Z_2^2 -ACTIONS WITH n -DIMENSIONAL FIXED POINT SET

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ABSTRACT. We describe the equivariant cobordism classification of smooth actions (M^m, Φ) of the group $G = Z_2^2$, considered as the group generated by two commuting involutions, on closed smooth m -dimensional manifolds M^m , for which the fixed point set of the action is a connected manifold of dimension n and $m = 4n - 1$ or $4n - 2$. For $m \geq 4n$, the classification is known.

1. INTRODUCTION

In [7], R. E. Stong determined all possible equivariant cobordism classes of smooth involutions $T : M \rightarrow M$, defined on closed smooth m -dimensional manifolds M , for which the fixed point set F has dimension n and $m = 2n - 1$; specifically, he showed that any such involution pair (M, T) is equivariantly cobordant to a union of involutions of the following two types:

i) Let $\pi : N \rightarrow S^1$ be a smooth fibering, where N is a closed n -dimensional manifold and S^1 is the 1-sphere, and let $M_1 = \{(x, y) \in N \times N \mid \pi(x) = \pi(y)\}$, with involution $T_1(x, y) = (y, x)$.

ii) Let P and Q be closed manifolds with dimensions p and q , respectively, where $p + q = n - 1$, and let $M_2(P, Q) = \frac{S^1 \times P \times P \times Q \times Q}{(x, p, p', q, q') \sim (-x, p', p, q, q')}$ with the involution $T_2[x, p, p', q, q'] = [x, p', p, q', q]$.

In general, for a given (M, T) , several summands of the second type, with different p values, may be needed; every such union of involutions is equivariantly cobordant to an involution with $(n$ -dimensional) connected fixed set.

Note. The equivariant cobordism classification for $\dim(F) = n$ and $\dim(M) \geq 2n$ was obtained in [1].

The purpose of this note is to extend the above result for Z_2^2 -actions; here, $G = Z_2^2$ is understood as the group generated by two commuting involutions T_1, T_2 . Specifically, we determine all possible equivariant cobordism classes of G -actions (M, Φ) , $\Phi = (T_1, T_2)$, for which F_Φ is connected, $\dim(F_\Phi) = n$ and $\dim(M) = 4n - 1$ or $4n - 2$; here, F_Φ is the fixed point set of Φ , that is, $F_\Phi = \{x \in M \mid T_i(x) = x, i = 1, 2\}$. We remark that, for $\dim(M) \geq 4n$, this type of classification was established in [3]. To state the results, we need to describe certain G -actions (M, Φ) with F_Φ connected and n -dimensional, and with $\dim(M) = 4n - 1$ and $4n - 2$. Set $T_3 = T_1 T_2$;

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given a G -action (M, Φ) , the *fixed data* of Φ is $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$, a closed manifold (the fixed set F_Φ) and a list of three vector bundles, $\varepsilon_1, \varepsilon_2$ and ε_3 , where ε_i is the normal bundle of F_Φ in F_{T_i} , $i = 1, 2, 3$. If $(F; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ is the fixed data of some G -action, then the same is true for $(F; \varepsilon_i, \varepsilon_j, \varepsilon_k)$, where (i, j, k) is any permutation of $(1, 2, 3)$; this is realized by new G -actions obtained from permutations of (T_1, T_2, T_3) . Denote by $R \rightarrow X$ the trivial one-dimensional vector bundle over any base space X and by $\tau(F) \rightarrow F$ the tangent bundle of F , where F is a manifold. In order to obtain the above-mentioned G -actions, the crucial point is the following *section theorem* of [4]: if a G -action (M^m, Φ) has fixed data $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ and ε_1 is isomorphic to $\varepsilon'_1 \oplus R$, then there is a G -action (N^{n-1}, Ψ) having $(F_\Phi; \varepsilon'_1, \varepsilon_2, \varepsilon_3)$ as fixed data. This will be combined with the following construction: from an involution (W, T) , we can form the G -action $(W \times W, \Phi)$, $\Phi = (T_1, T_2)$, where $T_1(x, y) = (T(x), T(y))$ and $T_2(x, y) = (y, x)$; we denote this G -action by $\Gamma(W, T)$. The fixed data of $\Gamma(W, T)$ is $(F_T; \tau(F_T), \eta, \eta)$, where $\eta \rightarrow F_T$ is the normal bundle of F_T in W .

