ON THE GROUND STATE ENERGY OF THE TRANSLATION INVARIANT PAULI-FIERZ MODEL

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ABSTRACT. In this note, we determine the ground state energy of the translation invariant Pauli-Fierz model to subleading order $O(\alpha^3)$ with respect to powers of the finestructure constant α and prove rigorous error bounds of order $O(\alpha^4)$. A main objective of our argument is its brevity.

1. INTRODUCTION

We study the translation invariant Pauli-Fierz model describing a spinless electron interacting with the quantized electromagnetic radiation field. We present a very short and simple method to determine the subleading terms of the ground state energy up to order $O(\alpha^3)$, where $0 < \alpha \ll 1$ denotes the finestructure constant, and to rigorously bound the error by $O(\alpha^4)$.

A well-known difficulty connected to this problem arises from the fact that the ground state energy is not an isolated eigenvalue of the Hamiltonian, and that the form factor in the interaction term of the Hamiltonian contains a critical frequency space singularity (the *infrared problem* of Quantum Electrodynamics (QED)). One of the most striking consequences is that the ground state energy does not exist as a convergent power series in α . It is inaccessible to ordinary perturbation theory and can only be written as a convergent expansion in powers of α of the form $\sum_n b_n(\alpha) \alpha^n$, with α -dependent coefficients $b_n(\alpha)$ that *diverge* in the limit $\alpha \rightarrow 0$ at a subpower rate, i.e., $|b_n(\alpha)| = o(\alpha^{-\delta})$ for any $\delta > 0$, [1, 3, 5].

It is possible to determine the ground state energy and the renormalized electron mass to any arbitrary precision in powers of α , with rigorous error bounds, by use of sophisticated rigorous renormalization group methods, [1, 3, 5]. However, these algorithms are highly complex, and explicitly computing the ground state energy to any subleading order in powers of α is a voluminous task.

Therefore, it remains a desirable goal to devise alternative methods that produce such results more directly for non-trivial, but reasonably low orders in powers of α with rigorous error bounds, and with much less effort.

The present paper is the first in a number of works investigating an approach to such problems based on perturbations around the *true ground state* of the interacting system. A key ingredient in our argument is a new estimate on the expected

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photon number in the ground state derived from [6] whose proof involves a bound on the renormalized electron mass uniform in the infrared cutoff, [1, 5].

Estimates on the ground state energy play an important role, for instance, in binding problems, e.g., the determination of the hydrogen binding energy. The leading term of the ground state energy was demonstrated to be of order $O(\alpha^2)$ (up to normal ordering) in [8], and explicitly determined, with a rigorous error bound of order $O(\alpha^3)$. A similar result was subsequently obtained for the spin $\frac{1}{2}$ case in [4]. In [7], the ground state energy for the Bogoliubov-transformed, translation-invariant Nelson model is explicitly determined up to $O(\alpha^3)$, with an error bound of order $O(\alpha^{\frac{7}{2}})$ (uniformly in the ultraviolet cutoff). While for a fixed ultraviolet cutoff, this result is analogous to ours, we present here a new and particularly short method, and our sharp error bound of order $O(\alpha^4)$ is necessary for a companion paper, where we address the hydrogen ground state energy.

2. DEFINITION OF THE MODEL

We study a non-relativistic electron interacting with the quantized electromagnetic field in the Coulomb gauge. The Hilbert space accounting for the pure states of the electron is given by $L^2(\mathbb{R}^3)$, where we neglect its spin. The Fock space of transverse photon states in the Coulomb gauge is given by

$$\mathfrak{F} = \bigoplus_{n \in \mathbb{N}} \mathfrak{F}_n,$$

where the n -photon space $\mathfrak{F}_n = \bigotimes_s^n (L^2(\mathbb{R}^3) \otimes \mathbb{C}^2)$ is the symmetric tensor product of n copies of one-photon Hilbert spaces $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$. The factor $\mathbb{C}^2$ accounts for the two independent transversal polarizations of the photon. On $\mathfrak{F}$, we introduce creation and annihilation operators $a_\lambda^*(k)$, $a_\lambda(k)$ satisfying the distributional commutation relations

