ON THE C^∞ WAVE-FRONT SET OF TRACES OF CR FUNCTIONS ON MAXIMALLY REAL SUBMANIFOLDS

Z. ADWAN AND G. HOEPFNER

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ABSTRACT. We prove that, in a locally integrable structure, the C^∞ wave-front set of the trace of a CR function at a point p in a totally real submanifold of maximal dimension is independent of the maximally real submanifold passing through the point p .

0. INTRODUCTION

In 1983, the authors Baouendi et al. introduced the notion of hypo-analyticity and the hypo-analytic wave-front set. Among other things, they were able to answer some questions about extendability of CR functions and they also characterized the hypo-analytic wave-front set in terms of the exponential decay of a modified FBI transform, which they introduced in that paper. Furthermore, they proved that the hypo-analytic wave-front set at a point p in a hypo-analytic manifold is independent of the maximally real submanifold passing through the point p . More precisely, here is the exact statement:

Theorem 4.1 [BCT]. *Let $(M, \mathcal{V})$ be a hypo-analytic structure, $p \in M$, U a neighborhood of p in M , h a distribution solution of $\mathcal{V}$ in U , $\xi^0 \in T_p^*M \setminus \{0\}$, X, Y two maximally real C^∞ submanifolds of U passing through p , and h_X, h_Y the traces of h on X and on Y respectively. If the pullback $i_X^*(\xi^0)$ does not belong to the hypo-analytic wave-front set of h_X , then $i_Y^*(\xi^0)$ does not belong to that of h_Y .*

In this paper, we will be working in the setting of a locally integrable structure and we will prove an analogous result for the C^∞ wave-front set.

This paper is presented as follows: in section 1, we fix some notation, recall some basic definitions and state the main result, Theorem 1.3; in section 2 we modify the change of coordinates made in [BCT], using the first integrals given in the recent work of Eastwood and Graham [EG]. The main tool in proving our result is the FBI transform, which can also be used to characterize microlocal smoothness (see Theorem 2.3). Finally, we reserve section 3 for the proof of our main result, Theorem 1.3.

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1. PRELIMINARIES

Let M be a C^∞ manifold, $\dim_{\mathbb{R}} M = m + n$ and $\mathcal{V} \subset \mathbb{C}TM$ a subbundle of rank n that is involutive, that is, the bracket is a closed operation in $\mathcal{V}$. The involutive structure $(M, \mathcal{V})$ is called locally integrable if the orthogonal of $\mathcal{V}$ in $\mathbb{C}T^*M$ is locally generated by exact forms. In this paper $(M, \mathcal{V})$ will be a locally integrable structure.

A distribution f on M is called a solution if $Lf = 0$ for all smooth sections L of $\mathcal{V}$.

Definition 1.1. *A smooth submanifold $\mathcal{X}$ of M is called maximally real if*

$$(1.1) \quad \mathbb{C}T_p M = \mathbb{C}T_p \mathcal{X} \oplus \mathcal{V}_p, \quad \forall p \in \mathcal{X}.$$

When $(M, \mathcal{V})$ is a CR structure, that is, $\mathcal{V} \cap \overline{\mathcal{V}} = \{0\}$, then a submanifold $\mathcal{X}$ is maximally real if and only if it is totally real of maximal dimension.

Let us denote by $\pi_{\mathcal{X}}$ the natural quotient map $T^*M|_{\mathcal{X}} \rightarrow T^*\mathcal{X}$, and by $\pi_{\mathcal{X}}^{\mathbb{C}}$ the analogous map for the complexified cotangent bundles. It is easy to see that Definition 1.1 is equivalent to the fact that $\pi_{\mathcal{X}}^{\mathbb{C}}$ induces a bijection of $\mathcal{V}^\perp|_{\mathcal{X}}$ onto $\mathbb{C}T^*\mathcal{X}$. As a consequence, $\pi_{\mathcal{X}}$ induces an injection of $T^0 = \mathcal{V}^\perp \cap T^*M$ into $T^*\mathcal{X}$. We recall that $T^0 \setminus \{0\}$ is the characteristic set of all sections of $\mathcal{V}$.

Definition 1.2. *We say that a point of $T^*\mathcal{X} \setminus \{0\}$ is characteristic if it belongs to $\pi_{\mathcal{X}}(T^0|_{\mathcal{X}})$, noncharacteristic otherwise.*

Even when $\mathcal{V}$ is a line bundle, the dimension of T_p^0 may not be constant as p varies. However, when $\mathcal{V}$ is a CR structure, then T^0 is a vector bundle.

