HOMEOMORPHISMS OF THE 3-SPHERE THAT PRESERVE A HEEGAARD SPLITTING OF GENUS TWO

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ABSTRACT. Let $\mathcal{H}_2$ be the group of isotopy classes of orientation-preserving homeomorphisms of $\mathbb{S}^3$ that preserve a Heegaard splitting of genus two. In this paper, we construct a tree in the barycentric subdivision of the disk complex of a handlebody of the splitting to obtain a finite presentation of $\mathcal{H}_2$.

1. INTRODUCTION

Let $\mathcal{H}_g$ be the group of isotopy classes of orientation-preserving homeomorphisms of $\mathbb{S}^3$ that preserve a Heegaard splitting of genus g , for $g \geq 2$. It was shown by Goeritz [3] in 1933 that $\mathcal{H}_2$ is finitely generated. Scharlemann [7] gave a modern proof of Goeritz's result, and Akbas [1] refined this argument to give a finite presentation of $\mathcal{H}_2$. In arbitrary genus, first Powell [6] and then Hirose [4] claimed to have found a finite generating set for the group $\mathcal{H}_g$, though serious gaps in both arguments were found by Scharlemann. Establishing the existence of such generating sets appears to be an open problem.

In this paper, we recover Akbas's presentation of $\mathcal{H}_2$ by a new argument. First, we define the complex $P(V)$ of primitive disks, which is a subcomplex of the disk complex of a handlebody V in a Heegaard splitting of genus two. Then we find a suitable tree T , on which $\mathcal{H}_2$ acts, in the barycentric subdivision of $P(V)$ to get a finite presentation of $\mathcal{H}_2$. In the last section, we will see that the tree T can be identified with the tree used in Akbas's argument [1].

Throughout the paper, $(V, W; \Sigma)$ will denote a Heegaard splitting of genus two of $\mathbb{S}^3$. That is, $\mathbb{S}^3 = V \cup W$ and $V \cap W = \partial V = \partial W = \Sigma$, where V and W are handlebodies of genus two. For essential disks D and E in a handlebody, the intersection $D \cap E$ is always assumed to be transverse and minimal up to isotopy. In particular, if D intersects E (indicated as $D \cap E \neq \emptyset$), then $D \cap E$ is a collection of pairwise disjoint arcs that are properly embedded in both D and E .

Finally, $\text{Nbd}(X)$ will denote a regular neighborhood of X and $\text{cl}(X)$ the closure of X for a subspace X of a polyhedral space, where the ambient space will always be clear from the context.

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FIGURE 1

2. PRIMITIVE DISKS IN A HANDLEBODY

We call an essential disk D in V *primitive* if there exists an essential disk D_d in W such that ∂D and ∂D_d have a single transverse intersection in Σ . Such a D_d is called a *dual disk* of D . Notice that any primitive disk is nonseparating. We call a pair of disjoint, nonisotopic primitive disks in V a *reducing pair* of V . Similarly, a triple of pairwise disjoint, nonisotopic primitive disks is a *reducing triple*.

A 2-sphere P in $\mathbb{S}^3$ is called a *reducing sphere* for $(V, W; \Sigma)$ if P intersects Σ transversally in a single essential circle and so intersects each handlebody in a single essential disk. It is clear that $V \cap P$ and $W \cap P$ are essential separating disks in V and W , respectively.

Lemma 2.1. *A disk D in V is primitive if and only if $\text{cl}(V - \text{Nbd}(D))$ is an unknotted solid torus, i.e. $W \cup \text{Nbd}(D)$ is a solid torus.*

Proof. Let D be a primitive disk with a dual disk D_d . Then the boundary P of a regular neighborhood of $D \cup D_d$ is a reducing sphere that is disjoint from D (see Figure 1). The sphere P splits V into two solid tori, X and Y , and we may assume that D is a meridian disk for X . Cutting $\mathbb{S}^3$ along P , we get two 3-balls, B_1 and B_2 , where X and Y are contained in B_1 and B_2 , respectively. Since the handlebody W is the boundary connected sum of $\text{cl}(B_1 - X)$ and $\text{cl}(B_2 - Y)$ along the disk $W \cap P$, we have

$$\mathbb{Z} * \mathbb{Z} = \pi_1(W) = \pi_1(B_1 - X) * \pi_1(B_2 - Y) = \pi_1(\mathbb{S}^3 - X) * \pi_1(\mathbb{S}^3 - Y).$$