$$[a_\lambda(k), a_{\lambda'}^*(k')] = \delta_{\lambda, \lambda'} \delta(k - k') \quad , \quad [a_\lambda^\sharp(k), a_{\lambda'}^\sharp(k')] = 0,$$

where $a_\lambda^\sharp$ denotes either a_λ or a_λ^* . There exists a unique unit ray $\Omega_f \in \mathfrak{F}$, the Fock vacuum, which satisfies $a_\lambda(k) \Omega_f = 0$ for all $k \in \mathbb{R}^3$ and $\lambda \in \{+, -\}$.

The Hilbert space of states of the system consisting of both the electron and the radiation field is given by

$$\mathcal{H} := L^2(\mathbb{R}^3) \otimes \mathfrak{F}.$$

We use units such that $\hbar = c = 1$, and where the mass of the electron equals $m = \frac{1}{2}$. The electron charge is then given by $e = \sqrt{\alpha}$, where the finestructure constant α will here be considered as a small parameter.

The Hamiltonian of the system is given by

$$(1) \quad H = \mathbf{1}_{el} \otimes H_f + : (i\nabla_x \otimes \mathbf{1}_f - \sqrt{\alpha} A(x))^2 :$$

where $: (\cdots) :$ denotes normal ordering (corresponding to the subtraction of a normal ordering constant proportional to α),

$$H_f = \sum_{\lambda=+,-} \int_{\mathbb{R}^3} dk |k| a_\lambda^*(k) a_\lambda(k)$$

is the Hamiltonian for the free photon field, and the operator

$$A(x) = \sum_{\lambda=+,-} \int_{\mathbb{R}^3} \frac{dk}{2\pi |k|^{1/2}} \kappa(|k|) \varepsilon_\lambda(k) \left(e^{ikx} \otimes a_\lambda(k) + e^{-ikx} \otimes a_\lambda^*(k) \right)$$

couples the electron to the quantized vector potential. The polarization vectors $\varepsilon_\lambda(k)$, $\lambda \in \{+, -\}$, are unit vectors orthogonal to one another and to $k \in \mathbb{R}^3$, in accordance with the Coulomb gauge condition, $\operatorname{div} A = 0$.

The C^1 function κ implements a fixed ultraviolet cutoff on the wavenumbers k . Via scaling, we may assume κ to be compactly supported in $\{|k| \leq 2\}$, monotone, and to satisfy $\kappa = 1$ for $|k| \leq 1$. For convenience, we shall write

$$A(x) = A^-(x) + A^+(x),$$

where

$$A^-(x) = \sum_{\lambda=+,-} \int_{\mathbb{R}^3} \frac{dk}{2\pi|k|^{1/2}} \kappa(|k|) \varepsilon_\lambda(k) e^{ikx} \otimes a_\lambda(k)$$

is the part of $A(x)$ containing the annihilation operators, and $A^+(x) = (A^-(x))^*$.

The system is translation invariant, and H commutes with the operator of total momentum

$$P_{tot} = i\nabla_x \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes P_f,$$

where $i\nabla_x$ and

$$(2) \quad P_f = \sum_{\lambda=+,-} \int dk k a_\lambda^*(k) a_\lambda(k)$$

denote the electron and the photon momentum operators, respectively. It therefore suffices to consider the restriction of H to the fiber Hilbert space $\mathcal{H}_P \cong \mathfrak{F}$ corresponding to the value $P \in \mathbb{R}^3$ of the conserved total momentum, given by

$$(3) \quad H(P) = H_f + : (P - P_f - \sqrt{\alpha} A(0))^2 : .$$

Henceforth, we will write $A^\pm \equiv A^\pm(0)$. From [2, 5], it is known that $\inf \sigma(H) = \inf \sigma(H(0))$, and we will in the sequel only consider $P = 0$.