A) G -actions (M^m, Φ) with $m = 4n - 1$: let F^n be a connected and closed n -dimensional manifold such that $\tau(F^n)$ is cobordant, as a bundle, to a vector bundle $\mu^n \rightarrow F^n$ that has a section, that is, there exists an $(n - 1)$ -dimensional vector bundle $\nu^{n-1} \rightarrow F^n$ so that $\nu^{n-1} \oplus R$ is equivalent to μ^n . From [2], one knows that there is an involution (W, T) , equivariantly cobordant to the twist involution on $F^n \times F^n$, with fixed set F^n and with μ^n being the normal bundle of F^n in W . The fixed data of $\Gamma(W, T)$ then is $(F^n; \tau(F^n), \mu^n, \mu^n)$, and from [4] one can remove one section; that is, there exists a G -action (N^{4n-1}, Ψ) with fixed data $(F^n; \tau(F^n), \mu^n, \nu^{n-1})$. Evidently, this includes the case in which $\tau(F^n)$ has itself a section; for example, when F^n is any closed manifold with n odd. In this case, these actions are obtained by removing one section from the fixed data of the action $(F^n \times F^n \times F^n \times F^n, T_1, T_2)$, $T_1(x, y, z, w) = (y, x, w, z)$, $T_2(x, y, z, w) = (z, w, x, y)$.

B) G -actions (M^m, Φ) with $m = 4n - 2$: we describe two types of such actions. First, take F^n as in A), and suppose that also ν^{n-1} has a section, $\nu^{n-1} = \theta^{n-2} \oplus R$. By removing this additional section, one obtains a G -action (Q^{4n-2}, φ) with fixed data $(F^n; \tau(F^n), \mu^n, \theta^{n-2})$. Second, one takes the G -actions $\Gamma(W, T)$, where (W, T) is any involution with $\dim(W) = 2n - 1$ and with F_T connected and n -dimensional, that is, equivariantly cobordant to the ones described by Stong. This includes the case where, taking F^n as in A), we remove two sections to get a G -action with fixed data $(F^n; \tau(F^n), \nu^{n-1}, \nu^{n-1})$.

The desired classification is given by the following.

Theorem. *Let (M^m, Φ) be a Z_2^2 -action whose fixed point set F_Φ is connected and n -dimensional, and where either $m = 4n - 1$ or $m = 4n - 2$. Then, either (M^m, Φ) bounds equivariantly, or it is, up to permutations, equivariantly cobordant to an action of type A) if $m = 4n - 1$ and of type B) if $m = 4n - 2$.*

Note. For involutions, the connectedness of the fixed point set is redundant, since any involution is equivariantly cobordant to an involution with the property that the p -dimensional part of the fixed set is connected. However, this is not true for Z_2^2 -actions, since in this case the fixed data over different components of the p -dimensional part of the fixed set may have different lists of dimensions.

2. PROOFS OF THE RESULTS

In order to prove the stated result, we need some preliminaries about G -actions. Let (M, Φ) , $\Phi = (T_1, T_2)$, be a G -action with fixed data $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$. Each

s -dimensional component of $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ can be considered as an element of $\mathcal{N}_s(BO(n_1) \times BO(n_2) \times BO(n_3))$, the bordism of s -dimensional manifolds with a map into $BO(n_1) \times BO(n_2) \times BO(n_3)$, where n_i is the dimension of ε_i over the component and $BO(n_i)$ is the classifying space for n_i -dimensional vector bundles (this is the *simultaneous cobordism* between lists of bundles). According to [6], two G -actions are equivariantly cobordant if and only if they have fixed data simultaneously cobordant. Also, if (M, Φ) has fixed data $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ is simultaneously cobordant to $(F; \mu_1, \mu_2, \mu_3)$, then there exists a G -action (N, Ψ) with fixed data $(F; \mu_1, \mu_2, \mu_3)$, hence equivariantly cobordant to (M, Φ) . The following result is the lemma found in [3, Section 3; page 108], particularized to Z_2^2 -actions.

Lemma 2.1. *Let (M, Φ) be a G -action with fixed data $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$, where F_Φ is connected and n -dimensional. Then,*

- a) *if $\dim(\varepsilon_1) > n$, $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ bounds simultaneously;*
- b) *if $\dim(\varepsilon_1) = n$, $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ is simultaneously cobordant to $(F_\Phi; \tau(F_\Phi), \varepsilon_2, \varepsilon_3)$.*

Now consider an n -dimensional vector bundle $\varepsilon \rightarrow F$, where F is a connected, closed and n -dimensional manifold, and let $RP(\varepsilon) \rightarrow F$ be the real projective space bundle associated to ε . Denote by $\lambda \rightarrow RP(\varepsilon)$ the line bundle of the double cover $S(\varepsilon) \rightarrow RP(\varepsilon)$, $S(\varepsilon)$ the sphere bundle of ε . Suppose that μ and ν are two additional vector bundles over F , with $\dim(\mu) = p$, $\dim(\nu) = q$ and $p \geq q$. The crucial point of the argument to be used is the following.

