WAVELET MULTIPLIERS ON $L^p(\mathbb{R}^n)$

YU LIU, ALIP MOHAMMED, AND M. W. WONG

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ABSTRACT. We give results on the boundedness and compactness of wavelet multipliers on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$.

1. WAVELET MULTIPLIERS

Let $\sigma \in L^\infty(\mathbb{R}^n)$. Then we define the linear operator $T_\sigma : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ by

$$T_\sigma u = (\sigma \hat{u})^\vee, \quad u \in L^2(\mathbb{R}^n),$$

where $\hat{u}$ is the Fourier transform of u defined by

$$\hat{u}(\xi) = \lim_{R \rightarrow \infty} (2\pi)^{-n/2} \int_{|x| \leq R} e^{-ix \cdot \xi} u(x) dx, \quad \xi \in \mathbb{R}^n,$$

the convergence is understood to take place in $L^2(\mathbb{R}^n)$ and $(\sigma \hat{u})^\vee$ is the inverse Fourier transform of $\sigma \hat{u}$. It is a consequence of Plancherel's theorem that $T_\sigma : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is a bounded linear operator.

Let $\pi : \mathbb{R}^n \rightarrow U(L^2(\mathbb{R}^n))$ be the unitary representation of the additive group $\mathbb{R}^n$ on $L^2(\mathbb{R}^n)$ defined by

$$(\pi(\xi)u)(x) = e^{ix \cdot \xi} u(x), \quad x, \xi \in \mathbb{R}^n,$$

for all functions u in $L^2(\mathbb{R}^n)$, where $U(L^2(\mathbb{R}^n))$ is the group of all unitary operators on $L^2(\mathbb{R}^n)$. Let φ be any function in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_2 = 1$, where $\|\cdot\|_p$ denotes the norm in $L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$. Then it is proved in [9] that

$$(1.1) \quad (\varphi u, \varphi v) = (2\pi)^{-n} \int_{\mathbb{R}^n} (u, \pi(\xi)\varphi)(\pi(\xi)\varphi, v) d\xi$$

for all functions u and v in the Schwartz space $\mathcal{S}$, where $(\cdot, \cdot)$ is the inner product in $L^2(\mathbb{R}^n)$.

Let $\sigma \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then for all functions u in $\mathcal{S}$, we define $P_{\sigma, \varphi} u$ by

$$(1.2) \quad (P_{\sigma, \varphi} u, v) = (2\pi)^{-n} \int_{\mathbb{R}^n} \sigma(\xi) (u, \pi(\xi)\varphi) (\pi(\xi)\varphi, v) d\xi, \quad v \in \mathcal{S}.$$

Then it can be proved easily that $P_{\sigma, \varphi} u \in L^2(\mathbb{R}^n)$ for all u in $\mathcal{S}$ and $P_{\sigma, \varphi}$, initially defined on $\mathcal{S}$, can be extended to a bounded linear operator from $L^2(\mathbb{R}^n)$ into

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$L^2(\mathbb{R}^n)$. In fact, we have the following theorem in [9], which is formulated as Proposition 19.6 in [12].

Theorem 1.1. *Let $\sigma \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. Then there exists a unique bounded linear operator $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ such that*

$$\|P_{\sigma,\varphi}\|_* \leq (2\pi)^{-n/p} \|\varphi\|_\infty^{2/p'} \|\sigma\|_p,$$

where $\|\cdot\|_*$ is the norm in the C^* -algebra of all bounded linear operators from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$, and for all functions u and v in $L^2(\mathbb{R}^n)$, $(P_{\sigma,\varphi}u, v)$ is given by (1.2) for all simple functions σ on $\mathbb{R}^n$ for which the Lebesgue measure of the set $\{\xi \in \mathbb{R}^n : \sigma(\xi) \neq 0\}$ is finite.

Remark 1.2. Let $\sigma \in L^\infty(\mathbb{R}^n)$. Then it is proved in [9] that the bounded linear operators $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ and $\varphi T_\sigma \overline{\varphi} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ are unitarily equivalent. Henceforth we denote $\varphi T_\sigma \overline{\varphi}$ by $P_{\sigma,\varphi}$. By (1.1) and (1.2), the bounded linear operator $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is a variant of a localization operator first studied in [4], [5], [6] and [8], and extensively in [12]. Had the “admissible wavelet” φ in (1.2) been replaced by the function φ_0 on $\mathbb{R}^n$ given by $\varphi_0(x) = 1$ for all x in $\mathbb{R}^n$, we would have obtained $T_\sigma : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ instead of $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$. In other words, the bounded linear operator $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ would have been a Fourier multiplier. Since the function φ in the bounded linear operator $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ plays the role of the admissible wavelet in a localization operator, it is natural to call the bounded linear operator $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ a wavelet multiplier.

The results on wavelet multipliers hitherto obtained are on $L^2(\mathbb{R}^n)$. See, for instance, the works [6], [8], [9] and [12]. In [11], the L^p -boundedness of localization operators associated to left regular representations is studied for $1 \leq p \leq \infty$. L^p -boundedness and L^p -compactness of two-wavelet localization operators on the Weyl-Heisenberg group can be found in [2] and [3]. A set of results on the L^p -boundedness of wavelet multipliers obtained by reducing the problem to the corresponding one on Fourier multipliers is given in [10]. The aim of this paper is to give another set of results on the L^p -boundedness and also L^p -compactness of wavelet multipliers.

In [12], a function φ in $L^2(\mathbb{R}^n)$ satisfying $\|\varphi\|_2 = 1$ and

$$\int_{\mathbb{R}^n} |(\varphi, \pi(\xi)\varphi)|^2 d\xi < \infty$$

is said to be an admissible wavelet of $\pi : \mathbb{R}^n \rightarrow U(L^2(\mathbb{R}^n))$. For every admissible wavelet φ , we define the wavelet constant c_φ by

$$c_\varphi = \int_{\mathbb{R}^n} |(\varphi, \pi(\xi)\varphi)|^2 d\xi.$$

It can be found on page 111 of [12] that the set of admissible wavelets for $\pi : \mathbb{R}^n \rightarrow U(L^2(\mathbb{R}^n))$ consists of all functions φ in $L^2(\mathbb{R}^n) \cap L^4(\mathbb{R}^n)$ for which $\|\varphi\|_2 = 1$, and for every admissible wavelet φ ,

$$c_\varphi = (2\pi)^n \|\varphi\|_4^4.$$

For the linear operators studied in this paper, we use functions φ , which are not admissible wavelets *per se*, but there is no danger of confusion in still calling them admissible wavelets.

In Section 2, we give two different proofs for the fact that wavelet multipliers associated to symbols in $L^1(\mathbb{R}^n)$ and admissible wavelets in $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ are bounded linear operators on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. If the symbols are in $L^r(\mathbb{R}^n)$, $1 \leq r \leq 2$, and the admissible wavelets are in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then the associated wavelet multipliers are proved in Section 3 to be bounded linear operators on $L^p(\mathbb{R}^n)$, $r \leq p \leq r'$. The boundedness results are then sharpened in Section 4 to results on compactness. In spite of the compactness results in Section 4, the boundedness results in Sections 2 and 3 are not completely superseded because the technique that we use to obtain compactness does not give explicit estimates for the norms of the wavelet multipliers.

2. SYMBOLS IN $L^1(\mathbb{R}^n)$

Let $\sigma \in L^1(\mathbb{R}^n)$ and let $\varphi \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then for all f in $L^1(\mathbb{R}^n)$, we define $P_{\sigma,\varphi}f$ by

$$(2.1) \quad (P_{\sigma,\varphi}f)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \sigma(\xi)(f, \pi(\xi)\varphi)(\pi(\xi)\varphi)(x) d\xi, \quad x \in \mathbb{R}^n,$$

where $(\cdot, \cdot)$ is defined by

$$(u, v) = \int_{\mathbb{R}^n} u(x) \overline{v(x)} dx$$

for all measurable functions u and v on $\mathbb{R}^n$ whenever the integral exists.