Now we are ready to state our main result:

Theorem 1.3. *Let h be a distribution solution in an open neighborhood U of a point p_0 of M , $\xi_0 \neq 0$ a characteristic cotangent vector at p_0 , $\mathcal{X}, \mathcal{Y}$ two maximally real C^∞ submanifolds of U passing through p_0 , and $h_{\mathcal{X}}$ and $h_{\mathcal{Y}}$ the traces of h on $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then $\pi_{\mathcal{X}}(\xi_0)$ does not belong to the C^∞ wave-front set of $h_{\mathcal{X}}$ if and only if $\pi_{\mathcal{Y}}(\xi_0)$ does not belong to the C^∞ wave-front set of $h_{\mathcal{Y}}$.*

Suppose that $(M, \mathcal{V})$ is locally integrable and $\mathcal{X}$ is a maximally real submanifold of M . Given $p \in \mathcal{X}$, set $d = \dim_{\mathbb{R}} T_p^0$ and $r = m - d$. According to [EG], Proposition 2.2, for each integer $N > 0$ there are i) a neighborhood U of p in M , ii) local coordinates $\{x_1, \dots, x_m, y_1, \dots, y_n\}$ vanishing at p and iii) smooth solutions Z_j , $j = 1, \dots, m$ for the locally integrable structure on U such that

$$\mathcal{X} \cap U = \{y = 0\}$$

and

$$(1.2) \quad Z_j(x, y) = x_j + iy_j + i\Psi_j(x), \quad 1 \leq j \leq r,$$

$$(1.3) \quad Z_j(x, y) = x_j + i\Phi_j(x, y), \quad r + 1 \leq j \leq m,$$

$$dZ_1(x, y) \wedge \dots \wedge dZ_m(x, y) \neq 0 \quad \text{for } (x, y) \in U,$$

$$\{dZ_1(x, y), \dots, dZ_m(x, y)\} \text{ generates } \mathcal{V}_{(x, y)}^\perp \text{ for } (x, y) \in U,$$

where the $\Psi_j(x)$ and $\Phi_k(x, y)$ are smooth real functions satisfying

$$(1.4) \quad |\Psi_j(x)| = O(|x|^N), \quad \Phi_k(0) = 0, \quad d_x \Phi_k(0, 0) = 0 \quad \text{and} \quad |\Phi_k(x, 0)| = O(|x|^N).$$

It is convenient to take U in the form of a product

$$U = V \times W,$$

where V (resp., W) is an open ball in the x -space $\mathbb{R}^m$ (resp., in the y -space $\mathbb{R}^n$), centered at the origin. The dual coordinates will be denoted by $\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n$, and $T^*M|_U$ will be identified with $U \times \mathbb{R}^{m+n}$.

If $\mathcal{X}$ is any maximally real submanifold of M passing through a point p_0 , we can adapt $\mathcal{X}$ in the sense that $\mathcal{X} \cap U$ is defined by the equation $y = 0$ as before. We write $\mathcal{X}_{y_0}$ for the submanifold $y = y_0$ of U ; thus $\mathcal{X}_0 = \mathcal{X} \cap U$. We will identify the cotangent bundle of $\mathcal{X}_{y_0}$ to $V \times \mathbb{R}^m$ and $\pi_{\mathcal{X}_{y_0}}$ to the coordinate projection $U \times \mathbb{R}^{m+n} \rightarrow V \times \mathbb{R}^m$.

For a vector in $x \in \mathbb{R}^m$ (resp. $y \in \mathbb{R}^n$) we define $x' = (x_1, \dots, x_r)$ and $x'' = (x_{r+1}, \dots, x_m)$ (resp. $y' = (y_1, \dots, y_r)$ and $y'' = (y_{r+1}, \dots, y_n)$). One can show that (see [BCT]):

$$(1.5) \quad T_0^0 = \{(\xi, \eta) \in T_0^*M : \xi' = 0, \eta = 0\},$$

$$(1.6) \quad \pi_{\mathcal{X}}(T_0^0) = \{\xi \in T_0^*\mathcal{X} : \xi' = 0\}.$$

Observe that in order that T_0^0 be $\neq 0$ we must have $r < m$. We make the hypothesis that this is indeed the case.

2. CHANGE OF COORDINATES

Suppose that $\mathcal{X}$ is a maximally real submanifold through 0 and that (x, y) are coordinates in a neighborhood U of the origin with $\mathcal{X} \cap U = \{y = 0\}$ and (1.2) and (1.3) are satisfied as in section 1. We recall Proposition 4.1 in [BCT].

Proposition 2.1. *If $\mathcal{Y}$ is any maximally real C^∞ submanifold of U containing the origin, then there are an invertible $r \times r$ complex matrix S and a C^∞ map from an open neighborhood $V_\# \subset V$ of the origin to $\mathbb{R}^n$, $f_\#$, with $f_\#(0) = 0$, such that, if we make the change of coordinates*

$$(2.1) \quad x'_\# + iy'_\# = Sz', \quad x''_\# = x'', \quad y''_\# = y'',$$

then $\mathcal{Y}$ is defined in an open neighborhood $U_\# \subset U$ of the origin by the equations

$$(2.2) \quad y_\# = f_\#(x_\#).$$

Conversely, any system of equations (2.1) and (2.2) defines a maximally real C^∞ submanifold in some open neighborhood of the origin, passing through that point.