Lemma 2.2. *Suppose that the list $(RP(\varepsilon); \lambda, \mu \oplus (\nu \otimes \lambda))$ bounds as an element of $\mathcal{N}_{2n-1}(BO(1) \times BO(p+q))$. Then the list $(F; \varepsilon, \mu, \nu)$ is simultaneously cobordant to the list $(F; \varepsilon, \nu \oplus (p-q)R, \nu)$. Here, $(p-q)R$ means the Whitney sum of $p-q$ copies of R , and the bundles μ and ν are considered over $RP(\varepsilon)$ via pullbacks by the projection.*

Proof. One lets $W(F) = 1 + w_1 + \dots + w_n$, $W(\varepsilon) = 1 + v_1 + \dots + v_n$, $W(\mu) = 1 + u_1 + \dots + u_p$, $W(\nu) = 1 + \theta_1 + \dots + \theta_q$ and $W(\lambda) = 1 + c$ be the Stiefel-Whitney classes of F , ε , μ , ν and λ . One knows that the Stiefel-Whitney class of $RP(\varepsilon)$ is $W(RP(\varepsilon)) = (1 + w_1 + \dots + w_n) \cdot \{(1 + c)^n + v_1(1 + c)^{n-1} + \dots + v_{n-1}(1 + c) + v_n\}$ and the Stiefel-Whitney class of the bundle $\mu \oplus (\nu \otimes \lambda)$ is

$$W(\mu \oplus (\nu \otimes \lambda)) = (1 + u_1 + \dots + u_p) \cdot \{(1 + c)^q + \theta_1(1 + c)^{q-1} + \dots + \theta_{q-1}(1 + c) + \theta_q\}.$$

Because the list $(RP(\varepsilon); \lambda, \mu \oplus (\nu \otimes \lambda))$ is a simultaneous boundary, any class of dimension $2n - 1$ given by a product of classes $w_i(RP(\varepsilon))$, c , $w_j(\mu \oplus (\nu \otimes \lambda))$, evaluated on the fundamental homology class $[RP(\varepsilon)]$, gives a zero characteristic number.

For any r , one lets

$$W[r] = \frac{W(RP(\varepsilon))}{(1 + c)^{n-r}} \quad \text{and} \quad V[r] = \frac{W(\mu \oplus (\nu \otimes \lambda))}{(1 + c)^{q-r}}.$$

That is,

$$W[r] = (1 + w_1 + \dots + w_n) \{ (1 + c)^r + v_1(1 + c)^{r-1} + \dots + v_r + \frac{v_{r+1}}{1 + c} + \dots + \frac{v_n}{(1 + c)^{n-r}} \}$$

and

$$V[r] = (1 + u_1 + \dots + u_p) \left\{ (1+c)^r + \theta_1(1+c)^{r-1} + \dots + \theta_r + \frac{\theta_{r+1}}{1+c} + \dots + \frac{\theta_q}{(1+c)^{q-r}} \right\}.$$

The classes $W[r]_t$ and $V[r]_l$ are polynomials in $w_i(RP(\varepsilon))$, c , $w_j(\mu \oplus (\nu \otimes \lambda))$; hence they can be used to give characteristic numbers. Also, for these classes, one has the following special properties (see [5]):

$$\begin{aligned} W[r]_{2r} &= w_r c^r + \text{terms with smaller } c \text{ powers,} \\ W[r]_{2r+1} &= (w_{r+1} + v_{r+1})c^r + \text{terms with smaller } c \text{ powers,} \\ W[r]_{2r+2} &= v_{r+1}c^{r+1} + \text{terms with smaller } c \text{ powers,} \end{aligned}$$

and in the same way,

$$\begin{aligned} V[r]_{2r} &= u_r c^r + \text{terms with smaller } c \text{ powers,} \\ V[r]_{2r+1} &= (u_{r+1} + \theta_{r+1})c^r + \text{terms with smaller } c \text{ powers,} \\ V[r]_{2r+2} &= \theta_{r+1}c^{r+1} + \text{terms with smaller } c \text{ powers.} \end{aligned}$$

