CONVEX SOLIDS WITH PLANAR MIDSURFACES

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ABSTRACT. We show that the boundary of an n -dimensional closed convex set $B \subset \mathbb{R}^n$, possibly unbounded, is a convex quadric surface if and only if the middle points of every family of parallel chords of B lie in a hyperplane. To prove this statement, we show that the boundary of B is a convex quadric surface if and only if there is a point $p \in \text{int } B$ such that all sections of $\text{bd } B$ by 2-dimensional planes through p are convex quadric curves. Generalizations of these statements that involve boundedly polyhedral sets are given.

1. INTRODUCTION AND MAIN RESULTS

By a *midsurface* of a convex set in $\mathbb{R}^n$ we mean the locus of middle points of a family of parallel chords of the set. A classical result of convex geometry states that the boundary of a convex body $B \subset \mathbb{R}^n$ is an ellipsoid provided every midsurface of B lies in a hyperplane (see Brunn [2, pp. 59–61] for $n = 2$, Blaschke [1, p. 159] for $n = 3$, and Busemann [3, p. 92] for $n \geq 3$). Kubota [8] (for $n = 2$) and Gruber [4] showed that the statement above remains true provided the locus of points that divide any family of parallel chords of B in the same ratio lies in a hyperplane. Replacing hyperplanes by polyhedral surfaces one can obtain a joint characterization of ellipsoids and polytopes (see [12]).

It is well-known (see, e.g., [11, pp. 74–75]) that every midsurface of a quadric surface in $\mathbb{R}^n$, given by the equation $\sum_{i,k=1}^n \alpha_{ik} x_i x_k + 2 \sum_{i=1}^n \beta_i x_i + \gamma = 0$, belongs to a hyperplane. Our purpose here is to expand the characterization of ellipsoids above by considering all convex quadric surfaces.

In what follows, by *convex solids* in $\mathbb{R}^n$ we mean n -dimensional closed convex sets distinct from the whole space. As usual, $\text{bd } B$ and $\text{int } B$ denote, respectively, the boundary and the interior of a convex solid B . A *convex surface* in $\mathbb{R}^n$ is the boundary of a convex solid. This definition includes a hyperplane or a pair of parallel hyperplanes. We say that a convex surface in $\mathbb{R}^n$ is a *convex quadric surface* provided it is a part of a quadric surface. The classification of quadric surfaces (see, for example, [10]) implies that a convex quadric surface in $\mathbb{R}^n$ that

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contains no line can be expressed in suitable coordinates by one of the equations:

$$\begin{aligned}\alpha_1 x_1^2 + \cdots + \alpha_n x_n^2 &= 1, & (\text{ellipsoid}) \\ \alpha_1 x_1^2 - \alpha_2 x_2^2 - \cdots - \alpha_n x_n^2 &= 1, \ x_1 \geq 0, & (\text{arm of elliptic hyperboloid on two sheets}) \\ \alpha_1 x_1^2 - \alpha_2 x_2^2 - \cdots - \alpha_n x_n^2 &= 0, \ x_1 \geq 0, & (\text{elliptic cone}) \\ \alpha_1 x_1^2 + \cdots + \alpha_{n-1} x_{n-1}^2 &= x_n, & (\text{elliptic paraboloid})\end{aligned}$$

where all scalars $\alpha_1, \dots, \alpha_n$ are positive. Convex quadric surfaces containing lines are both-way unbounded cylinders based on convex quadrics of the same type that lie in proper subspaces of $\mathbb{R}^n$. Our first result is given as follows.

Theorem 1. *For a convex solid $B \subset \mathbb{R}^n$ ($n \geq 2$) distinct from a halfspace, the following conditions are equivalent:*

- (a) *bd B is a convex quadric surface,*
- (b) *every midsurface of B lies in a hyperplane,*
- (c) *for every family $\mathcal{F}$ of parallel chords of B , there is a scalar $\lambda_{\mathcal{F}} \in]0, 1[$ such that the points dividing the chords from $\mathcal{F}$ in the ratio $\lambda_{\mathcal{F}}$ lie in a hyperplane.*

Analysis of the proof of Theorem 1 shows that the scalar $\lambda_{\mathcal{F}}$ in condition (c) above equals $1/2$. Here by a *chord* of B we mean any segment $[u, w]$ which equals the intersection of B and the line (u, w) . The condition that B be distinct from a halfspace is justified by the fact that a convex solid in $\mathbb{R}^n$ has chords if and only if it is not a halfspace (see [5]). Finally, we say that a segment $[u, w]$ is divided by its point v in the ratio $\lambda \in [0, 1]$ provided $\|u - v\| = \lambda\|u - w\|$.