Now we will adapt the changes of coordinates made in [BCT], using the modified first integrals given by [EG], for the locally integrable case. For the matrix S given by Proposition 2.1, we can define the powers S^t of the matrix S for all real t . Consider the following C^∞ map:

$$\Lambda : \mathbb{R}^m \times \mathbb{R} \longrightarrow \mathbb{C}^r \times \mathbb{R}^{(m-r)+(n-r)} \cong \mathbb{R}^{m+n}$$

given by

$$(2.3) \quad \Lambda(\tilde{x}, t) = (z' = S^{-t}\tilde{x}', x'' = \tilde{x}'', 0).$$

Select a ball $\tilde{V}$ centered at $0 \in \mathbb{R}^m$ with radius small enough that the image of $\tilde{V} \times (-2, 2)$ under the map Λ will be contained in U . Call $\tilde{Z}_j$ the pull-back of Z_j

($1 \leq j \leq m$) via Λ . We have:

$$(2.4) \quad \begin{aligned} \tilde{Z}_j(\tilde{x}, t) &= Z_j(\Lambda(\tilde{x}, t)) = Z_j(x' + iy' = S^{-t}\tilde{x}', \tilde{x}'', 0) \\ &= \begin{cases} (S^{-t}\tilde{x}')_j + i\Psi_j(\Re(S^{-t}\tilde{x}'), \tilde{x}''), & j = 1, \dots, r, \\ \tilde{x}_j + i\tilde{\Phi}_j(\tilde{x}, t), & j = r+1, \dots, m, \end{cases} \end{aligned}$$

where $\tilde{\Phi}_j(\tilde{x}, t) = \Phi_j(S^{-t}\tilde{x}', \tilde{x}'', 0)$. Write $\tilde{\Phi}'' = (\tilde{\Phi}_{r+1}, \dots, \tilde{\Phi}_m)$. By (1.4) and the definition of Λ we see that, for some $B > 0$,

$$|\tilde{\Phi}''(\tilde{x}, t)| \leq B|\tilde{x}|^2, \quad \forall \tilde{x} \in \tilde{V}, t \in (-2, 2).$$

The $\tilde{Z}_j$ define a locally integrable structure on $\tilde{V} \times (-2, 2)$. For each t , $|t| < 2$, $\tilde{X}_t = \tilde{V} \times \{t\}$ is a maximally real submanifold in this structure and Λ induces an isomorphism of $\tilde{X}_t$ onto an open neighborhood of the origin in the maximally real submanifold of U ,

$$\mathcal{X}_t = \{(x, y) \in U : \Im(S^t z') = 0, y'' = 0\}.$$

Now we will use the coordinates $x_\#, y_\#$ of Proposition 2.1 to define another C^∞ map of an open neighborhood of the origin in $\mathbb{R}^{m+1}$ into U . Assume that the neighborhood $V_\#$ in Proposition 2.1 and an open ball $W_\#$ in $\mathbb{R}^n$ are small enough that the following two properties hold:

- (i) $U_\# \doteq V_\# \times W_\# \subset U$;
- (ii) for all $t \in (-2, 2)$, $tf_\#(V_\#) \subset W_\#$.

Define $\Gamma : V_\# \times (-2, 2) \subset \mathbb{R}^{m+1} \rightarrow U_\# \subset \mathbb{R}^{m+n}$ by

$$\Gamma(x_\#, t) = (x_\#, tf_\#(x_\#)).$$

If we define $F : U \rightarrow U_\#$ by

$$F(x, y) = (Sz', x'', y''),$$

then $\{Z_j \circ F^{-1}\}$ defines a locally integrable structure in $U_\#$ isomorphic to the locally integrable structure defined by $\{Z_j\}$ on U (in the sense that F sends solutions in (U, Z) to solutions in $(U_\#, Z \circ F^{-1})$ and vice versa). We have

$$(2.5) \quad \begin{aligned} Z_j \circ F^{-1}(x_\#, y_\#) &= Z_j(S^{-1}z'_\#, x''_\#, y''_\#) \\ &= \begin{cases} (S^{-1}z'_\#)_j + i\Psi_j(\Re(S^{-1}z'_\#), x''_\#), & j = 1, \dots, r, \\ x_{\#j} + i\Phi_j(S^{-1}z'_\#, x''_\#, y''_\#), & j = r+1, \dots, m, \end{cases} \end{aligned}$$

with

$$|\Psi_j(\Re(S^{-1}z'_\#), x''_\#)| = O(|(x_\#, y'_\#)|^N).$$

Replacing $Z_1 \circ F^{-1}, \dots, Z_r \circ F^{-1}$ by

$$Z^* = S \begin{pmatrix} Z_1 \circ F^{-1} \\ \vdots \\ Z_r \circ F^{-1} \end{pmatrix}$$

we get new equivalent first integrals given by

$$(2.6) \quad Z_j^*(x_\#, y_\#) = \begin{cases} z'_{\#j} + \Psi_j^*(\Re(S^{-1}z'_\#), x''_\#), & j = 1, \dots, r, \\ x_{\#j} + i\Phi_j(S^{-1}z'_\#, x''_\#, y''_\#), & j = r+1, \dots, m, \end{cases}$$

where

$$\Psi^*(x_\#, y_\#) = S \begin{pmatrix} \Psi_1(\Re(S^{-1}z'_\#), x''_\#) \\ \vdots \\ \Psi_r(\Re(S^{-1}z'_\#), x''_\#) \end{pmatrix}.$$

Observe that

$$|\Psi^*(x_\#, y_\#)| = O(|(x_\#, y'_\#)|^N).$$

Call $Z_\#$ the pull-back of Z^* via Γ . We have

$$(2.7) \quad \begin{aligned} Z_{\#j}(x_\#, t) &= Z_j^*(x_\#, tf_\#(x_\#)) \\ &= \begin{cases} x_{\#j} + itf_{\#j}(x_\#) + \Psi_{\#j}(x_\#, t), & j = 1, \dots, r, \\ x_{\#j} + i\Phi_{\#j}(x_\#, t), & j = r+1, \dots, m, \end{cases} \end{aligned}$$

where

$$\Phi_\#(x_\#, t) = \Phi(S^{-1}(x'_\# + itf'_\#(x_\#)), x''_\#, tf''_\#(x_\#))$$

and

$$\Psi_\#(x_\#, t) = \Psi^*(x_\#, tf_\#(x_\#)).$$

Because $f_\#(0) = 0$ we have, for suitable constants $C, B > 0$,

$$(2.8) \quad |\Phi_\#(x_\#, t)| \leq B|x_\#|^2 \quad \text{and} \quad |\Psi_\#(x_\#, t)| \leq C|x_\#|^N, \quad x_\# \in V_\#, \quad |t| < 2.$$

The $Z_{\#j}$ define a locally integrable structure on $V_\# \times (-2, 2)$. The image of the maximally real submanifold $Y_{\#t} = V_\# \times \{t\}$ via Γ is an open neighborhood of the origin in the maximally real submanifold $\mathcal{Y}_t$ of $U_\#$ defined by the equation $y_\# = tf_\#(x_\#)$.

Remarks. (a) Since S is a constant matrix (SZ', Z'') , define on U the same locally integrable structure as Z . Thus, both of the locally integrable structures defined on $\tilde{V} \times (-2, 2)$ and $V_\# \times (-2, 2)$ are the pull-back of one and the same structure on U . In particular the pull-back of any distribution solution in U is a distribution solution in each of the locally integrable structures so defined.

(b) Both mappings Λ and Γ preserve x'' and therefore any form $\xi''dx''$: the pull-back of $\xi''dx''$ via Λ is $\xi''d\tilde{x}''$, which via Γ is $\xi''dx''_\#$. Thus, if $(0, \xi'', 0) \in T_0^*M$ is a characteristic point (see (1.5) and (1.6)), its pull-back to the maximally real submanifolds $\tilde{X}_\tau$ via Λ and to $Y_{\#t}$ via Γ is represented by $(0, \xi'')$ in the coordinates $\tilde{x}$ and $x_\#$ respectively.

As in [BCT], but using the coordinates given by [EG], we are going to change the coordinates of $\tilde{x}_j$, $j \leq r$, in $\tilde{V} \times (-2, 2)$. Since we shall not modify the coordinates $\tilde{x}_j$ for $j > r$, this change of coordinates will have no effect on the remarks just made. Cover the interval $[0, 1]$ with a finite number of open intervals $J_1, \dots, J_k$, all contained in $(-2, 2)$ and such that, if t_l is the central point of J_l , the matrix

$$\Re S^{t_l-t}$$

is invertible for any t in a neighborhood of $\overline{J_l}$. In $\tilde{V} \times J_l$ take as first integrals $(S^{t_l}\tilde{Z}', \tilde{Z}'')$. Then the new first integrals (which we will still call $\tilde{Z}$) are:

$$\tilde{Z}'(\tilde{x}, t) = S^{t_l-t}\tilde{x}' + iS^{t_l} \begin{pmatrix} \Psi_1(\Re(S^{-t}\tilde{x}'), \tilde{x}'') \\ \vdots \\ \Psi_r(\Re(S^{-t}\tilde{x}'), \tilde{x}'') \end{pmatrix} \doteq S^{t_l-t}\tilde{x}' + iS^{t_l}(\Psi(\Re(S^{-t}\tilde{x}'), \tilde{x}''))$$

with

$$(2.9) \quad |S^{t_l}(\Psi(\Re(S^{-t}\tilde{x}'), \tilde{x}''))| \leq D|\tilde{x}|^N.$$

Let $\tilde{\underline{x}}' = \Re(S^{t_l-t}\tilde{x}')$, $\tilde{\underline{x}}'' = \tilde{x}''$. In these new coordinates, for $j = 1, \dots, r$, we have

$$\tilde{Z}_j(\tilde{\underline{x}}, t) = \tilde{\underline{x}}_j + i\Im(S^{t_l-t}[\Re(S^{t_l-t})]^{-1}\tilde{\underline{x}}')_j + i[S^{t_l}(\tilde{\Psi}(\tilde{\underline{x}}, t))]_j,$$

where

$$\tilde{\Psi}(\tilde{\underline{x}}, t) = \Psi(\Re(S^{-t}[\Re(S^{t_l-t})]^{-1}\tilde{\underline{x}}'), \tilde{\underline{x}}'')$$

and, for $j = r+1, \dots, m$,

$$(2.10) \quad \tilde{Z}_j(\tilde{\underline{x}}, t) = \tilde{\underline{x}}_j + i\tilde{\Phi}_j(\Re([\Re(S^{t_l-t})]^{-1}\tilde{\underline{x}}'), \tilde{\underline{x}}'').$$

We see that in both cases we are in a situation that can be described as follows: let V be an open ball centered at the origin in $\mathbb{R}^m$, and let J be an open interval in the real line. On $V \times J$ we are given a locally integrable structure defined by the following m C^∞ functions:

$$(2.11) \quad Z_j(x, t) = \begin{cases} x_j + \Psi_j(x, t) + i\Phi_j(x, t), & j = 1, \dots, r, \\ x_j + i\Phi_j(x, t), & j = r+1, \dots, m, \end{cases}$$

with Φ_j and Ψ_j real-valued,

$$(2.12) \quad \Phi_j(0, t) \equiv 0, \quad j = 1, \dots, m, \quad \Psi_j(x, t) = O(|x|^N), \quad t \in J, x \in V, j = 1, \dots, r.$$

Moreover there is $B > 0$ such that

$$(2.13) \quad |\Phi''(x, t)| \leq B|x|^2, \quad x \in V, \quad t \in J.$$

Now we recall the definition of the FBI transform in the locally integrable structure described above: let $u \in C^\infty(J; \mathcal{E}'(V))$. The FBI transform of u , denoted F_u^κ , $\kappa > 0$, is given by the “integral” (which, in reality, is a duality bracket)

$$(2.14) \quad \begin{aligned} F_u^\kappa(t; y, \xi) &= \int_V e^{Q^\kappa(x, t, y, \xi)} u(x, t) dZ(x, t) \\ &= \int_V e^{Q^\kappa(x, t, y, \xi)} u(x, t) \det(Z_x(x, t)) dx, \end{aligned}$$

where

$$(2.15) \quad Q^\kappa(x, t, y, \xi) = i\xi \cdot (y - Z(x, t)) - \kappa |\xi| \langle y - Z(x, t) \rangle^2$$

and with $y \in \mathbb{R}^m$, $\xi \in \mathbb{R}^m \setminus \{0\}$ and t in a neighborhood of 0 in $\mathbb{R}$.

Theorem 2.2. *Assume (2.11), (2.12), and (2.13) hold. There are open balls $B_r(0) \subset B_{2r}(0) \subset V$ in $\mathbb{R}^m$, a neighborhood of the origin $O \subset \mathbb{R}^m$ and constants κ^* , $\delta > 0$ such that, if $\chi \in C_0^\infty(B_{2r}(0))$ is equal to one in $B_r(0)$, then to every solution h in $U = V \times (-\delta, \delta)$ there are constants $A_1, A_2 > 0$ such that*

$$|\mathcal{F}_{\chi_h}^\kappa(t; y, \xi) - \mathcal{F}_{\chi_h}^\kappa(0; y, \xi)| \leq A_1 e^{-A_2|\xi|}$$

in the region

$$|t| < \delta, \quad y \in O \subset \mathbb{R}^m, \quad \xi \in \mathbb{R}^m \setminus \{0\}, \quad \kappa \geq \kappa^*.$$

Proof. We shall begin by modifying the Z_j for $j \leq r$. Let $L^j(x, t)$ denote the linear part of the Taylor expansion of Z_j with respect to x about $x = 0$. Notice that by (2.12), we have

$$|Z_j(x, t) - L^j(0, x'', 0)| \leq \text{const.} \left(|x'| + |x|^2 + |t|^2 \right).$$

Thus, after substitution of $Z_j - L^j(0, x'', 0)$ for Z_j we may assume that $L^j(x, 0)$ is independent of x'' . But this means that, possibly after decreasing V , we have

$$\begin{aligned} |A^{-1}Z'(x, 0) - x'| &= \left| A^{-1} \left[Ax' + O(|x|^2) \right] - x' \right| \\ &= O(|x|^2), \quad x \in V, \end{aligned}$$

where $A \in \text{GL}(r, \mathbb{C})$ is given by

$$A = \left(\frac{\partial Z'}{\partial x'}(0, 0) \right).$$

Set

$$\begin{aligned} \tilde{Z}'(x, t) &= A^{-1}Z'(x, t) \\ &= A^{-1}(x' + \Psi(x, t) + i\Phi'(x, t)) \\ &= A^{-1} \left(\frac{\partial Z'}{\partial x'}(0, 0)x' + \Psi(x, t) + i \left[\Phi'(x, t) - \frac{\partial \Phi'}{\partial x'}(0, 0)x' \right] \right) \\ &= (x' + \tilde{\Psi}(x, t) + i\tilde{\Phi}'(x, t)), \end{aligned}$$

where

$$\tilde{\Psi}(x, t) = \Re \left(A^{-1}\Psi(x, t) \right) - \Im \left(A^{-1} \left(\Phi'(x, t) - \frac{\partial \Phi'}{\partial x'}(0, 0)x' \right) \right)$$

and

$$\tilde{\Phi}'(x, t) = \Im \left(A^{-1}\Psi(x, t) \right) + \Re \left(A^{-1} \left(\Phi'(x, t) - \frac{\partial \Phi'}{\partial x'}(0, 0)x' \right) \right).$$

Note now that, in view of (2.12),

$$\tilde{\Psi}(x, t) = O(|x|^2 + |t|^2) \quad \text{and} \quad \tilde{\Phi}'(x, t) = O(|x|^2 + |t|^2).$$

Hence, if we substitute

$$\tilde{Z}'(x, t) = A^{-1}Z'(x, t)$$

for $Z'(x, t)$ and take (2.13) into account, we may as well suppose that

$$\Psi(x, t) = O(|x|^2 + |t|^2) \quad \text{and} \quad \Phi(x, t) = O(|x|^2 + |t|^2).$$

From the definition of the FBI transform (2.14) and Proposition IX.5.2 in [T], we obtain

$$\begin{aligned} \mathcal{F}_{\chi h}^\kappa(t; y, \xi) - \mathcal{F}_{\chi h}^\kappa(0; y, \xi) &= \int_0^1 \frac{d}{ds} (\mathcal{F}_{\chi h}^\kappa(st; y, \xi)) ds \\ &= \int_0^1 \frac{\partial \mathcal{F}_{\chi h}^\kappa}{\partial t}(st; y, \xi) t ds \\ (2.16) \quad &= \int_0^1 \left[\int_{r \leq |x| \leq 2r} e^{Q^\kappa(x, st, y, \xi)} L\chi(x) h(x, st) \det(Z_x(x, st)) dx \right] t ds \end{aligned}$$

where $Q^\kappa(x, st, y, \xi)$ is given by (2.15). Hence, by (2.16) and the fact that $h(\cdot, t)$ is a bounded set of distributions for $|t| < 1$, there exists an integer $M > 0$ such that

$$(2.17) \quad |\mathcal{F}_{\chi h}^\kappa(t; y, \xi) - \mathcal{F}_{\chi h}^\kappa(0; y, \xi)| \leq A_0 |t| (1 + |\xi|)^M \left\| e^{\Re Q^\kappa(x, st, y, \xi)} \right\|_{L_x^\infty(B_{2r}(0) \setminus B_r(0))}$$

for all $|t| \leq 1$ and all y, ξ, κ . Now, note that for some $C > 0$ large enough, we have

$$\begin{aligned} \Re Q^\kappa(x, st, 0, \xi) &= \xi \cdot \Phi(x, st) - \kappa |\xi| |x + (\Psi(x, st), 0)|^2 + \kappa |\xi| |\Phi(x, st)|^2 \\ &\leq |\xi| |\Phi(x, st)| - \kappa |\xi| |x|^2 + \kappa |\xi| |\Psi(x, st)|^2 + \kappa |\xi| |\Phi(x, st)|^2 \\ &\leq C |\xi| (|st|^2 + |x|^2) - \kappa |\xi| |x|^2 + C \kappa |\xi| (|st|^4 + |x|^4) \\ &\leq C |\xi| [|t|^2 + \kappa |t|^4] + |\xi| [-\kappa |x|^2 + C |x|^2 + C \kappa |x|^4]. \end{aligned}$$

Thus, when $r \leq |x| \leq 2r$ and $|t| < 1$ we have

$$\Re Q^\kappa(x, st, 0, \xi) \leq C |\xi| (1 + \kappa) |t|^2 + |\xi| [-\kappa r^2 + 4Cr^2 + 16C\kappa r^4].$$