For a sequence $\omega = (i_1, \dots, i_s)$ of natural numbers, one lets $|\omega| = i_1 + \dots + i_s$, and for $w = 1 + w_1 + \dots + w_p$, one lets $w_\omega = w_{i_1} \dots w_{i_s}$ be the product of the classes w_i . Then given sequences $\omega = (i_1, \dots, i_s)$, $\omega' = (j_1, \dots, j_t)$, $\beta = (a_1, \dots, a_k)$ and $\beta' = (b_1, \dots, b_l)$, and a natural number $1 \leq r \leq p$ with $|\omega| + |\omega'| + |\beta| + |\beta'| + r = n$, one may form the class

$$X = \left(\prod_{i \in \omega} W[i]_{2i} \right) \cdot \left(\prod_{i \in \omega'} W[i-1]_{2i} \right) \cdot \left(\prod_{i \in \beta} V[i]_{2i} \right) \cdot \left(\prod_{i \in \beta'} V[i-1]_{2i} \right) \cdot V[r-1]_{2r-1}.$$

Since X has dimension $2n-1$, it gives the zero characteristic number $X[RP(\varepsilon)]$. From the properties listed above, one has

$$\begin{aligned} \prod_{i \in \omega} W[i]_{2i} &= W(F)_\omega \cdot c^{|\omega|} + \text{terms with smaller } c \text{ powers,} \\ \prod_{i \in \omega'} W[i-1]_{2i} &= W(\varepsilon)_{\omega'} \cdot c^{|\omega'|} + \text{terms with smaller } c \text{ powers,} \\ \prod_{i \in \beta} V[i]_{2i} &= W(\mu)_\beta \cdot c^{|\beta|} + \text{terms with smaller } c \text{ powers,} \\ \prod_{i \in \beta'} V[i-1]_{2i} &= W(\nu)_{\beta'} \cdot c^{|\beta'|} + \text{terms with smaller } c \text{ powers and} \\ V[r-1]_{2r-1} &= (u_r + \theta_r) \cdot c^r + \text{terms with smaller } c \text{ powers.} \end{aligned}$$

It follows that X has the form

$$X = W(F)_\omega \cdot W(\varepsilon)_{\omega'} \cdot W(\mu)_\beta \cdot W(\nu)_{\beta'} \cdot (u_r + \theta_r) \cdot c^{n-1} + \text{terms with smaller } c \text{ powers.}$$

Now if a term of dimension $2n-1$ involves a power of c less than $n-1$, it necessarily has a factor of dimension greater than n coming from the cohomology of F , which is zero. Also one knows that $H^*(RP(\varepsilon); Z_2)$ is the free $H^*(F; Z_2)$ module on $1, c, c^2, \dots, c^{n-1}$. Therefore

$$\begin{aligned} 0 = X[RP(\varepsilon)] &= W(F)_\omega \cdot W(\varepsilon)_{\omega'} \cdot W(\mu)_\beta \cdot W(\nu)_{\beta'} \cdot (u_r + \theta_r) \cdot c^{n-1} [RP(\varepsilon)] \\ &= W(F)_\omega \cdot W(\varepsilon)_{\omega'} \cdot W(\mu)_\beta \cdot W(\nu)_{\beta'} \cdot (u_r + \theta_r) [F] \end{aligned}$$

and thus

$$W(F)_\omega \cdot W(\varepsilon)_{\omega'} \cdot W(\mu)_\beta \cdot W(\nu)_{\beta'} \cdot u_r [F] = W(F)_\omega \cdot W(\varepsilon)_{\omega'} \cdot W(\mu)_\beta \cdot W(\nu)_{\beta'} \cdot \theta_r [F].$$

This says that any class u_r in a characteristic number of $(F; \varepsilon, \mu, \nu)$ can be replaced by θ_r without changing the value of the characteristic number. In particular, if $q < r \leq p$, the class u_r can be replaced by the zero class, and thus $(F; \varepsilon, \mu, \nu)$ and $(F; \varepsilon, \nu \oplus (p-q)R, \nu)$ have the same characteristic numbers, which gives the result. $\square$

Lemma 2.3. *Let (M, Φ) be a G -action with fixed data $(F; \varepsilon, \mu, \nu)$, where F is connected, $\dim(F) = \dim(\varepsilon) = n$, $\dim(\mu) = p$ and $\dim(\nu) = q$, with $p \geq q$. Then (M, Φ) is equivariantly cobordant to an action obtained by removing $p - q$ sections from the third bundle of the fixed data of the action $\Gamma(W, T)$, where (W, T) is an involution fixing F and with the normal bundle of F in W being $\nu \oplus (p - q)R$ (if $p = q$, (M, Φ) is equivariantly cobordant to the action $\Gamma(W, T)$, where (W, T) is an involution with fixed set and normal bundle $\nu \rightarrow F$).*