Thus $\pi_1(\mathbb{S}^3 - X) = \pi_1(\mathbb{S}^3 - Y) = \mathbb{Z}$, and consequently X and Y are unknotted. Since D is a meridian of X , then $\text{cl}(V - \text{Nbd}(D))$ is ambient isotopic to Y , and is thus unknotted. The converse is a special case of Theorem 1 in [3]. $\square$

Let E and D be nonseparating disks in V such that $E \cap D \neq \emptyset$, and let $C \subset D$ be a disk cut off from D by an outermost arc α of $D \cap E$ in D such that $C \cap E = \alpha$. The arc α cuts E into two disk components, say G and H . The two disks $E_1 = G \cup C$ and $E_2 = H \cup C$ are called the *disks obtained from surgery on E along C* . Since E and D are assumed to intersect minimally, E_1 and E_2 are isotopic to neither E nor D , and moreover have fewer arcs of intersection with D than E had.

Notice that E_1 and E_2 are not isotopic to each other, otherwise E would be a separating disk. Finally, observe that both E_1 and E_2 are isotopic to a meridian

disk of the solid torus $\text{cl}(V - \text{Nbd}(E))$, otherwise one of them would be isotopic to E . Thus E_1 and E_2 are all nonseparating disks in V .

Lemma 2.2. *For a reducing pair $\{E, E'\}$ of V , there exists, up to isotopy, a unique reducing pair $\{E_d, E'_d\}$ of W for which E_d and E'_d are dual disks of E and E' , and are disjoint from E' and E , respectively.*

Proof. Existence is also a special case of Theorem 1 in [3]. For uniqueness, assume that $\{F_d, F'_d\}$ is another reducing pair of W such that F_d and F'_d are dual disks of E and E' , and are disjoint from E' and E , respectively.

Suppose that F_d intersects E'_d . Then there exists a disk $C \subset F_d$ cut off from F_d by an outermost arc α of $F_d \cap E'_d$ in F_d such that $C \cap E'_d = \alpha$. Denote the arc $C \cap \Sigma$ by δ . The disks obtained from surgery on E'_d along C are isotopic to a meridian disk of the solid torus $\text{cl}(W - \text{Nbd}(E'_d))$. Hence δ must intersect ∂E which is a longitudinal circle of the solid torus.

Let $\tilde{\Sigma}$ be the 2-holed torus obtained by cutting Σ along E'_d . Denote by $l_{\pm}$ the boundary circles of $\tilde{\Sigma}$ that came from $\partial E'_d$. Then δ is an essential arc in $\tilde{\Sigma}$ with endpoints in a single boundary circle l_+ or l_- , say l_+ . Moreover, since $\partial F_d \cap l_+$ and $\partial F_d \cap l_-$ contain the same number of points, we also have an essential arc δ' of $\partial F_d \cap \tilde{\Sigma}$ with endpoints in the boundary circle l_- . The arc δ' also intersects ∂E , otherwise δ' would be inessential. Thus ∂F_d meets ∂E in at least two points, and this contradicts that F_d is a dual disk of E . Therefore, we conclude that F_d is disjoint from E'_d .

Now, both E_d and F_d are meridian disks of the solid torus $\text{cl}(W - \text{Nbd}(E'_d))$. Since both E_d and F_d are disjoint from $E' \cup E'_d$, they are isotopic to each other in the solid torus and hence in W . Similarly, F'_d is isotopic to E'_d in W . $\square$

Theorem 2.3. *Let E and D be primitive disks in V such that $E \cap D \neq \emptyset$ and let $C \subset D$ be a disk cut off from D by an outermost arc α of $D \cap E$ in D such that $C \cap E = \alpha$. Then both disks obtained from surgery on E along C are primitive.*

Proof. First, we consider the case when there exists a primitive disk E' such that $\{E, E'\}$ is a reducing pair and E' is disjoint from C . (The existence of such an E' will be established in the final paragraph.) Let E_d and E'_d be the dual disks of E and E' , respectively, given by Lemma 2.2. By isotopy of D , we may assume that ∂D intersects ∂E_d and $\partial E'_d$ minimally in Σ . Denote the arc $C \cap \Sigma$ by δ . It suffices to show that δ intersects $\partial E'_d$ in a single point, since then the resulting disks from surgery are both primitive with common dual disk E'_d .