Choosing $r < 1/(8\sqrt{C})$ and $\kappa > 16C$, we can insure that

$$\Re Q^\kappa(x, st, 0, \xi) \leq C |\xi| (1 + \kappa) |t|^2 - \frac{1}{2} \kappa r^2 |\xi|,$$

for all $|t| < 1$, $\xi \in \mathbb{R}^m \setminus \{0\}$, $0 \leq s \leq 1$, $\kappa \geq \kappa^* = 16C$ and $r \leq |x| \leq 2r$. We can now pick $\delta > 0$ small enough (and only dependent on C) such that

$$\Re Q^\kappa(x, st, 0, \xi) \leq -\frac{1}{4} \kappa r^2 |\xi|,$$

for all $|t| < \delta$, $\xi \in \mathbb{R}^m \setminus \{0\}$, $0 \leq s \leq 1$, $\kappa \geq \kappa^*$, and $r \leq |x| \leq 2r$. Thus, there is a neighborhood O in $\mathbb{R}^m$ so that for all $y \in O$,

$$(2.18) \quad \Re Q^\kappa(x, st, y, \xi) \leq -\frac{1}{8} \kappa r^2 |\xi| = -A |\xi|,$$

Hence, using (2.17) and (2.18) we obtain

$$|\mathcal{F}_{\chi h}^\kappa(t; y, \xi) - \mathcal{F}_{\chi h}^\kappa(0; y, \xi)| \leq A_1 |t| e^{-A_2 |\xi|}$$

in the region

$$|t| < \delta, \quad y \in O \subset \mathbb{R}^m, \quad \xi \in \mathbb{R}^m \setminus \{0\}, \quad \kappa \geq \kappa^*.$$

Hence, the proof is complete. $\square$

The following theorem states that microlocal smoothness can be characterized by the rapid decay in ξ of the FBI transform. The proof can be found in [BH1].

Theorem 2.3. *Let $h(x, t)$ be a solution in $U = V \times (-\delta, \delta)$, let $h_0(x) = h(x, 0)$ be the trace of h on $V \times \{0\}$, and let χ be defined as in the previous theorem. In order for a cotangent vector $\xi^0 \in T_0^*V$ not to be in $WF_0(h_0)$, it is necessary and sufficient that there exists a neighborhood O of the origin in $\mathbb{R}^m$, a conic neighborhood $\mathcal{C}$ of ξ^0 in $\mathbb{R}^m \setminus \{0\}$, and $\kappa^* > 0$ so that the following holds:*

$$\forall N \in \mathbb{N}, \exists C_N > 0 : |\mathcal{F}_{\chi h}^\kappa(0; y, \xi)| \leq \frac{C_N}{|\xi|^N} \quad \forall (y, \xi) \in O \times \mathcal{C}, \quad \kappa \geq \kappa^*.$$

We are now in a position to prove the following corollary of the previous two theorems:

Corollary 2.4. *Let $h(x, t)$ be a solution in $U = V \times J$, t_0 any point in J , $\xi^0 \in T_0^*V$ and let $h_{t_0} = h(x, t_0)$ be the trace of h on $V \times \{t_0\}$. Then*

$$\xi^0 \notin WF_0(h_{t_0}) \implies \xi^0 \notin WF_0(h_{t_1}) \text{ for any } t_1 \in J.$$

Proof. It suffices to show that to every closed subinterval K of J there is a number $\delta > 0$ such that the assertion is true if $t_0, t_1 \in K$ and $|t_0 - t_1| < \delta$. Define the new solution

$$\tilde{h}(x, t) = h(x, t + t_0).$$

Then $\tilde{h}(x, 0) = h(x, t_0)$, $\tilde{h}(x, t_1 - t_0) = h(x, t_1)$, $WF_0(h_{t_0}) = WF_0(\tilde{h}_0)$ and also $WF_0(h_{t_1}) = WF_0(\tilde{h}_{t_1-t_0})$. Hence, if $\xi^0 \notin WF_0(h_{t_0})$, then $\xi^0 \notin WF_0(\tilde{h}_0)$ and so, using Theorem 2.3 we get that there exists a neighborhood O of the origin in $\mathbb{R}^m$, a conic neighborhood $\mathcal{C}$ of ξ^0 in $\mathbb{R}^m \setminus \{0\}$, and $\kappa^* > 0$ so that the following holds:

$$\forall N \in \mathbb{N}, \exists C_N > 0 : \left| \mathcal{F}_{\chi \tilde{h}}^{\kappa}(0; y, \xi) \right| \leq \frac{C_N}{|\xi|^N} \quad \forall (y, \xi) \in O \times \mathcal{C}, \kappa \geq \kappa^*.$$

Now, since $|t_0 - t_1| < \delta$, we can use Theorem 2.2 to find constants $A_1, A_2 > 0$ such that

$$\left| \mathcal{F}_{\chi \tilde{h}}^{\kappa}(t_0 - t_1; y, \xi) - \mathcal{F}_{\chi \tilde{h}}^{\kappa}(0; y, \xi) \right| \leq A_1 e^{-A_2 |\xi|}$$

in the region

$$y \in B_{r/2}(0) \subset \mathbb{R}^m, \quad \xi \in \mathbb{R}^m \setminus \{0\}, \quad \kappa \geq \kappa^*.$$