Theorem 1 is a corollary of a sharper statement below, which considers families of parallel chords in a neighborhood of $\text{bd } B$. Let us recall that a line $l \subset \mathbb{R}^n$ is called *non-recession* for B if the intersection of B and any translate of l is either empty or a chord of B . For a scalar $\delta > 0$ and a non-recession line l of B that supports B , denote by $B_{\delta}(l)$ the set of points in $\text{int } B$ at a distance less than δ from l . Let $\mathcal{F}_{\delta}(l)$ be the family of chords of B each being parallel to l and intersecting $B_{\delta}(l)$. A 2-dimensional convex solid $C \subset \mathbb{R}^2$ will be called δ -*polygonal* provided it is locally polygonal, and for any non-recession line l of C that supports C the set $C_{\delta}(l)$ contains at most one vertex of C . Obviously, any convex polygon in $\mathbb{R}^2$ is δ -polygonal for a suitable $\delta > 0$.

Theorem 2. *For a convex solid $B \subset \mathbb{R}^n$ ($n \geq 2$) distinct from a halfspace, the following conditions are equivalent:*

- (a) *either $\text{bd } B$ is a convex quadric surface or B a direct sum of a subspace and a line-free closed convex set C of dimension m , $2 \leq m \leq 3$, such that C is a simplicial cone if $m = 3$ or C is δ -polygonal for a suitable $\delta > 0$ if $m = 2$,*
- (b) *there is a scalar $\delta > 0$ such that for any non-recession line l of B that supports B one can find a scalar $\lambda(l) \in [0, 1]$ with the following property: points that divide the chords $m \in \mathcal{F}_{\delta}(l)$ in the ratio $\lambda(l)$ lie in a hyperplane.*

Analysis of the proof of Theorem 2 reveals the following remarks.

Remark 1. The ratio $\lambda(l)$ in condition (b) of Theorem 2 can take any value from $[0, 1]$ if B is a direct sum of a subspace and a 3-dimensional simplicial cone or a planar δ -polygonal set; otherwise $\lambda(l) = 1/2$. Thus condition (b) of Theorem 2 becomes equivalent to conditions (a)-(c) of Theorem 1 if we replace $[0, 1]$ by $]0, 1[$.

Remark 2. If B is *strictly convex* (i.e., $\text{bd } B$ does not contain segments), then condition (b) of Theorem 2 can be relaxed by assuming that δ may also depend on the choice of l . This is not true if B is not strictly convex. Indeed, let B be the sum of a line segment and a circle in $\mathbb{R}^2$. Then for any line l tangent to B there is a scalar $\delta = \delta(l) > 0$ such that the middle points of chords $m \in \mathcal{F}_\delta(l)$ belong to a line.

The proof of Theorem 2 consists of two steps (see Sections 4 and 5 below): we consider the case $n = 2$ separately and use Theorems 3 and 4 for the case $n \geq 3$.

Theorem 3. *The boundary of a convex solid $B \subset \mathbb{R}^n$ ($n \geq 3$) is a convex quadric surface if and only if there is a point $p \in \text{int } B$ such that all sections of $\text{bd } B$ by 2-dimensional planes through p are convex quadric curves.*

A convex solid $B \subset \mathbb{R}^n$ is called *boundedly polyhedral* (boundedly polygonal if $n = 2$) provided its intersection with each polytope in $\mathbb{R}^n$ is a polytope (see [6]). The boundary of a boundedly polyhedral convex solid will be called a *boundedly polyhedral surface* (boundedly polygonal line if $n = 2$).

Theorem 4. *The boundary of a convex solid $B \subset \mathbb{R}^n$ ($n \geq 3$) is a convex quadric surface or a boundedly polyhedral surface if and only if there is a point $p \in \text{int } B$ such that every section of $\text{bd } B$ by a 2-dimensional plane through p is a convex quadric curve or a boundedly polygonal line.*

We remark here that Theorem 3 is a generalization of the following statement of Busemann [3, pp. 91–92], initially proved by Kubota [7] for $n = 3$: the boundary of a convex body $B \subset \mathbb{R}^n$ ($n \geq 3$) is an ellipsoid if and only if all sections of $\text{bd } B$ by 2-dimensional planes through a given point $p \in \text{int } B$ are ellipses. Theorem 4 comprises Theorem 3 above and Theorem 4.7 from [6], which states that a convex solid $B \subset \mathbb{R}^n$ is boundedly polyhedral if and only if all its sections by 2-dimensional planes through a given point $p \in \text{int } B$ are boundedly polygonal.

In what follows we will be using the fact that any convex solid $B \subset \mathbb{R}^n$ can be uniquely expressed as the vector sum $B = M + C$, where M is a subspace and C is a line-free closed convex set of some dimension m , $1 \leq m \leq n$. The affine span of a convex set S (that is, the smallest plane containing S) will be denoted $\text{aff } S$ (see, e.g., [13] for general references on convex sets).

2. PROOF OF THEOREM 3

Since the “if” part trivially holds, it remains to prove the “only if” part. Let $B \subset \mathbb{R}^n$ ($n \geq 3$) be a convex solid with the property that all sections of $\text{bd } B$ by 2-dimensional planes through a given point $p \in \text{int } B$ are convex quadric curves. Express B as a direct sum of a subspace M and a line-free closed convex set C of some dimension m . Obviously, $\text{bd } B$ is a convex quadric surface if $1 \leq m \leq 2$. Hence we assume that $m \geq 3$. Let q be the projection of p on $\text{aff } C$ in the direction of M . Then q belongs to the relative interior of C and all sections of C by 2-dimensional planes $L \subset \text{aff } C$ through q are convex quadratic curves. This shows that we can reduce the proof to the case when B is a line-free convex solid.

Since the case when B is compact is proved in [3], we assume, in what follows, that B is unbounded. Choose a hyperplane H through p such that the intersection $B \cap H$ is compact (this is possible because B is line-free). Then [3] implies that $B \cap H$ is a solid $(n - 1)$ -dimensional ellipsoid. Choose in H an $(n - 2)$ -dimensional

plane N through p such that $B \cap N$ is symmetric about p , and let N_1 and N_2 be the $(n-2)$ -dimensional planes in H both parallel to N and supporting $B \cap H$. Continuously rotating H about N (together with N_1 and N_2), we find a position of H such that N_1 and N_2 are at the same distance from N . This implies that the ellipsoid $B \cap H$ is symmetric about p .

Let G be a hyperplane parallel to H and supporting B . Choose a point $q \in B \cap G$ and consider the halfline $l = [q, p)$. We claim that $l \subset B$. Indeed, since B is unbounded, it contains a halfline m with apex q . Let L be a 2-dimensional plane containing $l \cup m$. The choice of H implies that p is the middle point of the chord $B \cap H \cap L$, which is parallel to the line $G \cap L$. Hence l belongs to the conjugate line of $G \cap L$ with respect to the quadric curve $\text{bd } B \cap L$. Thus $l \subset B \cap L \subset B$.

Applying a suitable affine transformation, we may suppose that $q = \theta$, l is perpendicular to G , and $B \cap H$ is an $(n-1)$ -dimensional ball of radius 1. We divide our proof into several cases.

Assume the existence of a 2-dimensional plane L_0 through l such that $B \cap L_0$ is a convex cone. Obviously, θ is the apex of $B \cap L_0$. Let P_1 and P_2 be hyperplanes that support B along the sides of this cone. Consider the $(n-2)$ -dimensional planes $M_1 = G \cap P_1$ and $M_2 = G \cap P_2$. If L is a 2-dimensional plane through l such that $G \cap L$ does not lie in $M_1 \cup M_2$, then, as is easily seen, the section $B \cap L$ has θ as a singular point; whence $B \cap L$ is a convex cone with apex θ . By a continuity argument, $B \cap L'$ is also a cone with apex θ for any 2-dimensional plane L' through l such that $G \cap L' \subset M_1 \cup M_2$. Hence B is the convex cone with apex θ generated by the $(n-1)$ -dimensional ball $B \cap H$. Thus B is an elliptic cone.

Now assume that no section of B by a 2-dimensional plane through l is a cone. Let K denote the recession cone of B . Then $K \subset B$ because $\theta \in B$. Consider separately the following two cases: $\dim K \leq n-1$ and $\dim K = n$.

1. Let $\dim K \leq n-1$. Then there is a hyperplane W that contains K . Choose a 2-dimensional plane L through l that does not lie in W . The section $B \cap L$ is a planar convex set whose recession cone belongs to the line $L \cap W$. Since $\text{bd } B \cap L$ is a convex quadric curve, it must be a parabola symmetric about l . By a continuity argument, the same is true for any 2-dimensional plane $L \subset W$ through l . If (x, y) are the orthogonal coordinates in L such that l is the positive halfline of the y -axis, then $\text{bd } B \cap L$ is given by an equation $y = \alpha_L x^2$, $\alpha_L > 0$. We have $\alpha_L = \|p\|$ because $\text{bd } B \cap L$ intersects the ball $B \cap H$ with center p and radius 1 at a pair of opposite points. Hence $\text{bd } B$ is an elliptic paraboloid.

2. Let $\dim K = n$. Then for any 2-dimensional plane L through l , the recession cone of $B \cap L$ is from a halfline. Hence $\text{bd } B \cap L$ cannot be a parabola, and it cannot be a cone by the assumption above. Furthermore, since $\text{bd } B \cap L$ intersects the ball $B \cap H$ with center p and radius 1 at a pair of opposite points, $\text{bd } B \cap L$ is an arm of hyperbola symmetric about l ; whence it can be expressed in suitable orthogonal coordinates (x, y) by an equation $y = (\alpha_L x^2 + 1)^{1/2} - 1$. Similar to the above, $\|p\| = (\alpha_L + 1)^{1/2} - 1$, which implies that $\text{bd } B$ is an arm of elliptic hyperboloid on two sheets. $\square$

3. PROOF OF THEOREM 4

Since the “if” part of the statement is trivial, it remains to prove the “only if” part. As in the proof of Theorem 3 we may assume that B is a line-free convex solid.

We proceed by induction on $n \geq 3$. Let $n = 3$ and $B \subset \mathbb{R}^3$ be a line-free convex solid such that every section of $\text{bd } B$ by a 2-dimensional plane through a given point $p \in \text{int } B$ is either a convex quadric curve or a boundedly polygonal line. Assume, for contradiction, that $\text{bd } B$ is neither quadratic nor polyhedral. Then there two planes through p , say L and M , such that $P = \text{bd } B \cap L$ is a boundedly polygonal line and $Q = \text{bd } B \cap M$ is a convex quadric curve. We may assume that $B \cap L \cap M$ is a line segment, since otherwise one of L, M can be replaced by a plane through p whose intersection with B is bounded. Because B is line-free, both sections P and Q are line-free. We may also assume that L is horizontal and M is vertical. Let the part of Q above L and the part of P in front of M be bounded (see the figure below). Denote by F the part of $\mathbb{R}^3$ situated above L and in front of M , and let a and b be the endpoints of the chord $B \cap L \cap M$.

Since P is locally polygonal, the arc $P \cap F$ contains segments $[a, c]$ and $[b, e]$. Choose any points $v \in]a, c[$ and $w \in]b, e[$. Consider a plane N through p and v . Obviously, $\text{bd } B \cap N$ approaches P when N approaches L . Since no boundedly polygonal line can be approximated by a convex quadric curve, $\text{bd } B \cap N$ should be a boundedly polygonal line when N is sufficiently close to L . Fix such a position of N and choose a segment $[v, v'] \subset \text{bd } B \cap N$ that lies in F . Clearly, the plane T containing both segments $[a, c]$ and $[v, v']$ supports B and $B \cap T$ is a 2-dimensional face of B . From this fact it easily follows that any section of B by a plane through p and v is boundedly polygonal. Similarly, any section of B by a plane through p and w is boundedly polygonal.

Given a pair of points $r, s \in Q \cap F$ such that r precedes s clockwise along Q , denote by $\gamma(r, s)$ the open arc in $Q \cap F$ that joins r and s . Let $H(r, w)$ and $H(s, v)$ denote the planes through p, r, w and p, s, v , respectively. We are going to recursively construct two sequences of points $r_k, s_k \in \gamma(r, s)$, $k = 1, 2, \dots$, with the properties described below. Put $r_1 = r$ and $s_1 = s$, and assume that r_k and s_k , $k \geq 1$, are chosen. Since the section $\text{bd } B \cap H(r_k, w)$ is boundedly polygonal, it contains a segment $[r_k, r'_k] \subset F$. Choose a point $s_{k+1} \in \gamma(r_k, s_k)$ so close to r_k that the plane $H(s_{k+1}, v)$ intersects the open segment $]r_k, r'_k[$. Similarly, the section $\text{bd } B \cap H(s_{k+1}, v)$ contains a segment $[s_{k+1}, s'_{k+1}] \subset F$. We claim that $[r_k, r'_k]$ and $[s_{k+1}, s'_{k+1}]$ are disjoint. Indeed, otherwise both r_k and s_{k+1} would lie in a plane that supports B , which is impossible because r_k and s_{k+1} belong to the boundary quadric curve Q . As above, there is a point $r_{k+1} \in \gamma(r_k, s_{k+1})$ so close to s_{k+1} that the plane $H(r_{k+1}, w)$ intersects $]s_{k+1}, s'_{k+1}[$ and any segment $[r_{k+1}, r'_{k+1}] \subset \text{bd } B \cap H(r_{k+1}, w) \cap F$ does not intersect $[s_{k+1}, s'_{k+1}]$.

Furthermore, the sequences $r_1, r_2, \dots$ and $s_1, s_2, \dots$ can be selected such that both of them converge to a point $t \in Q$. As is easily seen, none of the curves

$C_1 = \text{bd } B \cap H(t, v)$ and $C_2 = \text{bd } B \cap H(t, w)$ contain a segment $[t, t'] \subset F$. The last is in contradiction with the fact that both C_1 and C_2 are boundedly polygonal. Hence the case $n = 3$ is proved.

Now assume that Theorem 4 holds for all $m \leq n - 1$ ($n \geq 4$), and let B be a line-free convex solid in $\mathbb{R}^n$ such that every section of $\text{bd } B$ by a 2-dimensional plane through a given point $p \in \text{int } B$ is either a convex quadric curve or a boundedly polygonal line. If all these sections are convex quadric curves, then, by Theorem 3, $\text{bd } B$ is a convex quadric surface. Assume the existence of a 2-dimensional plane L_0 through p such that $B \cap L_0$ is boundedly polygonal, and let L be any 2-dimensional plane through p . Choose a hyperplane $H \subset \mathbb{R}^n$ through p that contains both L_0 and L . By the inductive assumption, $B \cap H$ is boundedly polyhedral; whence $B \cap L$ is boundedly polygonal. Thus all sections of B by 2-dimensional planes through p are boundedly polygonal, and, as proved in [6], B is boundedly polyhedral. $\square$

4. PROOF OF THEOREM 2 FOR $n = 2$

(a) $\Rightarrow$ (b) If $\text{bd } B$ is a convex quadric curve, then any midcurve of B lies on a line. Assume that B is δ -polygonal for a suitable scalar $\delta > 0$. Let l be a non-recession line that supports B , and put $B \cap l = [u, w]$ (possibly $u = w$). Since $B_\delta(l)$ contains at most one vertex of B , the arc $\Gamma = \text{bd } B \cap B_\delta(l)$ can be expressed, up to the orientation of $\text{bd } B$, as the union of two or three segments: $\Gamma =]r, u[\cup]w, s[$ or $\Gamma =]r, u[\cup]w, s[\cup [s, t[$. In either case the left endpoints of the chords $m \in \mathcal{F}_\delta(l)$ belong to $]r, u[$, and we can put $\lambda(l) = 0$.

(b) $\Rightarrow$ (a) We divide the proof into a sequence of lemmas, assuming that a convex solid $B \subset \mathbb{R}^2$ satisfies condition (b) of Theorem 2. Let us recall that B is called *regular* provided each boundary point of B belongs to a unique support line of B .

Lemma 1. *If B is not regular or is not strictly convex, then B is δ -polygonal.*

Proof. Given a boundary point p of B , denote by $h_1(p)$ and $h_2(p)$ the tangent halflines of B with common apex p such that $h_2(p)$ indicates the counterclockwise direction of $\text{bd } B$.

1. We first state that, given a point $p \in \text{bd } B$, each of the sets $h_1(p) \cap B$, $h_2(p) \cap B$ contains a segment, provided at least one of them does. Indeed, assume that $h_1(p) \cap B$ contain a segment $[q, p]$, while $h_2(p) \cap B = \{p\}$. By a continuity argument, there is a point $p_0 \in \text{bd } B \cap B_\delta(l)$ that can be reached from p counterclockwise along $\text{bd } B$ such that for any point $v \in \text{bd } B$ between p and p_0 the following conditions hold:

- 1) the tangent halflines $h_1(p)$ and $h_2(v)$ are not opposite and $[p, v] \not\subset \text{bd } B$,
- 2) if l_v is the line that contains $h_2(v)$, then the part of $\text{bd } B \cap B_\delta(l_v)$ that can be reached from v counterclockwise is not a segment,
- 3) $B_\delta(l_v)$ intersects the segment $[q, p]$ along a subsegment $]r, p[$ where $r \in [q, p[$ (this is possible because δ does not depend on l).

Given a point $v \in \text{bd } B$ between p and p_0 , choose a scalar $\lambda(l_v)$ that satisfies condition (b). Conditions 1) and 2) above imply that $0 \neq \lambda(l_v) \neq 1$. Denote by l'_v and l''_v the lines through r and p , respectively, both parallel to l_v . Then left endpoints of the chords $m \in \mathcal{F}_\delta(l_v)$ that lie between l'_v and l''_v belong to $]r, p[$, while the points that divide these chords in the ratio $\lambda(l_v)$ belong to a line given by condition (b). In this case right endpoints of the chords $m \in \mathcal{F}_\delta(l_v)$ fulfill a segment $[r', p'] \subset \text{bd } B$. If v converges to p_1 , then $[r', p']$ tends to a non-trivial

segment $[p, z] \subset \text{bd } B$, which contradicts the assumption. Hence $h_2(p) \cap B$ contains a segment.

2. Next we state that if p is a singular point of $\text{bd } B$ (i.e., if the halflines $h_1(p)$ and $h_2(p)$ are not opposite), then at least one of the sets $h_1(p) \cap B$, $h_2(p) \cap B$ contains a segment. Indeed, assume, for contradiction, that $h_1(p) \cap B = h_2(p) \cap B = \{p\}$. Let l be the line containing $h_2(p)$. Choose a scalar $\lambda(l)$ that satisfies condition (b). For a given chord $m = [u, w] \in \mathcal{F}_\delta(l)$, let $v \in]u, w[$ satisfy $\|u - v\| = \lambda(l)\|u - w\|$. Since none of the segments $[p, u]$, $[p, w]$ lie in $\text{bd } B$, the scalar $\lambda(l)$ cannot have any of the values 0 or 1. Hence $\lambda(l) \in]0, 1[$. Denote by l^* the line through v and p . Let the points u' and w' on l be such that the segments $[u, u']$, $[v, p]$, and $[w, w']$ are parallel. Then

$$\frac{\|u - v\|}{\|v - w\|} = \frac{\|u' - p\|}{\|p - w'\|} = \frac{\|w - w'\|}{\|p - w'\|} : \frac{\|u - u'\|}{\|u' - p\|}.$$

If $[u, w]$ approaches l , then, by condition (b), the point v converges to p along l^* . In this case the ratio $\|w - w'\|/\|p - w'\|$ tends to zero, while $\|u - u'\|/\|u' - p\|$ tends to a positive number by virtue of the fact that $h_1(p)$ and $h_2(p)$ are not opposite. Thus $\|u - v\|/\|v - w\| \rightarrow 0$ when $[u, w]$ approaches l , in contradiction with the equality $\|u - v\|/\|v - w\| = \lambda(l)/(1 - \lambda(l))$. Hence at least one of the sets $h_1(p) \cap B$, $h_2(p) \cap B$ contains a segment.

The two statements above obviously imply the existence of a locally polygonal arc $\Omega \subset \text{bd } B$. We state that if r, s, t are three consecutive vertices of Ω , then $\|r - s\| \geq \delta$ and $\|s - t\| \geq \delta$. Indeed, otherwise we can find a line l' with the following properties: (i) $B \cap l' = \{s\}$, (ii) $r, t \in B_\delta(l')$, (iii) the lines through r and t parallel to l' are distinct and do not contain any other vertex of B . Then the locus of points that divide the chords $m \in \mathcal{F}_\delta(l')$ in any given ratio μ is a polygonal line with at least three segments, which contradicts condition (b). This argument also shows that $\Omega = \text{bd } B$ and B is δ -polygonal. $\square$

If B is strictly convex, then any support line of B is non-recessional for B . In what follows, given a support line l of B and a scalar $\lambda(l)$ that satisfies condition (b), denote by $l^*(\lambda)$ the line containing the locus of points that divide the chords $m \in \mathcal{F}_\delta(l)$ in the ratio $\lambda(l)$ and call it a *conjugate* line of l .

Lemma 2. *If B is regular and strictly convex, then for any tangent line l of B , a scalar $\lambda(l)$ that satisfies condition (b) is uniquely defined and equals $1/2$.*

Proof. First we are going to prove the uniqueness of $\lambda(l)$ and $l^*(\lambda)$. Indeed, assume, for contradiction, that for a certain tangent line l of B there are two distinct scalars $\lambda(l)$ and $\lambda'(l)$ both satisfying condition (b). The strict convexity of B implies that both $\lambda(l)$ and $\lambda'(l)$ are distinct from 0 or 1. Let $\lambda(l) < \lambda'(l)$. Choose a chord $[x, z] \in \mathcal{F}_\delta(l)$, and let the points $y, y' \in [x, z]$ satisfy the conditions

$$\|x - y\| = \lambda(l)\|x - z\|, \quad \|x - y'\| = \lambda'(l)\|x - z\|.$$

When $[x, z]$ approaches l , the respective points y and y' fulfill some segments on the lines $l^*(\lambda)$ and $l^*(\lambda')$. Since y divides $[x, y']$ in the ratio $\lambda(l)/\lambda'(l)$, the endpoints z should lie on a line, in contradiction with strict convexity of B . Thus $\lambda(l)$ and $l^*(\lambda)$ are uniquely defined by l . In what follows, we write l^* for $l^*(\lambda)$.

The proof of equality $\lambda(l) = 1/2$ requires multiple steps. Choose a point $p \in \mathbb{R}^2 \setminus B$ separated by l from B such that B has two tangent lines l_1 and l_2 through p and both contact points $u_1 \in l_1 \cap B$ and $u_2 \in l_2 \cap B$ lie in $B_\delta(l)$.

1. We first state that p belongs to the conjugate line l^* if and only if $[u_1, u_2]$ is parallel to l and is divided by l^* in the ratio $\lambda(l)$. Indeed, assume that $p \in l^*$ and choose a point $z \in \text{bd } B$ such that $[u_1, z]$ is parallel to l . Then $[u_1, z]$ is divided by l^* in the ratio $\lambda(l)$. Choose a chord $[u', z']$ of B that is parallel to l and lies between the lines l and (u_1, z) . Since $[u', z']$ is divided by l^* in the same ratio $\lambda(l)$, the lines (u_1, u') and (z, z') have a common point on l^* . Tending $[u', z']$ toward $[u_1, z]$ we conclude that l_1 and the tangent line of B at z have a common point on l^* , which is p . Hence $z = u_2$.

Conversely, let $[u_1, u_2]$ be parallel to l such that l^* intersects $[u_1, u_2]$ at a point t with $\|u_1 - t\|/\|u_1 - u_2\| = \lambda(l)$. Choose any chord $[u'_1, u'_2]$ of B that is parallel to l and lies between the lines l and (u_1, u_2) . Let t' be the point in $[u'_1, u'_2]$ such that $\|u'_1 - t'\|/\|u'_1 - u'_2\| = \lambda(l)$. By condition (b), both t and t' belong to l^* . Then the lines (u_1, u'_1) and (u_2, u'_2) have a common point on l^* . Tending $[u'_1, u'_2]$ toward $[u_1, u_2]$ we obtain that l_1 and l_2 have a common point on l^* , which is p .

2. Next we state that $\lambda(l_1) + \lambda(l_2) = 1$ provided the point p belongs to l^* . Indeed, let v be the point at which l supports B . Then $l^* = (p, v)$ as proved above. Furthermore, l^* intersects $[u_1, u_2]$ at a point u_0 such that $\|u_1 - u_0\|/\|u_1 - u_2\| = \lambda(l)$. Let $[p_1, q_1]$ be a chord of B parallel to l_1 and so close to l_1 that $p_1, q_1 \in B_\delta(l)$. Choose points p_2 and q_2 in $\text{bd } B$ such that both chords $[p_1, p_2]$ and $[q_1, q_2]$ are parallel to l . Since both $[p_1, p_2]$ and $[q_1, q_2]$ are intersected by l^* in the ratio $\lambda(l)$, we easily obtain that (p_2, q_2) is parallel to l_2 and the lines (p_1, q_1) and (p_2, q_2) meet on l^* .

Denote by r_1 the point where $[p_1, q_1]$ intersects l_1^* , and let r_2 be the point of $[p_2, q_2]$ such that $[r_1, r_2]$ is parallel to l . Also, let r_0 be the point of intersection of $[r_1, r_2]$ and l^* . If the chord $[p_1, q_1]$ approaches l_1 , the respective chord $[p_2, q_2]$ approaches l_2 . During this movement, r_1 converges to u_1 along l_1^* and r_0 converges to u_0 along l^* . This easily implies that r_2 converges to u_2 along some line l' that divides $[p_2, q_2]$ in the ratio $1 - \lambda(l_1)$. Since the scalar $\lambda(l_2)$ and the respective line l_2^* are uniquely determined, we have $\lambda(l_2) = 1 - \lambda(l_1)$ and $l' = l_2^*$.

3. Finally, to show that $\lambda(l) = 1/2$, consider the points w_1 and w_2 at which l intersects the lines l_1 and l_2 , respectively. By the argument above (with w_1 and w_2 instead of p), we have $\lambda(l) = 1 - \lambda(l_1) = 1 - \lambda(l_2)$. This and $\lambda(l_1) + \lambda(l_2) = 1$ immediately imply the equality $\lambda(l) = 1/2$. $\square$

Lemma 3. *If B is smooth and strictly convex, then $\text{bd } B$ is a convex quadric curve.*

Proof. Choose a tangent line l of B . Keeping the notation of Lemma 2, select a point $p \in l^* \setminus B$ and tangent lines l_1, l_2 through p such that the points $u_1 \in B \cap l_1$ and $u_2 \in B \cap l_2$ belong to $B_\delta(l)$. We state that the conjugate lines l^*, l_1^*, l_2^* are either parallel or have a common point. Indeed, consider the lines l^* and l_1^* . If these lines have a common point z , then the similarity of triangles $\Delta(u_0, u_1, z)$ and $\Delta(r_0, r_1, z)$ immediately implies the similarity of triangles $\Delta(u_0, u_2, z)$ and $\Delta(r_0, r_2, z)$. Thus $z \in l_2^*$. If the lines l^* and l_1^* are parallel, then all three segments $[u_1, r_1]$, $[u_0, r_0]$, and $[u_2, r_2]$ are parallel, implying that l_2^* is parallel to each of l^* and l_1^* .

Denote by l_3 the line parallel to $[u_1, v]$ and supporting B at a point from the arc $\Gamma = \text{bd } B \cap B_\delta(l)$. As above, the lines l^*, l_1^*, l_3^* are either parallel or have z as a common point. This argument shows the existence of a dense subset S of Γ such that the conjugate lines of tangents to B at the points of S are either parallel or have z as a common point. By a continuity argument, this property holds for the entire arc Γ . We consider the following two cases separately.

1. Let the conjugate lines of tangents to B at Γ be parallel. Choose in $\mathbb{R}^2$ a new coordinate system (x, y) such that $v = 0$, the y -axis goes along l with positive direction corresponding to the clockwise orientation of $\text{bd } B$, and the x -axis goes along l^* with positive direction toward $\text{int } B$. Choose a point $b = (x, y) \in \Gamma \setminus \emptyset$, and let l_0 be the line parallel to the segment $[b, \emptyset]$ and tangent to B at a point of Γ . Then the conjugate line l_0^* bisects $[b, \emptyset]$ (because $\lambda(l_0) = 1/2$) and is parallel to l^* , i.e., to the x -axis. Thus the equation of l_0^* is $\eta = y/2$. Furthermore, the point $(0, y/2)$, which is a common point of l and l_0^* , should belong to the tangent line l_b at b , given by the equation

$$(4.1) \quad \eta - y = \frac{dy}{dx}(\xi - x).$$

This condition results in the equation

$$\frac{y}{2} - y = -\frac{dy}{dx}x \quad \text{or} \quad \frac{1}{x} = \frac{2}{y} \frac{dy}{dx},$$

which gives $x = Cy^2$. Hence Γ is a piece of parabola.

2. Let the conjugate lines of tangents to B at Γ have a common point z . Since $z \in l^*$, we can write $z = (\alpha, 0)$ with respect to the coordinate system above. For a point $b = (x, y) \in \Gamma \setminus \{\emptyset\}$, the tangent line l_b of B at this point is given by (4.1). If l_0 is the line parallel to the segment $[b, \emptyset]$ and tangent to B at a point of Γ , then the conjugate line l_0^* bisects $[b, \emptyset]$ and contains z . Thus the equation of l_0^* is

$$\eta = \frac{y}{x - 2\alpha}(\xi - \alpha),$$

whence l_0^* meets the y -axis at $q = (0, \alpha y / (2\alpha - x))$. As shown in Lemma 2, l_b also contains q . This gives

$$\frac{\alpha y}{2\alpha - x} - y = \frac{dy}{dx}(0 - x) \quad \text{or} \quad \frac{1}{x} + \frac{1}{x - 2\alpha} = \frac{2}{y} \frac{dy}{dx}.$$

This results in $Cy^2 = x|x - 2\alpha|$ with $C > 0$. Hence Γ is a piece of ellipse if $\alpha > 0$ and a piece of hyperbola if $\alpha < 0$.

Since $\text{bd } B$ can be covered by countably many pairwise overlapping arcs of the form $\text{bd } B \cap B_\delta(l)$, and since a piece of quadric curve is uniquely defined by a set of five points with no three on a line (see, e.g., [9, pp.369–371]), all the pieces represent the same convex quadric curve, which is $\text{bd } B$. $\square$

5. PROOF OF THEOREM 2 FOR $n \geq 3$

(a) $\Rightarrow$ (b) If $\text{bd } B$ is a convex quadric surface in $\mathbb{R}^n$ ($n \geq 3$), then every midsurface of B lies in a hyperplane. Assume that B is a direct sum of a subspace and a line-free closed convex set C of dimension m , $2 \leq m \leq 3$, such that C is a simplicial cone if $m = 3$ or C is δ -polygonal for a suitable $\delta > 0$ if $m = 2$. Obviously, B satisfies condition (b) if and only if C satisfies this condition with respect to its affine span. Since the case $m = 2$ is discussed in Section 4, we assume that C is a 3-dimensional simplicial cone. Denote by S_1, S_2, S_3 the 2-dimensional faces of C . A simple argument show that for any non-recession line l of C there is a face, say S_1 , of C such that every chord of C parallel to l has an endpoint on S_1 . Taking $\lambda(l) = 0$ or $\lambda(l) = 1$, we obtain that the points that divide the chords $m \in \mathcal{F}_\delta(l)$ in the ratio $\lambda(l)$ lie in S_1 .

(b) $\Rightarrow$ (a) Let $B \subset \mathbb{R}^n$ ($n \geq 3$) be a convex solid that satisfies condition (b) of the theorem. Choose a point $p \in \text{int } B$ and a 2-dimensional plane L through p . Select a line $l \subset L$ that is non-recessional for $B \cap L$ and supports $B \cap L$. Let $\delta > 0$ and $\lambda(l) \in [0, 1]$ be scalars that satisfy condition (b) for B , and let H be a hyperplane that divides the chords $m \in \mathcal{F}_\delta(l)$ in the ratio $\lambda(l)$. Clearly, $l = H \cap L$ is a line and each chord $m \in \mathcal{F}_\delta(l)$ that lies in L is divided by l in the ratio $\lambda(l)$. As proved in Section 4, the relative boundary of $B \cap L$ is a convex quadric curve or a δ -polygonal line.

If every section of $\text{bd } B$ by a 2-dimensional plane through p is a convex quadric curve, then Theorem 3 implies that $\text{bd } B$ is a convex quadric surface. Assume that at least one such section is a δ -polygonal line. Then, by Theorem 4, B is boundedly polyhedral; whence it can be expressed as a direct sum of a subspace M and a line-free boundedly polyhedral set C of some dimension m , $1 \leq m \leq n$.

First we are going to show that $m \leq 3$. Indeed, assume for a moment that $m \geq 4$ and choose a vertex p of C . Let P be an $(m-1)$ -dimensional plane in $\text{aff } C$ that separates p from the other vertices of C such that the section $C \cap P$ is a convex $(m-1)$ -dimensional polytope that lies in the open ball $\{x \in \mathbb{R}^n \mid \|x - p\| < \delta\}$. Since $\dim(C \cap P) \geq 3$, there is a line $l' \subset P$ with the property that for any scalar $\mu \in [0, 1]$ the locus of points that divide in the ratio μ the chords of $C \cap P$ parallel to l' does not lie in an $(m-2)$ -dimensional plane from P (see [4]). Let l be the line through p and parallel to l' . By the above, l is a non-recession line, $C \cap P \subset B_\delta(l)$, and for any scalar $\mu \in [0, 1]$ the locus of points that divide the chords $m \in \mathcal{F}_\delta(l)$ in the ratio μ does not lie in a hyperplane. Since the last contradicts condition (b), we conclude that $m \leq 3$.

It remains to show that C is a simplicial cone provided $m = 3$. Indeed, by an argument similar to the above, any section of C by a plane $L \subset \text{aff } C$ close to a vertex of C should be a triangle. Hence each vertex of C belongs to exactly three edges of C . To show that C is a cone, assume for a moment that C has two adjacent vertices, say u and w . Then the section of C by a plane Q parallel and close to $[u, w]$ is a polygon with at least four sides. Let v be the middle point of $[u, w]$. An easy argument shows the existence of a support non-recession line l of B that contains v and has direction close to that of $[u, w]$ such that $C \cap Q \subset B_\delta(l)$ and for any scalar $\mu \in [0, 1]$ the locus of points that divide in the ratio μ the chords $m \in \mathcal{F}_\delta(l)$ does not lie in a hyperplane. Hence C is a simplicial cone provided $m = 3$. $\square$

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