightarrow K$ be continuous and such that for any $x \in \partial\Omega$ and $\gamma > 1$, $F(x) \neq \gamma x$. Assume also that each subset $C \subset \Omega$ such that $C = \Omega \cap P$, where $P \subset K$ is a hyperconvex hull of the set $\bigcup_{\lambda \in [0,1]} \lambda F(C)$ in K , is relatively compact. Then F has a fixed point in $\bar{\Omega}$.*

Proof. Put $H(\lambda, x) := \lambda F(x)$ and $V := \{0\}$. It is now enough to apply Theorem 2 with K in place of X . $\square$

Corollary 5. *Let X be a metrizable topological vector space, $K \subset X$ be hyperconvex, $\Omega \subset K$ be open in K . Let $r: X \rightarrow K$ be a continuous retraction onto K such that $r(0) \in \overline{\Omega}$. Let $F: \overline{\Omega} \rightarrow K$ be continuous and such that each $C \subset \Omega$ satisfying $C = \Omega \cap P$, where $P \subset K$ is a hyperconvex hull of $r(\bigcup_{\lambda \in [0,1]} \lambda F(C))$ in K , is relatively compact. Assume also that $r(\lambda F(x)) \neq x$ for $x \in \partial\Omega$ and $\lambda \in [0,1]$. Then F has a fixed point in $\overline{\Omega}$.*

Proof. Again let $V := \{0\}$, but now let $H(\lambda, x) := r(\lambda F(x))$. The corollary follows immediately from Theorem 2. $\square$

The result given below (Corollary 6) is one of the most important results in this note. It is a metric analogue of the well-known topological nonlinear alternative.

Corollary 6 (A nonlinear alternative). *Let X be a metrizable topological vector space, $K \subset X$ be hyperconvex, $\Omega \subset K$ be open in K . Let $r: X \rightarrow K$ be a continuous retraction onto K such that $r(0) \in \overline{\Omega}$. Then each continuous, compact mapping $F: \overline{\Omega} \rightarrow K$ has at least one of the following properties:*

- (1) *F has a fixed point;*
- (2) *there exists an $x \in \partial\Omega$ with $x = r(\lambda F(x))$ for some $\lambda \in [0,1]$.*

Proof. Assume that (2) is not satisfied. We will show that the assumptions of Theorem 2 are satisfied with $H(\lambda, x) := r(\lambda F(x))$. Conditions (1) and (2) from that theorem are obviously true. Let $C \subset \Omega$ be such that $C = \Omega \cap P$ for some $P \in \mathcal{H}_X(H(I \times C) \cup V) = \mathcal{H}_X(\bigcup_{\lambda \in I} r(\lambda F(C)))$. Note that from the well-known properties of the Kuratowski measure of noncompactness it follows that $\bigcup_{\lambda \in I} r(\lambda F(C))$ is relatively compact, which completes the proof. $\square$

4. APPENDIX

We will now turn our attention to the proof of the following theorem on $\mathfrak{m}$ -hyperconvexity, proved by Aronszajn and Panitchpakdi in [1]. Note that by *m-separability* the authors mean the existence of a dense subset of cardinality less than $\mathfrak{m}$. (For the convenience of the reader, we use the same notation as in the rest of our paper, although it differs from that used by the cited authors.)

Theorem 7 ([1, p. 413, Theorem 2]). *Let (Y, d) be a metric space and $\mathfrak{m}$ be a cardinal number greater than 2. If for every nonexpansive mapping $f: A \rightarrow Y$ and for any $(\mathfrak{m} + 1)$ -separable metric space X including A there exists a nonexpansive extension $\tilde{f}: X \rightarrow Y$ of the mapping f , then the space Y is $\mathfrak{m}$ -hyperconvex.*

In the course of the proof, the authors consider a family of closed balls $\{\overline{B}(x_i, r_i)\}_{i \in I}$ with $\text{card } I < \mathfrak{m}$ such that $d(x_i, x_j) \leq r_i + r_j$ for $i, j \in I$ and define $A := \{x_i\}_{i \in I}$, $X := A \cup \{\xi\}$ where $\xi \notin Y$. Then a distance function is introduced in X by the formula $d(x, \xi) := r'(x) := \inf\{r > 0 : \exists i \in I \overline{B}(x_i, r_i) \subset \overline{B}(x, r)\}$ for $x \in A$; the distances between other pairs of points are inherited from Y . It is then claimed that a distance so defined is a metric. In fact, it need not be; as the following example shows, it may be a pseudometric.

Example 4. Let $\mathfrak{m} := 2^c$, $Y := [0, 1]$, $I := (0, 1]$, $x_i := r_i := i$ for $i \in I$. Obviously $d(x_i, x_j) \leq r_i + r_j$ for $i, j \in I$. On the other hand, for any $r > 0$ there exists an $i \in I$ such that $\overline{B}(x_i, r_i) \subset \overline{B}(0, r)$. (Indeed, it is enough to take $i := \min\{1, \frac{1}{2}r\}$.) This means that $r'(0) = 0$.

Of course, the proof remains valid, as it is enough to notice that if $r'(x) = 0$ for some $x \in A$, then $x \in \bigcap_{i \in I} \overline{B}(x_i, r_i)$, which yields the $\mathbf{m}$ -hyperconvexity of Y . In fact, if $r'(x) = 0$, but $x \notin \overline{B}(x_{i_0}, r_{i_0})$ for some $i_0 \in I$, we would have $\overline{B}(x, \varepsilon) \cap \overline{B}(x_{i_0}, r_{i_0}) = \emptyset$ for some $\varepsilon > 0$; but then, taking $j_0 \in I$ such that $\overline{B}(x_{j_0}, r_{j_0}) \subset \overline{B}(x, \varepsilon)$, we would obtain $\overline{B}(x_{i_0}, r_{i_0}) \cap \overline{B}(x_{j_0}, r_{j_0}) = \emptyset$, which contradicts the hypothesis.

Note also that a similar flaw can be found in the proof of Theorem 4.2 in [10], which states that injectivity is equivalent to hyperconvexity (recall that a metric space Z is *injective* if for any subspace Y of any metric space X every nonexpansive $f: Y \rightarrow Z$ has a nonexpansive extension $\tilde{f}: X \rightarrow Z$). The method of proof is analogous. At some point the authors consider a family of closed balls $\{\overline{B}(x_i, r_i)\}_{i \in I}$ in some space Y such that $d(x_i, x_j) \leq r_i + r_j$ for $i, j \in I$ and define $A := \{x_i\}_{i \in I}$, $X := A \cup \{\xi\}$ where $\xi \notin Y$. Then a family $\mathcal{F}$ of nonnegative functions $f: A \rightarrow [0, +\infty)$ satisfying $d(x_i, x_j) \leq f(x_i) + f(x_j)$ and $f(x_i) \leq r_i$ for all $i, j \in I$ is considered; from the Kuratowski–Zorn lemma there exists a minimal element f' in $(\mathcal{F}, \leq)$, where the partial order denoted by $\leq$ is a usual pointwise order. It is then claimed that putting $d(x_i, \xi) := f'(x_i)$ allows us to extend a metric from A to a metric in X . Once again it does not have to be metric, which is proved by a similar example as before.

Example 5. Let $Y := \mathbb{R}$, $I := \{1, 2\}$, $x_1 := 0$, $x_2 := 1$, $r_0 := r_1 := 1$. Notice that a function $f': \{0, 1\} \rightarrow [0, +\infty)$ given by the formulae $f'(0) := 0$, $f'(1) := 1$ is minimal in $\mathcal{F}$, but it vanishes at x_1 and thus d as defined earlier is not a metric.

An identical argument as before shows that the proof is valid provided the case when f' equals zero at some $x \in A$ is considered separately. It is straightforward that in such a case $x \in \bigcap_{i \in I} \overline{B}(x_i, r_i)$, which finishes the proof.

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OPTIMIZATION AND CONTROL THEORY DEPARTMENT, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, ADAM MICKIEWICZ UNIVERSITY, UL. UMULTOWSKA 87, 61-614 POZNAŃ, POLAND
E-mail address: mbork@amu.edu.pl

DEPARTMENT OF MATHEMATICS, MORGAN STATE UNIVERSITY, 1700 E. COLD SPRING LANE, BALTIMORE, MARYLAND 21251
E-mail address: dbugajew@jewel.morgan.edu