Hence,

$$\left| \mathcal{F}_{\chi \tilde{h}}^{\kappa}(t_0 - t_1; y, \xi) \right| \leq A_1 e^{-A_2 |\xi|} + \frac{C_N}{|\xi|^N} \leq \frac{D_N}{|\xi|^N} \quad \forall (y, \xi) \in O \times \mathcal{C}, \kappa \geq \kappa^*.$$

This shows, using Theorem 2.3 again, that $\xi^0 \notin WF_0(\tilde{h}_{t_0-t_1})$ or, equivalently, ξ^0 is not in $WF_0(h_{t_1})$. $\square$

3. PROOF OF THE MAIN THEOREM

We return to the case where $(M, \mathcal{V})$ is a locally integrable structure of dimension m and codimension n , locally defined by the system of coordinates $(x, y) \in U = V \times W \subset \mathbb{R}^{m+n}$ and first integrals Z_j , $j = 1, \dots, m$ defined in U satisfying (1.2), (1.3) and (1.4) of section 1.

Now we will prove our main result.

Proof of Theorem 1.3. We assume that $\mathcal{X}$ is defined in U by the equation $y = 0$. On the other hand we take $\mathcal{Y}$ to be defined (in $U_\#$) by (2.1) and (2.2). Using the definitions of $\mathcal{X}_t$ and $\mathcal{Y}_t$ in section 2, we see that

$$\mathcal{X} = \mathcal{X}_0, \quad U_\# \cap F(\mathcal{X}_1) = \mathcal{Y}_0, \quad \mathcal{Y}_1 = \mathcal{Y} \cap U_\#.$$

Now we apply Corollary 2.4 to the locally integrable structure in $\tilde{V} \times J_l$, defined by the functions $\tilde{Z}_j$, $j = 1, \dots, m$, given by (2.4) and (2.6). Since the intervals J_l , $l = 1, \dots, k$, form an open covering of $[0, 1]$, we see that if the C^∞ wave-front set of the trace of a distribution solution $\tilde{h}$ on $\tilde{V} \times \{0\}$ does not contain $\pi_{\mathcal{Y}}(\xi_0) = \pi_{\mathcal{X}}(\xi_0) = (0, \xi_0'')$, then the same is true for its trace in $\tilde{V} \times \{1\}$; i.e., the C^∞ wave-front set of the trace of $\tilde{h}$ on $\tilde{V} \times \{1\}$ does not contain $(0, \xi_0'')$.

Apply Corollary 2.4 again to the locally integrable structure in $V_\# \times (-2, 2)$, defined by the functions $Z_{\#j}$, $j = 1, \dots, m$, given by (2.5). So, we conclude that if the C^∞ wave-front set of the trace of a distribution solution $\tilde{h}$ on $V_\# \times \{0\}$ does

not contain $(0, \xi_0'')$, the same is true of its trace in $\tilde{V} \times \{1\}$; i.e., the C^∞ wave-front set of the trace of $\tilde{h}$ on $\tilde{V} \times \{1\}$ does not contain $(0, \xi_0'')$.

Let h be a distribution solution in an open neighborhood U of a point p_0 of M , $\xi_0 \neq 0$ a characteristic cotangent vector to M at p_0 such that $\pi_{\mathcal{X}}(\xi_0) = (0, \xi_0'')$ does not belong to the C^∞ wave-front set of $h_{\mathcal{X}}$. We take $\tilde{h}$ to be the pull-back via Λ of h and $h_{\#}$ to be the pull-back of $h \circ F^{-1}$ via Γ . Since Λ induces an isomorphism of $\tilde{X}_0$ onto an open neighborhood of the origin in $\mathcal{X}_0 = \mathcal{X}$, and Γ induces an isomorphism of $Y_{\#1}$ onto an open neighborhood of the origin in $\mathcal{Y}_1 = \mathcal{Y} \cap U_{\#}$, and these isomorphisms (together with F) preserve $(0, \xi_0'')$, the theorem is proved. $\square$

REFERENCES

- [BCT] M.S. Baouendi, C. H. Chang, and F. Trèves, *Microlocal hypo-analyticity and extension of CR functions*, J. Differential Geom. **18** (1983), 331–391. MR723811 (85h:32030)
- [BH1] S. Berhanu and J. Hounie, *An F. and M. Riesz theorem for planar vector fields*, Math. Ann. **320** (2001), 463–485. MR1846773 (2002f:35057)
- [EG] M. G. Eastwood and C. R. Graham, *Edge of the wedge theory in hypo-analytic manifolds*, Commun. Partial Differ. Equations **28** (2003), 2003–2028. MR2015410 (2004j:32038)
- [T] F. Trèves, *Hypo-analytic structures*, Local theory, Princeton University Press, 1992. MR1200459 (94e:35014)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT BROWNSVILLE, BROWNSVILLE, TEXAS 77520

E-mail address: `adwan@utb.edu`

DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PENNSYLVANIA 19122

E-mail address: `hoepfner@temple.edu`