Proposition 2.1. *For $p = 1$ or $p = \infty$, $P_{\sigma,\varphi} : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ is a bounded linear operator and*

$$\|P_{\sigma,\varphi}\|_{B(L^p(\mathbb{R}^n))} \leq (2\pi)^{-n} \|\varphi\|_1 \|\varphi\|_\infty \|\sigma\|_1,$$

where $\|\cdot\|_{B(L^p(\mathbb{R}^n))}$ is the norm in the Banach space $B(L^p(\mathbb{R}^n))$ of all bounded linear operators from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$.

Proposition 2.1 follows from very elementary estimates, which are omitted.

Remark 2.2. In Proposition 2.1, the case for $p = \infty$ follows from that for $p = 1$ and the fact that the adjoint of $P_{\sigma,\varphi} : L^1(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$ is $P_{\sigma,\varphi} : L^\infty(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n)$.

Using Proposition 2.1 and an interpolation, we get the following theorem.

Theorem 2.3. *For $1 \leq p \leq \infty$, there exists a unique bounded linear operator $P_{\sigma,\varphi} : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ such that*

$$\|P_{\sigma,\varphi}\|_{B(L^p(\mathbb{R}^n))} \leq (2\pi)^{-n} \|\varphi\|_1 \|\varphi\|_\infty \|\sigma\|_1,$$

where $\|\cdot\|_{B(L^p(\mathbb{R}^n))}$ is the norm in the Banach space $B(L^p(\mathbb{R}^n))$ of all bounded linear operators from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ and for all f in $L^p(\mathbb{R}^n)$, $P_{\sigma,\varphi}f$ is given by (2.1) for all simple functions σ on $\mathbb{R}^n$ for which the Lebesgue measure of the set $\{\xi \in \mathbb{R}^n : \sigma(\xi) \neq 0\}$ is finite.

We can give another proof of Theorem 2.3 by means of the following lemma, which is the well-known Schur's lemma and can be found in [7].

Lemma 2.4. *Let X be a σ -finite measure space, and let K be a measurable function on $X \times X$ such that there exists a positive constant C for which*

$$\sup_{x \in X} \int_X |K(x, y)| dy \leq C$$

and

$$\sup_{y \in X} \int_X |K(x, y)| dx \leq C.$$

Then for $1 \leq p \leq \infty$, the mapping $A \in L^p(X) \rightarrow L^p(X)$ defined by

$$(Au)(x) = \int_X K(x, y)u(y) dy, \quad x \in X,$$

for all u in $L^p(X)$, is a bounded linear operator and

$$\|Au\|_{L^p(X)} \leq C\|u\|_{L^p(X)}, \quad u \in L^p(X).$$

We call the bounded linear operator $A : L^p(X) \rightarrow L^p(X)$ in Lemma 2.4 the integral operator on $L^p(X)$ with kernel K .

Another Proof of Theorem 2.3. Let K be the function on $\mathbb{R}^n \times \mathbb{R}^n$ defined by

$$(2.2) \quad K(x, y) = \int_{\mathbb{R}^n} \sigma(\xi) \overline{(\pi(\xi)\varphi)(x)} (\pi(\xi)\varphi)(y) d\xi, \quad x, y \in \mathbb{R}^n.$$

Then, by Minkowski's inequality, we get

$$\begin{aligned} \int_{\mathbb{R}^n} |K(x, y)| dy &\leq \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \sigma(\xi) \overline{(\pi(\xi)\varphi)(x)} (\pi(\xi)\varphi)(y) d\xi \right| dy \\ &\leq \int_{\mathbb{R}^n} |\sigma(\xi)| |(\pi(\xi)\varphi)(x)| \left(\int_{\mathbb{R}^n} |(\pi(\xi)\varphi)(y)| dy \right) d\xi \\ &\leq \|\varphi\|_1 \int_{\mathbb{R}^n} |\sigma(\xi)| |(\pi(\xi)\varphi)(x)| d\xi \\ &\leq \|\varphi\|_1 \|\varphi\|_\infty \int_{\mathbb{R}^n} |\sigma(\xi)| d\xi \\ &= \|\varphi\|_1 \|\varphi\|_\infty \|\sigma\|_1 \end{aligned}$$

for all x in $\mathbb{R}^n$. Therefore

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |K(x, y)| dy \leq \|\varphi\|_1 \|\varphi\|_\infty \|\sigma\|_1.$$

Similarly,

$$\sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} |K(x, y)| dx \leq \|\varphi\|_1 \|\varphi\|_\infty \|\sigma\|_1.$$

Hence, by Lemma 2.4, $P_{\sigma, \varphi} : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ is a bounded linear operator for $1 \leq p \leq \infty$ and

$$\|P_{\sigma, \varphi}\|_{B(L^p(\mathbb{R}^n))} \leq (2\pi)^{-n} \|\varphi\|_1 \|\varphi\|_\infty \|\sigma\|_1.$$

Remark 2.5. The fringe benefit of the alternative proof is that the unique bounded linear operator on $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, obtained by interpolation in Theorem 2.3 is in fact the integral operator on $L^p(\mathbb{R}^n)$ with kernel given by (2.2).

3. SYMBOLS IN $L^r(\mathbb{R}^n)$, $1 \leq r \leq 2$

We begin with the following result, which is a new and interesting supplement to Proposition 19.6 in [12].

Proposition 3.1. *Let $\varphi \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be such that $\|\varphi\|_2 = 1$. Then for all $\sigma \in L^2(\mathbb{R}^n)$, $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is a bounded linear operator, where $P_{\sigma,\varphi}f$ is defined by (2.1) for all f in $L^2(\mathbb{R}^n)$, and*

$$\|P_{\sigma,\varphi}\|_{B(L^2(\mathbb{R}^n))} \leq (2\pi)^{-n/2} \|\varphi\|_\infty \|\sigma\|_2.$$

Proof. For all f and g in $L^2(\mathbb{R}^n)$, the function $G : \mathbb{R}^n \rightarrow \mathbb{C}$ defined by

$$(3.1) \quad G(\xi) = (f, \pi(\xi)\varphi)(\pi(\xi)\varphi, g), \quad \xi \in \mathbb{R}^n,$$

is in $L^2(\mathbb{R}^n)$. Indeed, using Schwarz' inequality and $\|\varphi\|_2 = 1$, we get

$$(3.2) \quad |(\pi(\xi)\varphi, g)| \leq \|g\|_2, \quad \xi \in \mathbb{R}^n.$$

So, by (3.2),

$$(3.3) \quad \int_{\mathbb{R}^n} |(f, \pi(\xi)\varphi)(\pi(\xi)\varphi, g)|^2 d\xi \leq \left(\int_{\mathbb{R}^n} |(f, \pi(\xi)\varphi)|^2 d\xi \right) \|g\|_2^2.$$

By Plancherel's formula, we get

$$(3.4) \quad \begin{aligned} \int_{\mathbb{R}^n} |(f, \pi(\xi)\varphi)|^2 d\xi &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} e^{ix \cdot \xi} \varphi(x) \overline{f(x)} dx \right|^2 d\xi \\ &= (2\pi)^n \|\varphi \overline{f}\|_2^2 \leq (2\pi)^n \|\varphi\|_\infty^2 \|f\|_2^2. \end{aligned}$$

Thus, by (3.1), (3.3), (3.4) and Schwarz' inequality, we get

$$\begin{aligned} |(P_{\sigma,\varphi}f, g)| &\leq (2\pi)^{-n} \|\sigma\|_2 \|G\|_2 \\ &\leq (2\pi)^{-n/2} \|\varphi\|_\infty \|\sigma\|_2 \|f\|_2 \|g\|_2 \end{aligned}$$

for all f and g in $L^2(\mathbb{R}^n)$. □

The following result is a new and interesting supplement to Theorem 19.11 in [12].

Theorem 3.2. *Let $\varphi \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be such that $\|\varphi\|_2 = 1$. Then for all $\sigma \in L^2(\mathbb{R}^n)$, $P_{\sigma,\varphi} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is a Hilbert-Schmidt operator, where $P_{\sigma,\varphi}f$ is defined by (2.1) for all f in $L^2(\mathbb{R}^n)$, and*

$$\|P_{\sigma,\varphi}\|_{S_2} \leq (2\pi)^{-n/2} \|\varphi\|_\infty \|\sigma\|_2.$$

Proof. Let $\sigma \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Let $\{\varphi_k : k = 1, 2, \dots\}$ be an orthonormal basis for $L^2(\mathbb{R}^n)$. Then, by Fubini's theorem and Parseval's identity,

$$\begin{aligned}
 & \sum_{k=1}^{\infty} \|P_{\sigma, \varphi} \varphi_k\|_2^2 \\
 &= \sum_{k=1}^{\infty} (P_{\sigma, \varphi} \varphi_k, P_{\sigma, \varphi} \varphi_k) \\
 &= (2\pi)^{-2n} \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} \sigma(\xi) (\varphi_k, \pi(\xi) \varphi) (\pi(\xi) \varphi, P_{\sigma, \varphi} \varphi_k) d\xi \\
 &= (2\pi)^{-2n} \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} \sigma(\xi) (\varphi_k, \pi(\xi) \varphi) (P_{\bar{\sigma}, \varphi} \pi(\xi) \varphi, \varphi_k) d\xi \\
 (3.5) \quad &= (2\pi)^{-2n} \int_{\mathbb{R}^n} \sigma(\xi) (P_{\bar{\sigma}, \varphi} \pi(\xi) \varphi, \pi(\xi) \varphi) d\xi.
 \end{aligned}$$

So, by (3.5), Fubini's theorem and Schwarz' inequality, we get

$$\begin{aligned}
 & \sum_{k=1}^{\infty} \|P_{\sigma, \varphi} \varphi_k\|_2^2 \\
 &= (2\pi)^{-2n} \int_{\mathbb{R}^n} \sigma(\xi) \left(\int_{\mathbb{R}^n} \bar{\sigma}(\eta) |(\pi(\eta) \varphi, \pi(\xi) \varphi)|^2 d\eta \right) d\xi \\
 &= (2\pi)^{-2n} \int_{\mathbb{R}^n} \sigma(\xi) \left(\int_{\mathbb{R}^n} \bar{\sigma}(\eta) |(\pi(\xi - \eta) \varphi, \varphi)|^2 d\eta \right) d\xi \\
 &= (2\pi)^{-2n} \int_{\mathbb{R}^n} \sigma(\xi) \left(\int_{\mathbb{R}^n} \bar{\sigma}(\xi - \eta) |(\pi(\eta) \varphi, \varphi)|^2 d\eta \right) d\xi \\
 &= (2\pi)^{-2n} \int_{\mathbb{R}^n} |(\pi(\eta) \varphi, \varphi)|^2 \left(\int_{\mathbb{R}^n} \sigma(\xi) \bar{\sigma}(\xi - \eta) d\xi \right) d\eta \\
 (3.6) \quad &\leq (2\pi)^{-n} \|\varphi\|_{\infty}^2 \|\sigma\|_2^2.
 \end{aligned}$$

Thus, by (3.6), we get

$$(3.7) \quad \|P_{\sigma, \varphi}\|_{S_2} \leq (2\pi)^{-n/2} \|\varphi\|_{\infty} \|\sigma\|_2$$

for all $\sigma \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Now, let $\sigma \in L^2(\mathbb{R}^n)$. Then there exists a sequence $\{\sigma_l\}_{l=1}^{\infty}$ in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that

$$(3.8) \quad \sigma_l \rightarrow \sigma$$

in $L^2(\mathbb{R}^n)$ as $l \rightarrow \infty$. By (3.8) and Proposition 3.1, $P_{\sigma_l, \varphi} \rightarrow P_{\sigma, \varphi}$ in $B(L^2(\mathbb{R}^n))$ as $l \rightarrow \infty$. By (3.7) and (3.8),

$$(3.9) \quad \|P_{\sigma_l, \varphi} - A\|_{S_2} \rightarrow 0$$

for some Hilbert-Schmidt operator A as $l \rightarrow \infty$. Now, we claim that $P_{\sigma_l, \varphi} \rightarrow A$ in $B(L^2(\mathbb{R}^n))$ as $l \rightarrow \infty$. If we assume that this is true for a moment, then, by (3.9), $P_{\sigma, \varphi} = A$. Hence $P_{\sigma, \varphi} : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is a Hilbert-Schmidt operator, and by (3.7) and (3.8),

$$\|P_{\sigma, \varphi}\|_{S_2} = \lim_{l \rightarrow \infty} \|P_{\sigma_l, \varphi}\|_{S_2} \leq (2\pi)^{-n/2} \|\varphi\|_{\infty} \lim_{l \rightarrow \infty} \|\sigma_l\|_2 = (2\pi)^{-n/2} \|\varphi\|_{\infty} \|\sigma\|_2.$$

It remains to prove that $P_{\sigma_l, \varphi} \rightarrow A$ in $B(L^2(\mathbb{R}^n))$ as $l \rightarrow \infty$. By Schwarz' inequality, Fubini's theorem and Parseval's identity,

$$\begin{aligned}
 & \| (P_{\sigma_l, \varphi} - A)f \|_2^2 \\
 &= \left((P_{\sigma_l, \varphi} - A) \sum_{j=1}^{\infty} (f, \varphi_j) \varphi_j, (P_{\sigma_l, \varphi} - A) \sum_{k=1}^{\infty} (f, \varphi_k) \varphi_k \right) \\
 &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (f, \varphi_j) \overline{(f, \varphi_k)} ((P_{\sigma_l, \varphi} - A) \varphi_j, (P_{\sigma_l, \varphi} - A) \varphi_k) \\
 &\leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |(f, \varphi_j)| |(f, \varphi_k)| \| (P_{\sigma_l, \varphi} - A) \varphi_j \|_2 \| (P_{\sigma_l, \varphi} - A) \varphi_k \|_2 \\
 &= \left(\sum_{j=1}^{\infty} |(f, \varphi_j)| \| (P_{\sigma_l, \varphi} - A) \varphi_j \|_2 \right)^2 \\
 &\leq \left(\sum_{j=1}^{\infty} |(f, \varphi_j)|^2 \right) \left(\sum_{j=1}^{\infty} \| (P_{\sigma_l, \varphi} - A) \varphi_j \|_2^2 \right) \\
 &= \| P_{\sigma_l, \varphi} - A \|_{S_2}^2 \| f \|_2^2
 \end{aligned}$$

for all f in $L^2(\mathbb{R}^n)$. Hence

$$(3.10) \quad \| P_{\sigma_l, \varphi} - A \|_{B(L^2(\mathbb{R}^n))} \leq \| P_{\sigma_l, \varphi} - A \|_{S_2}$$

for $l = 1, 2, \dots$. Thus, by (3.9) and (3.10), $P_{\sigma_l, \varphi} \rightarrow A$ in $B(L^2(\mathbb{R}^n))$ as $l \rightarrow \infty$. $\square$

We can now state and prove a theorem on the L^p -boundedness of wavelet multipliers.

Theorem 3.3. *Let φ be a function in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\|\varphi\|_2 = 1$. Let $\sigma \in L^r(\mathbb{R}^n)$, $1 \leq r \leq 2$. Then there exists a unique bounded linear operator $P_{\sigma, \varphi} : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ for all p in $[r, r']$, where r' is the conjugate index of r , such that*

$$(3.11) \quad \| P_{\sigma, \varphi} \|_{B(L^p(\mathbb{R}^n))} \leq (2\pi)^{-n/r} \|\varphi\|_1^{1-\frac{2}{r'}} \|\varphi\|_\infty \|\sigma\|_r$$

and for all functions f in $L^p(\mathbb{R}^n)$, $P_{\sigma, \varphi} f$ is given by (2.1) for all simple functions σ on $\mathbb{R}^n$ for which the Lebesgue measure of the set $\{\xi \in \mathbb{R}^n : \sigma(\xi) \neq 0\}$ is finite.

Proof. By an interpolation of Propositions 2.1 and 3.1, we get a unique bounded linear operator $P_{\sigma, \varphi} : L^r(\mathbb{R}^n) \rightarrow L^r(\mathbb{R}^n)$ such that (3.11) is satisfied and for every f in $L^r(\mathbb{R}^n)$, $P_{\sigma, \varphi} f$ is given by (2.1) for every simple function σ on $\mathbb{R}^n$ for which the Lebesgue measure of the set $\{\xi \in \mathbb{R}^n : \sigma(\xi) \neq 0\}$ is finite. By duality, the same is true when r is replaced by r' . By an interpolation of the results for $L^r(\mathbb{R}^n)$ and $L^{r'}(\mathbb{R}^n)$, the same is true for all p in (r, r') and this completes the proof. $\square$

4. COMPACT OPERATORS

We begin with the following result on compact wavelet multipliers.

Proposition 4.1. *Let $\sigma \in L^1(\mathbb{R}^n)$ and let $\varphi \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then the bounded linear operator $P_{\sigma, \varphi} : L^1(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$ is compact.*

Proof. We begin with the wavelet multiplier $P_{\sigma,\varphi,\psi} : L^1(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$ given by

$$(P_{\sigma,\varphi,\psi}u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \sigma(\xi)(u, \pi(\xi)\varphi)(\pi(\xi)\psi)(x) d\xi, \quad x \in \mathbb{R}^n,$$

for all u in $L^1(\mathbb{R}^n)$, where $\psi \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then as in the case for Proposition 2.1, it can be proved easily that $P_{\sigma,\varphi,\psi} : L^1(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$ is a bounded linear operator such that

$$(4.1) \quad \|P_{\sigma,\varphi,\psi}\|_{B(L^1(\mathbb{R}^n))} \leq (2\pi)^{-n} \|\varphi\|_\infty \|\psi\|_1 \|\sigma\|_1.$$

Let $\{u_j\}_{j=1}^\infty$ be a sequence of functions in $L^1(\mathbb{R}^n)$ such that

$$\|u_j\|_1 \leq 1, \quad j = 1, 2, \dots$$

Then

$$\|P_{\sigma,\varphi,\psi}u_j\|_\infty \leq (2\pi)^{-n} \|\varphi\|_\infty \|\psi\|_\infty \|\sigma\|_1, \quad j = 1, 2, \dots$$

So, $\{P_{\sigma,\varphi,\psi}u_j\}_{j=1}^\infty$ is uniformly bounded. Now, we consider first the case when the support of σ is compact and ψ is in the set $C_0^\infty(\mathbb{R}^n)$ of all C^∞ and compactly supported functions. Then the function g on $\text{supp}(\psi) \times \text{supp}(\sigma)$ defined by

$$g(x, \xi) = (\pi(\xi)\psi)(x), \quad (x, \xi) \in \text{supp}(\psi) \times \text{supp}(\sigma),$$

is uniformly continuous. So, for every positive number ε , there exists a positive number δ such that for all ξ in $\text{supp}(\sigma)$ and for all x and y in $\text{supp}(\psi)$ with $|x-y| < \delta$, we get

$$|(\pi(\xi)\psi)(x) - (\pi(\xi)\psi)(y)| < \varepsilon$$

and hence for $j = 1, 2, \dots$,

$$\begin{aligned} & |(P_{\sigma,\varphi,\psi}u_j)(x) - (P_{\sigma,\varphi,\psi}u_j)(y)| \\ & \leq \int_{\mathbb{R}^n} |\sigma(\xi)| |(u_j, \pi(\xi)\varphi)| |(\pi(\xi)\psi)(x) - (\pi(\xi)\psi)(y)| d\xi \\ & \leq \varepsilon \int_{\mathbb{R}^n} |\sigma(\xi)| d\xi \|u_j\|_1 \|\varphi\|_\infty \\ & = \varepsilon \|\sigma\|_1 \|\varphi\|_\infty. \end{aligned}$$

So, $\{P_{\sigma,\varphi,\psi}u_j\}_{j=1}^\infty$ is equicontinuous on $\mathbb{R}^n$. Therefore for every compact subset K of $\mathbb{R}^n$, the Ascoli-Arzelà theorem ensures that $\{P_{\sigma,\varphi,\psi}u_j\}_{j=1}^\infty$ has a subsequence that converges uniformly on K . Thus, by the Cantor diagonal procedure, we can find a subsequence $\{u_{j_k}\}_{k=1}^\infty$ of $\{u_j\}_{j=1}^\infty$ such that $\{P_{\sigma,\varphi,\psi}u_{j_k}\}_{k=1}^\infty$ converges pointwise to a function on $\mathbb{R}^n$. But

$$|(P_{\sigma,\varphi,\psi}u_{j_k})(x)| \leq (2\pi)^{-n} \|\sigma\|_1 \|\varphi\|_\infty |\psi(x)|, \quad x \in \mathbb{R}^n,$$

and $\psi \in L^1(\mathbb{R}^n)$. So, by Lebesgue's dominated convergence theorem, the sequence $\{P_{\sigma,\varphi,\psi}u_{j_k}\}_{k=1}^\infty$ converges in $L^1(\mathbb{R}^n)$ as $k \rightarrow \infty$. Therefore $P_{\sigma,\varphi,\psi} : L^1(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$ is compact. Now, let $\psi \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then let $\{\psi_j\}_{j=1}^\infty$ be a sequence of functions in $C_0^\infty(\mathbb{R}^n)$ such that $\psi_j \rightarrow \psi$ in $L^1(\mathbb{R}^n)$ as $j \rightarrow \infty$. So, by (4.1),

$$\|P_{\sigma,\varphi,\psi_j} - P_{\sigma,\varphi,\psi}\|_{B(L^1(\mathbb{R}^n))} \leq (2\pi)^{-n} \|\varphi\|_\infty \|\psi_j - \psi\|_1 \|\sigma\|_1 \rightarrow 0$$

as $j \rightarrow \infty$. Therefore $P_{\sigma,\varphi,\psi} : L^1(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$ is compact provided that the support of σ is compact. Finally, let $\sigma \in L^1(\mathbb{R}^n)$. Then let $\{\sigma_j\}_{j=1}^\infty$ be a sequence in $C_0^\infty(\mathbb{R}^n)$ such that $\sigma_j \rightarrow \sigma$ in $L^1(\mathbb{R}^n)$ as $j \rightarrow \infty$. Then, by (4.1) again,

$$\|P_{\sigma_j,\varphi,\psi} - P_{\sigma,\varphi,\psi}\|_{B(L^1(\mathbb{R}^n))} \leq (2\pi)^{-n} \|\varphi\|_\infty \|\psi\|_1 \|\sigma_j - \sigma\|_1 \rightarrow 0$$

as $j \rightarrow \infty$. Therefore $P_{\sigma,\varphi,\psi} : L^1(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$ is compact and the proposition is proved if we let $\psi = \varphi$. $\square$

We can now give the following result on the L^p -compactness of wavelet multipliers.

Theorem 4.2. *Under the hypotheses of Theorem 2.3, the bounded linear operator $P_{\sigma,\varphi} : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ is compact for $1 \leq p \leq \infty$.*

Proof. Using a duality argument such as that in Remark 2.2, we conclude that $P_{\sigma,\varphi} : L^\infty(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n)$ is compact. By an interpolation of compactness on $L^1(\mathbb{R}^n)$ and on $L^\infty(\mathbb{R}^n)$ such as the one given on pages 202–203 of [1], we complete the proof. $\square$

The following result is an analogue of Theorem 3.3 for compact operators. It is an immediate consequence of an interpolation of Theorem 3.3 and Proposition 4.1. See again pages 202 and 203 of [1] for the interpolation used.

Theorem 4.3. *Under the hypotheses of Theorem 3.3, the bounded linear operator $P_{\sigma,\varphi} : L^p(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)$ is compact for all p in $[r, r']$.*

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DEPARTMENT OF MATHEMATICS AND STATISTICS, YORK UNIVERSITY, 4700 KEELE STREET,
TORONTO, ONTARIO M3J 1P3, CANADA

DEPARTMENT OF MATHEMATICS AND STATISTICS, YORK UNIVERSITY, 4700 KEELE STREET,
TORONTO, ONTARIO M3J 1P3, CANADA

DEPARTMENT OF MATHEMATICS AND STATISTICS, YORK UNIVERSITY, 4700 KEELE STREET,
TORONTO, ONTARIO M3J 1P3, CANADA