3. STATEMENT OF THE MAIN RESULTS

On $\mathfrak{F}$, we define the positive bilinear form

$$(4) \quad \langle v, w \rangle_* := \langle v, (H_f + P_f^2) w \rangle$$

and its associated semi-norm $\|v\|_* = \langle v, v \rangle_*^{1/2}$.

Theorem 3.1 (Ground state energy of $H(0)$). *We have*

$$(5) \quad \begin{aligned} \inf \operatorname{spec} (H(0)) &= -\alpha^2 \|\Phi_2\|_*^2 \\ &+ \alpha^3 (2\|A^- \Phi_2\|^2 - 4\|\Phi_3\|_*^2 - 4\|\Phi_1\|_*^2) + O(\alpha^4) \end{aligned}$$

where

$$(6) \quad \Phi_2 := -(H_f + P_f^2)^{-1} A^+ \cdot A^+ \Omega_f,$$

$$(7) \quad \Phi_3 := -(H_f + P_f^2)^{-1} P_f \cdot A^+ \Phi_2,$$

$$(8) \quad \Phi_1 := -(H_f + P_f^2)^{-1} P_f \cdot A^- \Phi_2.$$

Remark 3.1. An easy computation shows that

$$(9) \quad C > \|\Phi_j\|_* > C^{-1} > 0$$

for a constant $C < \infty$ and $j = 1, 2, 3$.

Let $\Psi \in \mathfrak{F}$ be the minimizer of $H(0)$, normalized by $\langle \Psi, \Omega_f \rangle = 1$. Taking the $\langle \cdot, \cdot \rangle_*$ -orthonormal projections of Ψ along the vectors Φ_j , $j = 1, 2, 3$, and denoting the component in the $\langle \cdot, \cdot \rangle_*$ -orthogonal complement of their span by R , we get

$$(10) \quad \Psi = \Omega_f + 2\eta_1 \alpha^{\frac{3}{2}} \Phi_1 + \eta_2 \alpha \Phi_2 + 2\eta_3 \alpha^{\frac{3}{2}} \Phi_3 + R$$

where for $j = 1, 2, 3$,

$$(11) \quad \langle \Phi_i, \Phi_j \rangle_* = \|\Phi_j\|_*^2 \delta_{ij},$$

$$(12) \quad \langle \Phi_j, R \rangle_* = 0, \quad \langle \Omega_f, R \rangle = 0 = \langle \Omega_f, \Phi_j \rangle.$$

The coefficients η_j remain to be determined.

Theorem 3.2 (Ground state of $H(0)$). *Let Ψ be the ground state (10) of $H(0)$, normalized by $\langle \Psi, \Omega_f \rangle = 1$, [1, 5]. Then,*

$$(13) \quad \Psi = \Omega_f + \alpha \Phi_2 + 2\alpha^{\frac{3}{2}} \Phi_1 + 2\alpha^{\frac{3}{2}} \Phi_3 + \tilde{R},$$

where

$$(14) \quad \tilde{R} := R + 2(\eta_1 - 1)\alpha^{\frac{3}{2}} \Phi_1 + (\eta_2 - 1)\alpha \Phi_2 + 2(\eta_3 - 1)\alpha^{\frac{3}{2}} \Phi_3.$$

The coefficients η_j satisfy $|\eta_{1,3} - 1| \leq c\alpha$ and $|\eta_2 - 1| \leq c\alpha^2$. Moreover,

$$(15) \quad \|\tilde{R}\|, \|R\| \leq c\alpha \text{ and } \|\tilde{R}\|_*, \|R\|_* \leq c\alpha^2.$$

A key ingredient in our proof is the following estimate on the expected photon number.

Proposition 3.1. *The expected photon number in the ground state is bounded by*

$$(16) \quad \langle \Psi, N_f \Psi \rangle \leq C\alpha^2 \langle \Psi, \Psi \rangle$$

for a positive constant $C < \infty$ independent of α , where

$$(17) \quad N_f = \sum_{\lambda=+,-} \int dk a_\lambda^*(k) a_\lambda(k)$$

is the photon number operator.

Proof. For $\sigma > 0$, let $H_\sigma(P)$ denote the fiber Hamiltonian regularized by an infrared cutoff implemented by replacing the function κ by a C^1 function κ_σ with $\kappa_\sigma = \kappa$ on $[\sigma, \infty)$, $\kappa_\sigma(0) = 0$, and κ_σ monotonically increasing on $[0, \sigma]$. Then, $E_\sigma(P) := \inf \text{spec}(H_\sigma(P))$ is a simple eigenvalue with eigenvector $\Psi_\sigma(P) \in \mathfrak{F}$, [1, 5]. If $P = 0$, one has $\nabla_P E_\sigma(P = 0) = 0$, [1, 5]. In formula (6.11) of [6], it is shown that

$$(18) \quad a_\lambda(k) \Psi_\sigma(0) = (I) + (II)$$

where from (6.12), (6.22), (6.23), (6.25), (6.27) in [6] it follows that

$$(19) \quad \|(I)\| \leq C(k) |\nabla_P E_\sigma(0)| = 0$$

and that

$$(20) \quad (II) = \sqrt{\alpha} \frac{\kappa_\sigma(|k|)}{|k|^{1/2}} \frac{1}{H_\sigma(k) - E_\sigma(0) + |k|} (H_\sigma(0) - E_\sigma(0)) \epsilon_\lambda(k) \cdot \nabla_P \Psi_\sigma(0)$$

if the electron spin is zero. Thus, it follows immediately from (6.19) in [6] that

$$\|a_\lambda(k) \Psi_\sigma(0)\| \leq c\sqrt{\alpha} \frac{\kappa_\sigma(|k|)}{|k|} \left| \frac{1}{m_{ren,\sigma}} - 1 \right| \|\Psi_\sigma(0)\| \leq c\alpha \frac{\kappa_\sigma(|k|)}{|k|} \|\Psi_\sigma(0)\|,$$

for spin zero, where $m_{ren,\sigma}$ is the renormalized electron mass for $P = 0$, [1, 5], defined by

$$(21) \quad \frac{1}{m_{ren,\sigma}} = 1 - 2 \frac{\langle \nabla_P \Psi_\sigma(0), (H_\sigma(0) - E_\sigma(0)) \nabla_P \Psi_\sigma(0) \rangle}{\langle \Psi_\sigma(0), \Psi_\sigma(0) \rangle}.$$

As proved in [1, 5], $1 < m_{ren,\sigma} < 1 + c\alpha$ uniformly in $\sigma \geq 0$. Correspondingly, we obtain

$$\langle \Psi, N_f \Psi \rangle = \lim_{\sigma \searrow 0} \int dk \|a_\lambda(k) \Psi_\sigma(0)\|^2 \leq C \alpha^2 \|\Psi\|^2$$

as claimed, where $\Psi = s\text{-}\lim_{\sigma \searrow 0} \Psi_\sigma(0)$ (see [1]). $\square$

4. PROOF OF THEOREMS 3.1 AND 3.2

To prove Theorems 3.1 and 3.2, we derive the following upper and lower bounds on $\inf \text{spec}(H(0))$.

4.1. The upper bound. We define the trial function

$$(22) \quad \Psi^{\text{trial}} := \Omega_f + 2\alpha^{\frac{3}{2}} \Phi_1 + \alpha \Phi_2 + 2\alpha^{\frac{3}{2}} \Phi_3.$$

Then, the variational upper bound on the ground state energy

$$(23) \quad \langle \Psi^{\text{trial}}, H(0) \Psi^{\text{trial}} \rangle = -\alpha^2 \|\Phi_2\|_*^2 + \alpha^3 (2\|A^- \Phi_2\|^2 - 4\|\Phi_3\|_*^2 - 4\|\Phi_1\|_*^2) + O(\alpha^4)$$

is obtained from a straightforward computation.

4.2. The lower bound. Substituting the expression (10) for Ψ , and exploiting $\langle \cdot, \cdot \rangle_*$ -orthogonality, straightforward calculations explained in Section 5 yield

$$(24) \quad \begin{aligned} \langle \Psi, H(0) \Psi \rangle \geq & -\alpha^2 \|\Phi_2\|_*^2 + \alpha^3 |\eta_2|^2 (2\|A^- \Phi_2\|^2 - 4\|\Phi_1\|_*^2 - 4\|\Phi_3\|_*^2) \\ & + \frac{1}{2} \|R\|_*^2 \\ & + \alpha^2 |\eta_2 - 1|^2 \|\Phi_2\|_*^2 \\ & + 4\alpha^3 (|\eta_1 - \eta_2|^2 \|\Phi_1\|_*^2 + |\eta_3 - \eta_2|^2 \|\Phi_3\|_*^2) \\ & - c\alpha^4 (1 + |\eta_1|^2 + |\eta_2|^2 + |\eta_3|^2). \end{aligned}$$

We recall from (9) that $C > \|\Phi_j\|_* > C^{-1} > 0$ for some finite C independent of α and $j = 1, 2, 3$. We use

$$(25) \quad |\eta_2 - \eta_j|^2 \geq \frac{1}{2} |\eta_j - 1|^2 - |\eta_2 - 1|^2 \quad \text{and} \quad |\eta_j|^2 \leq 2|\eta_j - 1|^2 + 2$$

to show that the last three lines in (24) are bounded below by

$$(26) \quad |\eta_1 - 1|^2 \alpha^3 B_1 + |\eta_2 - 1|^2 \alpha^2 B_2 + |\eta_2 - 1|^2 \alpha^3 B_3 - c\alpha^4,$$

where the constants $B_j > 0$ are defined by

$$(27) \quad \begin{aligned} B_2 &= \|\Phi_2\|_*^2 - 4\alpha (\|\Phi_1\|_*^2 + \|\Phi_3\|_*^2) - c\alpha^2, \\ B_1 &= 2\|\Phi_1\|_*^2 - c\alpha, \\ B_3 &= 2\|\Phi_3\|_*^2 - c\alpha. \end{aligned}$$

Clearly, (26) $\geq -c\alpha^4$, and the minimizing triple (η_1, η_2, η_3) satisfies

$$(28) \quad |\eta_{1,3} - 1| \leq c\alpha \quad \text{and} \quad |\eta_2 - 1| \leq c\alpha^2.$$

Therefore, (24) is bounded from below by

$$(29) \quad -\alpha^2 \|\Phi_2\|_*^2 + \alpha^3 (2\|A^-\Phi_2\|^2 - 4\|\Phi_3\|_*^2 - 4\|\Phi_1\|_*^2) + \frac{1}{2}\|R\|_*^2 - c\alpha^4.$$

Combined with (23), we find $\|R\|_* \leq c\alpha^2$. This proves Theorems 3.1 and 3.2. $\square$

5. PROOF OF INEQUALITY (24)

Clearly,

$$(30) \quad \langle \Psi, H(0)\Psi \rangle = \langle \Psi, (H_f + P_f^2)\Psi \rangle$$

$$(31) \quad + 4\sqrt{\alpha} \Re \langle \Psi, P_f A^-\Psi \rangle$$

$$(32) \quad + 2\alpha \Re \langle \Psi, A^- A^-\Psi \rangle$$

$$(33) \quad + 2\alpha \langle \Psi, A^+ A^-\Psi \rangle.$$

We estimate the quadratic form of $H(0)$ on the true ground state Ψ .

5.1. Some auxiliary estimates. Since

$$(34) \quad \|A^-\psi\| \leq c\|H_f^{\frac{1}{2}}\psi\|,$$

one has

$$(35) \quad \|A^-\psi\|, \|P_f\psi\| \leq c\|\psi\|_*$$

for all $\psi \in \mathfrak{F}$ in the intersection of the domains of $H_f^{\frac{1}{2}}$ and P_f . For brevity let

$$(36) \quad \tilde{\Psi} := \Omega_f + 2\eta_1 \alpha^{\frac{3}{2}} \Phi_1 + \eta_2 \alpha \Phi_2 + 2\eta_3 \alpha^{\frac{3}{2}} \Phi_3$$

so that $\Psi = \tilde{\Psi} + R$. We observe that from (9), (35), and the Schwarz inequality,

$$(37) \quad \|A^-\tilde{\Psi}\|^2, \|P_f\tilde{\Psi}\|^2 \leq c\alpha^2 |\eta_2|^2 + c\alpha^3 (|\eta_1|^2 + |\eta_3|^2).$$

In addition, we recall that $A^\pm P_f = P_f A^\pm$ by the Coulomb gauge.

5.2. The term (30). First of all, the term (30) gives

$$(38) \quad \langle \Psi, (H_f + P_f^2)\Psi \rangle = \alpha^2 \|\Phi_2\|_*^2 + 4\alpha^3 (\|\Phi_1\|_*^2 + \|\Phi_3\|_*^2) + \|R\|_*^2$$

due to the orthogonality relations (11), (12).

5.3. The term (31). We expand the term (31) according to $\Psi = \tilde{\Psi} + R$. Clearly,

$$(39) \quad \begin{aligned} 4\sqrt{\alpha} \Re \langle \tilde{\Psi}, P_f A^-\tilde{\Psi} \rangle &= 8\alpha^3 \Re \eta_1 \eta_2 \langle P_f \Phi_1, A^-\Phi_2 \rangle \\ &+ 8\alpha^3 \Re \eta_2 \eta_3 \langle A^+\Phi_2, P_f \Phi_3 \rangle. \end{aligned}$$

Moreover, we have

$$(40) \quad 4\sqrt{\alpha} \Re \langle R, P_f A^-R \rangle \geq -\frac{1}{8}\|R\|_*^2.$$

Here, we used $\|A^-R\| \leq c\|H_f^{\frac{1}{2}}R\|$ combined with (37), and applied the Schwarz inequality in the form $|AB| \leq \frac{\delta}{2}A^2 + \frac{1}{2\delta}B^2$ for any $0 < \delta < \infty$.

Furthermore, we find

$$(41) \quad 8\sqrt{\alpha} \Re \langle \tilde{\Psi}, P_f A^-R \rangle \geq -\frac{1}{8}\|R\|_*^2 - c\alpha^4 (\eta_1^2 + \eta_3^2)$$

by similar arguments, using

$$(42) \quad \langle \Phi_2, P_f A^-R \rangle = \langle \Phi_3, R \rangle_* = 0 = \langle \Omega_f, P_f A^-R \rangle$$

(see (12)), combined with (37).

5.4. The term (32). We have

$$2\alpha \Re \langle \Psi, A^- A^- \Psi \rangle = 2\alpha \Re \langle A^+ \Psi, A^- \Psi \rangle.$$

Clearly,

$$(43) \quad \|A^+ \psi\|^2 = [A^-, A^+] \|\psi\|^2 + \|A^- \psi\|^2,$$

where $0 < [A^-, A^+] < c$, for all $\psi \in \mathfrak{F}$. Using the Schwarz inequality and similar arguments as above, one finds

$$(44) \quad \begin{aligned} 2\alpha \Re \langle \Psi, A^- A^- \Psi \rangle &\geq 2\alpha^2 \Re \eta_2 \langle A^+ A^+ \Omega_f, \Phi_2 \rangle \\ &\quad - \frac{1}{8} \|R\|_*^2 - c\alpha^2 \|R\|^2 \\ &\quad - c\alpha^4 (|\eta_1|^2 + |\eta_2|^2 + |\eta_3|^2), \end{aligned}$$

where we used that if $\psi = \Omega_f, \Phi_1, \Phi_3$, or R , then by (11), (12),

$$(45) \quad \langle A^+ A^+ \Omega_f, \psi \rangle = \langle \Phi_2, \psi \rangle_* = 0.$$

At this point, we apply Proposition 3.1 in

$$(46) \quad \|R\| \leq \|N_f^{\frac{1}{2}} R\| \leq \left| \|N_f^{\frac{1}{2}} \Psi\| - \|N_f^{\frac{1}{2}} \tilde{\Psi}\| \right| \leq c\alpha.$$

Here, we use the fact that R has a vanishing projection on the Fock vacuum, (12), and $\|N_f^{\frac{1}{2}} \tilde{\Psi}\| \leq c\alpha$, which one easily verifies.

5.5. The term (33). This term is estimated by

$$(47) \quad 2\alpha \|A^- \Psi\|^2 \geq 2\alpha^3 \|A^- \Phi_2\|^2 - c\alpha^4 (|\eta_1|^2 + |\eta_3|^4) - \frac{1}{8} \|R\|_*^2.$$

5.6. Collecting all estimates. Combining the $O(\alpha^3)$ -terms of (38) with (39), we find

$$(48) \quad \begin{aligned} &4\alpha^3 |\eta_1|^2 \|\Phi_1\|_*^2 + 8\alpha^3 \Re \eta_1 \eta_2 \langle \Phi_1, P_f A^- \Phi_2 \rangle \\ &= -4\alpha^3 |\eta_2|^2 \|\Phi_1\|_*^2 + 4\alpha^3 |\eta_1 - \eta_2|^2 \|\Phi_1\|_*^2, \end{aligned}$$

since from (8),

$$(49) \quad \langle \Phi_1, P_f A^- \Phi_2 \rangle = -\|\Phi_1\|_*^2.$$

Likewise,

$$(50) \quad \begin{aligned} &4\alpha^3 |\eta_3|^2 \|\Phi_3\|_*^2 + 8\alpha^3 \Re \eta_2 \eta_3 \langle P_f A^+ \Phi_2, \Phi_3 \rangle \\ &= -4\alpha^3 |\eta_2|^2 \|\Phi_3\|_*^2 + 4\alpha^3 |\eta_2 - \eta_3|^2 \|\Phi_3\|_*^2, \end{aligned}$$

since from (7),

$$(51) \quad \langle P_f A^+ \Phi_2, \Phi_3 \rangle = -\|\Phi_3\|_*^2.$$

Combining the $O(\alpha^2)$ -term in (38) with the first term on the r.h.s. of (44),

$$(52) \quad \begin{aligned} &\alpha^2 |\eta_2|^2 \|\Phi_2\|_*^2 + 2\alpha^2 \Re \eta_2 \langle A^+ A^+ \Omega_f, \Phi_2 \rangle \\ &= -\alpha^2 \|\Phi_2\|_*^2 + \alpha^2 |\eta_2 - 1|^2 \|\Phi_2\|_*^2, \end{aligned}$$

since from (6),

$$(53) \quad \langle A^+ A^+ \Omega_f, \Phi_2 \rangle = -\|\Phi_2\|_*^2.$$

Moreover, we use half of $\|R\|_*^2$ in (38) to compensate all four terms $-\frac{1}{8}\|R\|_*^2$ appearing in the above estimates, and we are left with the term $\frac{1}{2}\|R\|_*^2$ in (24).

All remaining terms are bounded below by $-c\alpha^4(1 + |\eta_1|^2 + |\eta_2|^2 + |\eta_3|^2)$. Collecting all bounds, we arrive at (24). $\square$

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