Proof. From the argument outlined in [3, Section 2; pages 107 and 108], particularized to Z_2^2 -actions, one has that the list $(RP(\varepsilon); \lambda, \mu \oplus (\nu \otimes \lambda))$ bounds as an element of $\mathcal{N}_{2n-1}(BO(1) \times BO(p+q))$. First using Lemma 2.2 for $(F; \varepsilon, \mu, \nu)$, and next using part b) of Lemma 2.1 for $(F; \varepsilon, \nu \oplus (p - q)R, \nu)$, one concludes that (M, Φ) is equivariantly cobordant to a G -action (N, Ψ) , $\Psi = (T_1, T_2)$, with fixed data $(F; \tau(F), \nu \oplus (p - q)R, \nu)$. Let $W \subset N$ be the component of F_{T_2} that contains F . Then the involution (W, T_1) fixes F with normal bundle $\nu \oplus (p - q)R \rightarrow F$, and thus (N, Ψ) (hence (M, Φ)) is equivariantly cobordant to a G -action obtained by removing $p - q$ sections from the third bundle of the fixed data of $\Gamma(W, T_1)$. $\square$

Now we prove the result. Let $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ denote the fixed data of the G -action (M^m, Φ) . If some ε_i has dimension greater than n , then part a) of Lemma 2.1, used up to permutation, says that $(F_\Phi; \varepsilon_1, \varepsilon_2, \varepsilon_3)$ bounds simultaneously, and thus (M^m, Φ) bounds equivariantly. Therefore we can suppose that $\dim(\varepsilon_i) \leq n$ for $i = 1, 2, 3$. By making permutations if necessary, we can suppose that $(\dim(\varepsilon_1), \dim(\varepsilon_2), \dim(\varepsilon_3))$ then is $(n, n, n - 1)$ if $m = 4n - 1$, and either $(n, n, n - 2)$ or $(n, n - 1, n - 1)$ if $m = 4n - 2$. In the case $(n, n, n - 1)$ (resp. $(n, n, n - 2)$), Lemma 2.3 says that (M^m, Φ) is equivariantly cobordant to an action obtained by removing one section (resp. two sections) from the third bundle of the fixed data of the action $\Gamma(W, T)$, where (W, T) is an involution with fixed set and normal bundle $\varepsilon_3 \oplus R \rightarrow F_\Phi$ ($\varepsilon_3 \oplus 2R \rightarrow F_\Phi$). Since $\dim(F_\Phi) = n$ and $\dim(W) = 2n$, one has from [1] that (W, T) is equivariantly cobordant to $(F_\Phi \times F_\Phi, \text{twist})$, and thus $\varepsilon_3 \oplus R \rightarrow F_\Phi$ (resp. $\varepsilon_3 \oplus 2R \rightarrow F_\Phi$) is cobordant to $\tau(F_\Phi)$. This means that (M^m, Φ) is equivariantly cobordant to an action of type A (resp. of type B).

In the case $(n, n - 1, n - 1)$, Lemma 2.3 says that (M^m, Φ) is equivariantly cobordant to the action $\Gamma(W, T)$, where (W, T) is an involution with fixed set and normal bundle $\varepsilon_3 \rightarrow F_\Phi$. Since $\dim(F_\Phi) = n$ and $\dim(W) = 2n - 1$, (W, T) is of the Stong type, and thus (M^m, Φ) is equivariantly cobordant to an action of type B. This ends the proof.

REFERENCES

- [1] C. Kosniowski and R. E. Stong, *Involutions and characteristic numbers*, Topology 17, (1978), 309-330. MR516213 (82a:57036)
- [2] P. E. Conner and E. E. Floyd, *Differentiable Periodic Maps*, Springer-Verlag, Berlin, (1964). MR0176478 (31:750)
- [3] P. L. Q. Pergher, *On Z_2^k actions*, Topology Appl. 117, (2002), 105-112. MR1874087 (2002j:57063)
- [4] P. L. Q. Pergher, *$(Z_2)^k$ -actions whose fixed data has a section*, Trans. Amer. Math. Soc. 353, (2001), 175-189. MR1783791 (2001j:57044)
- [5] P. L. Q. Pergher and R. E. Stong, *Involutions fixing (point) $\cup F^n$* , Transformation Groups 6, (2001), 79-86. MR1825169 (2002a:57054)

- [6] R. E. Stong, *Equivariant bordism and $(Z_2)^k$ -actions*, Duke Math. J. 37, (1970), 779-785. MR0271966 (42:6847)
- [7] R. E. Stong, *Involutions with n -dimensional fixed set*, Math. