HYPERSURFACES WHOSE TANGENT GEODESICS DO NOT COVER THE AMBIENT SPACE

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We dedicate this work to our beloved wives Cristina and Fabiola

ABSTRACT. Let $x : \Sigma^n \rightarrow M^{n+1}$ be an immersion of an n -dimensional connected manifold Σ in an $(n + 1)$ -dimensional connected complete Riemannian manifold M without conjugate points. Assume that the union of geodesics tangent to x does not cover M . Under these hypotheses we have two results. The first one states that M is simply connected provided that the universal covering of Σ is compact. The second result says that if x is a proper embedding and M is simply connected, then $x(\Sigma)$ is a normal graph over an open subset of a geodesic sphere. Furthermore, there exists an open star-shaped set $A \subset M$ such that $\bar{A}$ is a manifold with the boundary $x(\Sigma)$.

1. INTRODUCTION

Let $x : \Sigma^n \rightarrow M^{n+1}$ be an immersion of a connected n -dimensional manifold Σ in a connected complete $(n + 1)$ -dimensional Riemannian manifold M . A very strong condition would be to assume that x is totally geodesic. Here we will make a much weaker assumption, namely, that the union of geodesics tangent to x does not cover M . More precisely, let $W = W(x)$ be the set of points of M that do not lie on any geodesic tangent to x . We could ask in what situations we could have $W \neq \emptyset$. The first important answer to this question was given by Halpern [Hp]. To state it we first recall a well-known definition.

Definition 1.1. A subset B of a complete manifold M is said to be star-shaped with respect to x_0 if $x_0 \in B$ and for any point $p \in B$ there exists a unique minimal normal geodesic joining x_0 and p , and the image of that unique geodesic is contained in B .

Theorem A (Halpern [Hp]). *Let $x : \Sigma^n \rightarrow \mathbf{R}^{n+1}$, $n \geq 2$, be an immersion of a closed manifold Σ with $W \neq \emptyset$. Then, the following conclusions hold:*

- (1) x is an embedding;
- (2) Σ^n is diffeomorphic to the Euclidean sphere;
- (3) there exists a diffeomorphic disk D with $\partial D = x(\Sigma)$; furthermore D and its interior $\text{int}(D)$ are star-shaped with respect to any point of W .

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Beltagy [Be] extended this result to the case where M is a complete and simply connected Riemannian manifold without conjugate points, using the fact that the exponential map is a diffeomorphism and applying Theorem A to the tangent space at any point of W . Alexander [Al] modified Theorem A, restricting the attention to the union V of tangent spaces at saddle points of Σ , and obtained a weaker conclusion.

Remark 1. Theorem A and Alexander's Theorem can be extended to the case where M is the standard sphere S^{n+1} by using the stereographic projection associated with the antipodal point $-P$ where $P \in W$ in the case of Theorem A and $P \in S^{n+1} - V$ in the case of Alexander's Theorem.

Alencar and Frensel proved in [AF] that, if M is a space form $Q^{n+1}(c)$ and x is minimal with W being a nonempty open set, then x is totally geodesic. This result for a minimal immersion $x: M^2 \rightarrow Q_c^3$, $c \geq 0$, with nonempty W has been proved by Hasanis and Koutroufiotis [HK]. Another result in [AF] says that if $\Sigma^n \subset Q^{n+1}(c)$ is closed and has constant mean curvature with $W \neq \emptyset$, then Σ is a round sphere.

Now we state our first result.

Theorem 1.1. *Let $x: \Sigma^n \rightarrow M^{n+1}$, $n \geq 2$, be an immersion, where M is a connected complete Riemannian manifold without conjugate points and with $W \neq \emptyset$. If the universal covering of Σ is compact, then M is simply connected.*

Remark 2. The inclusion $T^n \subset T^{n+1}$ of flat tori shows that the hypothesis that $\tilde{\Sigma}$ is compact is essential in Theorem 1.1.

The following result is new even in the case that $M = \mathbf{R}^{n+1}$. First we recall the definition of a normal graph.

Definition 1.2. Let S, X be contained in a manifold M , where S is an embedded submanifold. We say that X is a normal graph over S if there exists a homeomorphism $h: S \rightarrow X$ such that for any point $x \in S$, the image $h(x)$ is connected to x by a unique minimal geodesic, and that unique geodesic is orthogonal to S .

Theorem 1.2. *Let Σ be a connected properly embedded hypersurface in a simply connected complete Riemannian manifold M without conjugate points with $W \neq \emptyset$. Then we have:*

- (1) Σ is a normal graph over an open subset of a geodesic sphere;
- (2) there exists an open set A such that A and its closure $\bar{A}$ are star-shaped with respect to any point of W ; furthermore, $\bar{A}$ is a manifold that has Σ as its boundary.

Remark 3. If we consider the spiral S given by $r = 1 + 2^{-\theta}$ in polar coordinates, the product $S \times \mathbf{R}^{n-1} \subset \mathbf{R}^{n+1}$ shows that the condition that Σ is properly embedded is essential in Theorem 1.2.

2. PROOF OF THEOREM 1.1

Let $x: \Sigma^n \rightarrow M^{n+1}$ be an immersion where M is a connected Riemannian manifold without conjugate points. Let $\pi: \tilde{M} \rightarrow M$ be the universal covering of M with the induced metric and let $\nu: \tilde{\Sigma} \rightarrow \Sigma$ be the universal covering of Σ . By the Fundamental Lifting Theorem, for any $p \in \Sigma$, $\tilde{p} \in \nu^{-1}(p)$ and $\tilde{q} \in \pi^{-1}(x(p))$,

there exists an immersion $y : \widetilde{\Sigma} \rightarrow \widetilde{M}$, namely the lifting of x , satisfying $y(\bar{p}) = \tilde{q}$ and such that the diagram below commutes:

$$(1) \quad \begin{array}{ccc} \widetilde{\Sigma} & \xrightarrow{y} & \widetilde{M} \\ \downarrow \nu & & \downarrow \pi \\ \Sigma & \xrightarrow{x} & M. \end{array}$$

Let $W(y) \subset \widetilde{M}$ be the set of points that do not lie on any geodesic tangent to the immersion y . Let $\mathcal{A}$ be the set of liftings of x .

Claim 2.1. For $W = W(x)$ it follows that

$$\pi^{-1}(W) = \bigcap_{y \in \mathcal{A}} W(y).$$

Proof. Assume that $\tilde{q} \notin \pi^{-1}(W)$. Since $q = \pi(\tilde{q}) \notin W$, there exists a geodesic $\gamma : [0, a] \rightarrow M$ with $\gamma(0) = q$ that is tangent to the immersion x at $t = a$. The geodesic γ can be lifted to a geodesic $\tilde{\gamma} : [0, a] \rightarrow \widetilde{M}$ with $\tilde{\gamma}(0) = \tilde{q}$ and such that $\tilde{\gamma}$ is tangent to some immersion $y \in \mathcal{A}$ at $t = a$. Thus $\tilde{q} \notin W(y)$.

Conversely, assume that $\tilde{q} \notin W(y)$ for some $y \in \mathcal{A}$. Then there exists a geodesic $\tilde{\gamma}$ tangent to the immersion y containing $\tilde{q}$. The geodesic $\gamma := \pi \circ \tilde{\gamma}$ is tangent to the immersion x and contains $\pi(\tilde{q})$. Therefore $\pi(\tilde{q}) \notin W$; hence $\tilde{q} \notin \pi^{-1}(W)$. $\square$

To prove Theorem 1.1 we assume that $\widetilde{\Sigma}$ is compact. Take $x_0 \in W$ and fix some lifting $y \in \mathcal{A}$. By Claim 2.1, $\pi^{-1}(x_0) \subset W(y)$. It follows from Beltagy's Theorem [Be] that $y(\widetilde{\Sigma})$ is an embedded sphere that is the boundary of a compact topological disk D containing the set $\pi^{-1}(x_0)$. We also have that D and its interior $\text{int}(D)$ are star-shaped with respect to any point of $W(y)$, and thus they are star-shaped with respect to any point of $\pi^{-1}(x_0)$. Since $\pi^{-1}(x_0)$ is discrete and contained in a compact disk, the set $\pi^{-1}(x_0)$ and the group $\text{Aut}(\pi)$ of automorphisms of π are both finite. For any $\phi \in \text{Aut}(\pi)$, we have that $\bar{y} := \phi \circ y$ is also a lifting of x since $\pi = \pi \circ \phi$.

$$(2) \quad \begin{array}{ccccc} & & \bar{y} & & \\ & \nearrow & & \searrow & \\ \widetilde{\Sigma} & \xrightarrow{y} & \widetilde{M} & \xrightarrow{\phi} & \widetilde{M} \\ & & \downarrow \pi & \nearrow \pi & \\ & & M & & \end{array}$$

Again we have that $\pi^{-1}(x_0) \subset W(\bar{y})$ and that $\bar{y}(\widetilde{\Sigma})$ is embedded as the boundary of the compact topological disk $\phi(D)$ containing the set $\pi^{-1}(x_0)$. We also have that $\phi(D)$ and $\text{int}(\phi(D))$ are star-shaped with respect to any point of $\pi^{-1}(x_0)$. Thus we have that

$$(3) \quad E = \bigcap_{\phi \in \text{Aut}(\pi)} \phi(D)$$

is invariant under $\text{Aut}(\pi)$ and that $\text{int}(E)$ is star-shaped with respect to any point of $\pi^{-1}(x_0)$. In fact, since $\text{Aut}(\pi)$ is finite, it is easy to see that $\text{int}(E) = \bigcap_{\phi \in \text{Aut}(\pi)} \text{int}(\phi(D))$. Thus $\text{int}(E)$ is star-shaped with respect to any point in $\pi^{-1}(x_0)$, since it is the intersection of star-shaped sets.

Claim 2.2. E is a compact topological disk.

Proof. Fix $\tilde{x}_0 \in \pi^{-1}(x_0)$. We know that $\text{int}(E)$ is star-shaped with respect to $\tilde{x}_0$. Fix a unit vector $v \in T_{\tilde{x}_0} \widetilde{M}$ and $\phi \in \text{Aut}(\pi)$. Since $\tilde{x}_0 \in W(\phi \circ y)$, we claim that the geodesic $\gamma : t \mapsto \exp_{\tilde{x}_0} tv$ meets transversely the hypersurface $\phi \circ y(\widetilde{\Sigma}) = \partial\phi(D)$ at a unique point. In fact, there exists a first time $t_{v,\phi}$ such that γ intersects $\partial\phi(D)$ since $\phi(D)$ is star-shaped with respect to $\tilde{x}_0$. Now we prove that this intersection is unique. Indeed, since $W(\phi \circ y) \neq \emptyset$, the geodesic γ intersects $\partial\phi(D)$ transversely. Since $\phi(D)$ is a manifold with boundary $\partial\phi(D)$, we have that $\gamma(t) \notin \phi(D)$ for $t > t_{v,\phi}$ sufficiently close to $t_{v,\phi}$. If γ intersects $\partial\phi(D)$ a second time this will contradict the fact that $\phi(D)$ is star-shaped with respect to $\tilde{x}_0$.

Since the intersection is a transversal at $t_{v,\phi}$, the time $t_{v,\phi}$ will depend smoothly on v . Let

$$t(v) = \min_{\phi \in \text{Aut}(\pi)} t_{v,\phi}.$$

Thus $t(v)$ depends continuously on v . Given $w \in E$, set

$$v(w) = \frac{(\exp_{\tilde{x}_0})^{-1}w}{\|(\exp_{\tilde{x}_0})^{-1}w\|}.$$

Let $B \subset T_{\tilde{x}_0} \widetilde{M}$ be the compact unit disk centered at 0. Now we define $F : B \rightarrow E$ given by

$$F(z) = \exp_{\tilde{x}_0} t(z/\|z\|) z,$$

for $z \neq 0$, and $F(0) = \tilde{x}_0$. It is not difficult to see that F has a continuous inverse $G : E \rightarrow B$ given by

$$G(w) = \frac{1}{t(v(w))} (\exp_{\tilde{x}_0})^{-1}w,$$

for $w \neq \tilde{x}_0$ and $G(\tilde{x}_0) = 0$. □

Finally, for any $\phi \in \text{Aut}(\pi)$, we have that ϕ must have a fixed point in E , since $\phi(E) \subset E$, and E is a compact disk by Claim 2.2. Thus we have that $\text{Aut}(\pi)$ is trivial and M is simply connected. Theorem 1.1 is proved.

3. PROOF OF THEOREM 1.2

Let M be a complete and simply connected Riemannian manifold without conjugate points. Let $\Sigma \subset M$ be a connected and properly embedded hypersurface satisfying $W \neq \emptyset$.

Fix $x_0 \in W$. In particular $x_0 \notin \Sigma$. Since Σ is properly embedded there exists some small geodesic sphere S centered at x_0 that does not intersect Σ . We define $F : \Sigma \rightarrow S$ as follows. Given $p \in \Sigma$, there exists a unique normal geodesic $\gamma := [x_0, p]$ joining x_0 and p . We know that γ is orthogonal to S at a unique intersection point, which we will call $F(p)$. Since $x_0 \in W$ it follows that γ is also transverse to Σ ; hence $F : \Sigma \rightarrow S$ is a local diffeomorphism onto its open image $F(\Sigma) \subset S$. Thus, to show that $F : \Sigma \rightarrow F(\Sigma)$ is a diffeomorphism it is sufficient to show that F is injective.

Define the set

$$\mathcal{C} := \left\{ p \in \Sigma \mid \text{the cardinality } \#([x_0, p] \cap \Sigma) = 1 \right\}.$$

We need to prove that $\mathcal{C} = \Sigma$.

Claim 3.1. $\mathcal{C} \neq \emptyset$. In fact, using that Σ is a properly embedded hypersurface, we obtain that there exists a point $p_0 \in \Sigma$, satisfying $d(x_0, p_0) = \min_{p \in \Sigma} d(x_0, p)$. Thus $p_0 \in \mathcal{C}$.

Claim 3.2. $\Sigma - \mathcal{C}$ is open as a subset of Σ .

To prove this we take $x_1 \in \Sigma - \mathcal{C}$. So there exists $x_2 \in \Sigma$ with $x_2 \neq x_1$ and $x_2 \in [x_0, x_1]$. In particular we have $F(x_1) = F(x_2) = q$. Since F is a local diffeomorphism, it is not difficult to see that there exist disjoint neighborhoods of x_1 and x_2 in Σ mapped by F onto the same neighborhood of $q \in S$. Thus we conclude that $\Sigma - \mathcal{C}$ is open in Σ .

Claim 3.3. $\Sigma - \mathcal{C}$ is closed as a subset of Σ .

In fact, take a sequence $x_k \rightarrow a \in \Sigma$, with $x_k \in \Sigma - \mathcal{C}$. Since Σ is properly embedded, there exists an open neighborhood U of $a \in M$ such that the intersection $\Sigma \cap U$ is connected and the restriction $F|_{\Sigma \cap U}$ is a diffeomorphism onto its open image. By definition of $\mathcal{C}$, for each k there exists a point $y_k \neq x_k$ with $y_k \in [x_0, x_k] \cap \Sigma$. Since (x_k) is bounded, we have that (y_k) is also bounded. So we can assume by passing to a subsequence that (y_k) converges to some point $b \in [x_0, a]$. Since Σ is properly embedded we have that $b \in \Sigma$. Since $F|_{\Sigma \cap U}$ is injective we have that $y_k \notin U$; hence $b \neq a$. So we conclude that $a \in \Sigma - \mathcal{C}$.

Thus $\mathcal{C} = \Sigma$ by connectedness of Σ and $F : \Sigma \rightarrow F(\Sigma)$ is a diffeomorphism.

Now we will prove that Σ is the boundary of an open star-shaped set with respect to x_0 . Consider the set

$$A := \left\{ z \in M \mid \#([x_0, z] \cap \Sigma) = 0 \right\}.$$

Given $z \in A$, the distance between $[x_0, z]$ and Σ is positive, since Σ is properly embedded. This implies that A is open.

We state that $\bar{A} - A = \Sigma$. In fact, given $p \in \Sigma$, the geodesic segment $[x_0, p] - \{p\} \subset A$; hence $\Sigma \subset \bar{A}$. Clearly, we have $A \cap \Sigma = \emptyset$; hence $\Sigma \subset \bar{A} - A$. Now take $p \in \bar{A} - A$. Assume by contradiction that $p \notin \Sigma$. Since $p \notin A$ the geodesic $[x_0, p]$ intersects Σ transversely at a unique point $q \neq p$. Consider the unit vector $v \in T_{x_0}M$ such that $\exp_{x_0} t_0 v = p$ for some $t_0 > 0$. Since F is a diffeomorphism onto its open image, there exists an open neighborhood U of v in the sphere $S^{n-1} \subset T_{x_0}M$ such that geodesic $t \mapsto \exp_{x_0} tw$ meets Σ at a unique point for any $w \in U$. Since Σ is properly embedded we can choose U and $\epsilon > 0$ sufficiently small such that $\exp_{x_0} tw \notin \Sigma$ if $|t - t_0| < \epsilon$. This defines a neighborhood V of p with $V \subset M - A$. This contradicts the fact that $p \in \bar{A}$.

Clearly we have that A and $\bar{A} = A \cup \Sigma$ are star-shaped with respect to x_0 . To conclude the proof we need to show that $\bar{A}$ is a manifold with boundary Σ . In fact, take a point $p \in \Sigma$, with $p = \exp_{x_0} t_0 v$ for some unit vector $v \in T_{x_0}M$. Again there exists an open neighborhood U of v in $S^{n-1} \subset T_{x_0}M$ such that for each $w \in U$ the geodesic $t \mapsto \exp_{x_0} tw$ meets Σ transversely at a unique point $q_w = \exp_{x_0} t_w w$. So the time t_w depends smoothly on w . Thus a small neighborhood W of p in $\bar{A}$ can be defined as

$$W = \exp_{x_0} \{ tw \mid w \in U, t_w - \epsilon < t \leq t_w \},$$

where $\epsilon > 0$ is small enough. Thus we have proved that $\bar{A}$ is a smooth manifold with boundary Σ . $\square$

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