Let Σ' be the 4-holed sphere obtained by cutting Σ along $\partial E \cup \partial E'$. Denote by $\partial E_{\pm}$ (resp. $\partial E'_{\pm}$) the boundary circles of Σ' that came from ∂E (resp. $\partial E'$). Then δ is an essential arc in Σ' with endpoints in a single boundary circle ∂E_+ or ∂E_- , say ∂E_- , and δ cuts off an annulus from Σ' . The boundary circle of the annulus that does not contain δ cannot be ∂E_+ , otherwise one of the disks obtained from surgery on E along C would be isotopic to E . Thus it must be $\partial E'_+$ or $\partial E'_-$, say $\partial E'_+$. Let γ be a spanning arc of the annulus connecting ∂E_- and $\partial E'_+$. Then δ can be regarded as the frontier of $\text{Nbd}(\partial E'_+ \cup \gamma)$ in Σ' (see Figure 2(a)). In Σ' , the boundary circle ∂E_d (resp. $\partial E'_d$) appears as an arc connecting ∂E_+ and ∂E_- (resp. $\partial E'_+$ and $\partial E'_-$). We observe that δ intersects $\partial E'_d$ in at least one point.

Suppose, for contradiction, that δ intersects $\partial E'_d$ in more than one point. Then γ intersects ∂E_d (and $\partial E'_d$) at least once. In particular, there exists an arc component

FIGURE 2

of $\gamma \cap (\Sigma' - \partial E_d)$, which connects $\partial E'_+$ and ∂E_d . Consequently, δ cuts off an annulus A from $\Sigma' - \partial E_d$, having one boundary circle $\partial E'_+$ (see Figure 2(b)).

Now, ∂E and $\partial E'$ represent the two generators x and y , respectively, of the free group $\pi_1(W) = \langle x, y \rangle$, and ∂D represents a word w in terms of x and y . Each such word w can be read from the intersections of ∂D with ∂E_d and $\partial E'_d$. In particular, the arc $\delta \cap \partial A$ determines a sub-word of a word w of the form $xy^{\pm 1}x^{-1}$ or $x^{-1}y^{\pm 1}x$, and hence each word w contains both x and x^{-1} (see Figure 2(b)).

We claim that each word w is reduced, and therefore cyclically reduced. Since $\partial D \cap \partial E_+$ and $\partial D \cap \partial E_-$ contain the same number of points, we also have an essential arc δ' of $\partial D \cap \Sigma'$ whose endpoints lie in ∂E_+ . The arc δ' cuts off an annulus from Σ' , and the boundary circle of the annulus that does not contain δ' must be $\partial E'_-$ (see Figure 2(a)). Since δ' also intersects $\partial E'_d$ in more than one point, the above argument holds for δ' . In particular, δ' cuts off an annulus A' from $\Sigma' - \partial E_d$ having one boundary circle $\partial E'_-$. Observe that the annuli A and A' meet ∂E_d on opposite sides from each other, as in Figure 2(b), since otherwise ∂D would not have minimal intersection with $\partial E_d \cup \partial E'_d$.

Let Σ'' be the 4-holed sphere obtained by cutting Σ along $\partial E_d \cup \partial E'_d$. Then $\partial A \cup \partial A'$ cuts Σ'' into two disks, and each disk meets each boundary circle of Σ'' in a single arc. Consequently, we see there exists no arc component of ∂D in Σ'' that meets only one of ∂E_d and $\partial E'_d$ in the same side. Thus we conclude that w contains neither $x^{\pm 1}x^{\mp 1}$ nor $y^{\pm 1}y^{\mp 1}$. Since this is true for each word w , then each is cyclically reduced.

It is well known that a cyclically reduced word w in the free group $\langle x, y \rangle$ of rank two cannot be a generator if w contains x and x^{-1} simultaneously. Therefore, $\pi_1(W \cup \text{Nbd}(D)) = \langle x, y \mid w \rangle$ cannot be the infinite cyclic group, and consequently $W \cup \text{Nbd}(D)$ is not a solid torus. This contradicts, by Lemma 2.1, that D is primitive in V .

It remains to show that such a primitive disk E' does exist. Choose a primitive disk E' so that $\{E, E'\}$ is a reducing pair. If C is disjoint from E' , we are done. Thus suppose C intersects E' . Then we have a disk $F \subset C$ cut off from C by an outermost arc β of $C \cap E'$ in C such that $F \cap E' = \beta$. By the above argument, the disks obtained from surgery on E' along F are primitive. One of them, say E'' , forms a reducing pair $\{E, E''\}$ with E . It is clear that $|C \cap E''| < |C \cap E'|$ since

at least β no longer counts. Then repeating the process for finding E'' , we get the desired primitive disk. $\square$

3. A SUFFICIENT CONDITION FOR CONTRACTIBILITY

In this section, we introduce a sufficient condition for contractibility of certain simplicial complexes. Let K be a simplicial complex. A vertex w is said to be *adjacent* to a vertex v if it is equal to v or if w spans a 1-simplex in K with v . The *star* $st(v)$ of v is the maximal subcomplex spanned by all vertices adjacent to v .

A *multiset* is a pair (A, m) , typically abbreviated to A , where A is a set and $m : A \rightarrow \mathbb{N}$ is a function. In other words, a multiset is a set with multiplicity. An *adjacency pair* is a pair (X, v) , where X is a finite multiset whose elements are vertices of K which are adjacent to v .

A *remoteness function* on K for a vertex v_0 is a function r from the set of vertices of K to $\mathbb{N} \cup \{0\}$ such that $r^{-1}(0) \subset st(v_0)$.

A function b from the set of adjacency pairs of K to $\mathbb{N} \cup \{0\}$ is called a *blocking function* for the remoteness function r if it has the following properties whenever (X, v) is an adjacency pair with $r(v) > 0$:

- (1) if $b(X, v) = 0$, then there exists a vertex w of K which is adjacent to v such that $r(w) < r(v)$ and (X, w) is also an adjacency pair (see Figure 3(a)), and
- (2) if $b(X, v) > 0$, then there exist $v' \in X$ and a vertex w' of K which is adjacent to v' such that
 - (a) $r(w') < r(v')$,
 - (b) every element of X that is adjacent to v' is also adjacent to w' , and
 - (c) if $Y = (X - \{v'\}) \cup \{w'\}$, then $b(Y, v) < b(X, v)$, where $Y = (X - \{v'\}) \cup \{w'\}$ means removing one instance of v' from X and adding one instance of w' to X (see Figure 3(b)).

A simplicial complex K is said to be *flag* if any collection of $k+1$ pairwise distinct vertices of K spans a k -simplex whenever any two span a 1-simplex. The proof of the following proposition is based on the proof of Theorem 5.3 in [5].

Proposition 3.1. *Let K be a flag complex with base vertex v_0 . If K has a remoteness function r for v_0 that admits a blocking function b , then K is contractible.*

Proof. It suffices to show that the homotopy groups are all trivial. Let $f : \mathbb{S}^q \rightarrow K$, $q \geq 0$, be a map carrying the base point of $\mathbb{S}^q$ to v_0 . We may assume that f

FIGURE 3

is simplicial with respect to a triangulation Δ of $\mathbb{S}^q$. If $r(f(u)) = 0$ for every vertex u of Δ , then the image of f lies in $st(v_0)$. Since K is a flag complex, f is null-homotopic.

Then let us assume that there exists a vertex u of Δ such that $r(f(u)) > 0$. Choose such a vertex u so that $r(f(u)) = n$ is maximal among all vertices of Δ . Our goal is to find a simplicial map $g : \mathbb{S}^q \rightarrow K$ with respect to some subdivision Δ' of Δ such that:

- g is homotopic to f ,
- $r(g(u)) < n$,
- $r(g(z)) = r(f(z))$ for every vertex z of Δ with $z \neq u$, and
- $r(g(u')) < n$ for every vertex u' of $\Delta' - \Delta$.

Then, repeating the process for other vertices whose values by $r \circ f$ are also n , we obtain a simplicial map h with respect to some subdivision of Δ such that h is homotopic to f and $r(h(u)) < n$ for every vertex u . We thus complete the proof inductively.

Let $u_1, u_2, \dots, u_s$ be the vertices in the link of u in Δ and let $f(u) = v$, $f(u_j) = v_j$ for $j = 1, \dots, s$, and $X = \{v_1, v_2, \dots, v_s\}$. Then X is a finite multiset and (X, v) is an adjacency pair. If $b(X, v) = 0$, then there exists a vertex w in $st(v)$ such that $r(w) < r(v)$ and (X, w) is an adjacency pair. Define a simplicial map $g : \mathbb{S}^q \rightarrow K$ with respect to the same triangulation Δ by $g(u) = w$ and $g(z) = f(z)$ for all vertices $z \neq u$. Since K is a flag complex, g is homotopic to f and we have $r(g(u)) < n$.

Now suppose $b(X, v) > 0$. Then there exist $v_j \in X$, and a vertex w_j of K which is adjacent to v_j such that (1) $r(w_j) < r(v_j)$, (2) every element of X that is adjacent to v_j is also adjacent to w_j , and (3) $b(Y, v) < b(X, v)$ where $Y = (X - \{v_j\}) \cup \{w_j\}$. Construct a subdivision Δ' of Δ by introducing the barycenter u'_j of the simplex $\langle u, u_j \rangle$ as a vertex, and replacing each simplex of the form $\langle u, u_j, z_1, z_2, \dots, z_r \rangle$ by the two simplices $\langle u, u'_j, z_1, z_2, \dots, z_r \rangle$ and $\langle u'_j, u_j, z_1, z_2, \dots, z_r \rangle$. Define a simplicial map $f' : \mathbb{S}^q \rightarrow K$ with respect to Δ' by $f'(u'_j) = w_j$ and $f'(z) = f(z)$ for every vertex z of Δ . Since K is a flag complex, f' is homotopic to f . Now Y is the image of the vertices of the link of u in Δ' , and $r(f'(u'_j)) = r(w_j) < r(v_j) \leq r(v) = n$.

Repeating it finitely many times, we obtain a subdivision Δ'' of Δ and a simplicial map f'' with respect to Δ'' so that f'' is homotopic to f and $b(Y'', v) = 0$, where Y'' is the image of the vertices of the link of u in Δ'' . Observe that $r(f''(u'')) < n$ for every vertex u'' of $\Delta'' - \Delta$ and $f''(z) = f(z)$ for every vertex z of Δ . Since $b(Y'', v) = 0$, we obtain a simplicial map $g : \mathbb{S}^q \rightarrow K$ with respect to Δ'' as above such that g is homotopic to f , $r(g(u)) < n$, and $r(g(u'')) < n$ for every vertex u'' of $\Delta'' - \Delta$. $\square$

4. THE COMPLEX OF PRIMITIVE DISKS

The *disk complex* $D(V_g)$ of a handlebody V_g of genus g , for $g \geq 2$, is a simplicial complex defined as follows. The vertices of $D(V_g)$ are isotopy classes of essential disks in V_g , and a collection of $k + 1$ vertices spans a k -simplex if and only if it admits a collection of representative disks which are pairwise disjoint. When V_g is a handlebody in a Heegaard splitting $(V_g, W_g; \Sigma_g)$ of $\mathbb{S}^3$, the *complex of primitive disks* $P(V_g)$ of V_g is defined to be the full subcomplex of $D(V_g)$ spanned by vertices whose representatives are primitive disks in V_g . As before, we write V and $(V, W; \Sigma)$ for V_2 and $(V_2, W_2; \Sigma_2)$, respectively. Notice that $D(V)$ and $P(V)$ are 2-dimensional.

It is a standard fact that any collection of isotopy classes of essential disks in V_g can be realized by a collection of representative disks that have pairwise minimal intersection. One way to see this is to choose the disks in their isotopy classes so that their boundaries are geodesics with respect to some hyperbolic structure on the boundary surface Σ_g and then remove simple closed curve intersections of the disks by isotopy. In particular, for a collection $\{v_0, v_1, \dots, v_k\}$ of vertices of $D(V_g)$, if v_i and v_j bound a 1-simplex for each $i < j$, then $\{v_0, v_1, \dots, v_k\}$ is realized by a collection of pairwise disjoint representatives. Thus we have

Lemma 4.1. *$D(V_g)$ is a flag complex, and consequently any full subcomplex of $D(V_g)$ is also a flag complex.*

Theorem 4.2. *If K is a full subcomplex of $D(V_g)$ satisfying the following condition, then K is contractible:*

- Suppose E and D are any two disks in V_g which represent vertices of K such that $E \cap D \neq \emptyset$. If $C \subset D$ is a disk cut off from D by an outermost arc α of $D \cap E$ in D such that $C \cap E = \alpha$, then at least one of the disks obtained from surgery on E along C also represents a vertex of K .

Proof. Since K is a flag complex, by Lemma 4.1, it suffices to find a remoteness function that admits a blocking function as in Proposition 3.1. Fix a base vertex v_0 of K . Define a remoteness function r on the set of vertices of K by putting $r(w)$ equal to the minimal number of intersection arcs of disks representing v_0 and w .

Let (X, v) be an adjacency pair in K where $r(v) > 0$ and $X = \{v_1, v_2, \dots, v_n\}$. Choose representative disks $E, E_1, \dots, E_n$ and D of $v, v_1, \dots, v_n$ and v_0 , respectively, so that they have transversal and pairwise minimal intersection. Since $r(v) > 0$, so $D \cap E \neq \emptyset$. Let $C \subset D$ be a disk cut off from D by an outermost arc α of $D \cap E$ in D such that $C \cap E = \alpha$. Observe that each E_i is disjoint from the arc α .

Let $b_0 = b_0(E, E_1, \dots, E_n, D)$ be the minimal number of arcs $\{C \cap E_1\} \cup \{C \cap E_2\} \cup \dots \cup \{C \cap E_n\}$ in C as we vary over such disks C cut off from D . Define $b = b(X, v)$ to be the minimal number b_0 as we vary over such representative disks of $v, v_1, \dots, v_n$ and v_0 . We verify that b is a blocking function for the remoteness function r as follows.

Suppose, first, that $b(X, v) = 0$. Then there exist representative disks $E', E'_1, \dots, E'_n$ and D' of $v, v_1, \dots, v_n$ and v_0 , respectively, and exists, as above, a disk $C' \subset D'$ cut off from D' so that $C' \cap (E'_1 \cup \dots \cup E'_n) = \emptyset$. By the assumption, a disk obtained from surgery on E' along C' represents a vertex w' of K again, and (X, w') is an adjacent pair. We have $r(w') < r(v)$ since the arc $C' \cap E'$ no longer counts.

Next, suppose $b(X, v) > 0$. Then choose representative disks $E'', E''_1, \dots, E''_n$ and D'' of $v, v_1, \dots, v_n$ and v_0 , respectively, and, similarly, a disk $C'' \subset D''$ cut off from D'' so that they realize $b(X, v)$. Choose an outermost arc β of $C'' \cap E''_k$ in C'' that cuts off a disk $F'' \subset C''$ such that (1) $F'' \cap E''_k = \beta$, (2) F'' is disjoint from the arc $C'' \cap E''$, and (3) F'' contains no arc of intersection of any E''_i which is disjoint from E''_k (note that β may still intersect an arc of some $C'' \cap E''_i$). Then a disk obtained from surgery on E''_k along F'' represents a vertex w'' of K by the assumption. By the construction, every element of X that is adjacent to v_k is also adjacent to w'' . We have $r(w'') < r(v_k)$ and $b(Y, v) < b(X, v)$, where $Y = (X - \{v_k\}) \cup \{w''\}$, since β no longer counts. $\square$

Theorem 4.2 shows that $D(V_g)$ is contractible since a disk obtained from surgery on an essential disk is also essential. Now we return to the genus two case.

Corollary 4.3. *$P(V)$ is contractible, and the link of any vertex of $P(V)$ is contractible.*

Proof. From Theorem 2.3, $P(V)$ and the link of a vertex v in $P(V)$ satisfy the condition of Theorem 4.2. For the case of the link of v , observe that one of the primitive disks from surgery represents v again, but the other one represents a vertex of the link. $\square$

Let $P(V)'$ be the first barycentric subdivision of $P(V)$ and let T be the graph formed by removing all of the open stars of the vertices of $P(V)$ in $P(V)'$. Each edge of T has one endpoint which represents the isotopy class of a reducing triple (the barycenter of a 2-simplex of $P(V)$) and one endpoint which represents the isotopy class of a reducing pair (the barycenter of an edge of that 2-simplex) in that reducing triple. By Corollary 4.3, we see that T is contractible. Thus we have

Corollary 4.4. *The graph T is a tree.*

5. A FINITE PRESENTATION OF $\mathcal{H}_2$

In this section, we will use our description of T to recover Akbas's presentation of $\mathcal{H}_2$ given in [1]. The tree T is invariant under the action of $\mathcal{H}_2$. In particular, $\mathcal{H}_2$ acts transitively on the set of vertices of T which are represented by reducing triples. The disks in a reducing triple are permuted by representative homeomorphisms of $\mathcal{H}_2$. It follows that the quotient of T by the action of $\mathcal{H}_2$ is a single edge. For convenience, we will not distinguish disks and homeomorphisms from their isotopy classes in their notation.

Fix two vertices $P = \{D_1, D_2\}$ and $Q = \{D_1, D_2, D_3\}$ that are endpoints of an edge e of T . Denote by $\mathcal{H}_P$, $\mathcal{H}_Q$ and $\mathcal{H}_e$ the subgroups of elements of $\mathcal{H}_2$ that preserve P , Q and e , respectively. By the theory of groups acting on trees due to Bass and Serre [8], $\mathcal{H}_2$ can be expressed as the free product of $\mathcal{H}_P$ and $\mathcal{H}_Q$ with an amalgamated subgroup $\mathcal{H}_e$.

It is not difficult to describe these subgroups of $\mathcal{H}_2$. We sketch the argument as follows. Let D'_1 and D'_2 be the disjoint dual disks of D_1 and D_2 , respectively, and let D'_3 be the dual disk of both D_1 and D_2 which is disjoint from D'_1 , D'_2 and D_3 (see Figure 4(a)). The reducing triple $\{D'_1, D'_2, D'_3\}$ of W is uniquely determined by Lemma 2.2. Denote ∂D_i and $\partial D'_i$ by d_i and d'_i , for $i \in \{1, 2, 3\}$, respectively.

Consider the group $\mathcal{H}_P$. The elements of $\mathcal{H}_P$ also preserve $D'_1 \cup D'_2$. Let $\mathcal{H}'_P$ be the subgroup of elements of $\mathcal{H}_P$ which preserve each of D_1 and D_2 . Since the union of Σ with the four disks D_i and D'_i , for $i \in \{1, 2\}$, separates S^3 into (two) 3-balls, $\mathcal{H}'_P$ can be identified with the group of isotopy classes of orientation preserving homeomorphisms of the annulus obtained by cutting Σ along $d_1 \cup d'_1 \cup d_2 \cup d'_2$, which preserve each of d_1 and d_2 . This group is generated by two elements β and β' (π -twists of each boundary circle) and has relations $(\beta\beta')^2 = 1$ and $\beta\beta' = \beta'\beta$. As generators of $\mathcal{H}'_P$, β and β' are shown in Figure 4(b) and Figure 4(c).

Let γ be the order two element of $\mathcal{H}_P$ which interchanges D_1 and D_2 , as shown in Figure 4(d). Then $\mathcal{H}_P$ is an extension of $\mathcal{H}'_P$ by $\langle \gamma \rangle$ with relations $\gamma\beta\gamma = \beta'^{-1}$ and $\gamma\beta'\gamma = \beta^{-1}$.

FIGURE 4

Next, let V' be a 3-ball cut off from V by $D_1 \cup D_2 \cup D_3$, and let $\mathcal{H}'_Q$ be the subgroup of elements of $\mathcal{H}_Q$ which preserve V' . Since the union of Σ with the six disks D_i and D'_i , for $i \in \{1, 2, 3\}$, separates S^3 into (four) 3-balls, $\mathcal{H}'_Q$ can be identified with the group of isotopy classes of orientation preserving homeomorphisms of the 3-holed sphere $\partial V' \cap \partial V$, which preserve $d_1 \cup d_2 \cup d_3$ and $d'_1 \cup d'_2 \cup d'_3$, respectively. This group is a dihedral group generated by γ and δ , where δ is the order three element shown in Figure 4(e).

Let α be the order two element of $\mathcal{H}_Q$ which preserves each D_1 , D_2 and D_3 but takes V' to another component cut off from V by $D_1 \cup D_2 \cup D_3$, as shown in Figure 4(f). Then $\mathcal{H}_Q$ is an extension of $\mathcal{H}'_Q$ by $\langle \alpha \rangle$ with relations $\alpha\gamma\alpha = \gamma$ and $\alpha\delta\alpha = \delta$. Finally, it is easy to see that $\mathcal{H}_e$ is generated by γ and α . Observing that $\beta' = \gamma\beta^{-1}\gamma$ and $\alpha = \beta\beta'$, we get a finite presentation of $\mathcal{H}_2$ with generators β , γ , and δ .

6. THE SCHARLEMANN-AKBAS TREE

Scharlemann [7] constructed a connected simplicial 2-complex Γ which deformation retracts to a certain graph $\tilde{\Gamma}$ on which $\mathcal{H}_2$ acts, and gave a finite generating set of $\mathcal{H}_2$. In this section, we describe Γ and $\tilde{\Gamma}$ briefly and compare them with $P(V)$ and T . For details about Γ and $\tilde{\Gamma}$, we refer the reader to [7] and [1].

Let P and Q be reducing spheres for a genus two Heegaard splitting $(V, W; \Sigma)$ of S^3 . Then $P \cap V$ and $Q \cap V$ are essential separating disks in V , and each of them cuts V into two unknotted solid tori. Define the *intersection number* $P \cdot Q$ to be the minimal number of arcs of $P \cap Q \cap V$ up to isotopy. Then we observe that $P \cdot Q \geq 2$ if $P \cap V$ and $Q \cap V$ are not isotopic to each other in V .

FIGURE 5

The simplicial complex Γ for $(V, W; \Sigma)$ is defined as follows. The vertices are the isotopy classes of reducing spheres relative to V , and a collection $P_0, P_1, \dots, P_k$ of $k+1$ vertices bounds a k -simplex if and only if $P_i \cdot P_j = 2$ for all $i < j$. It turns out that Γ is a 2-complex and each edge of Γ lies on a single 2-simplex. Thus Γ deformation retracts naturally to a graph $\tilde{\Gamma}$ in which each 2-simplex in Γ is replaced by the cone on its three vertices, and Akbas [1] showed that $\tilde{\Gamma}$ is a tree.

We observe that the reducing spheres for $(V, W; \Sigma)$ correspond exactly to the reducing pairs in V up to isotopy. That is, a reducing sphere cuts V into two unknotted solid tori, whose meridian disks are primitive in V . Conversely, a reducing pair has a unique pair of disjoint dual disks, as in Lemma 2.2, and the corresponding reducing sphere is the boundary of a small regular neighborhood of the union of either of the disks with its dual.

It is routine to check that two reducing spheres represent two vertices connected by an edge in Γ if and only if the corresponding reducing pairs are contained in a reducing triple. So the correspondence from reducing spheres to reducing pairs determines a natural embedding of Γ into $P(V)$ so that the image of $\tilde{\Gamma}$ is identified with the tree T (see Figure 5). Therefore, Corollary 4.4 can be considered as an alternate proof that $\tilde{\Gamma}$ is a tree.

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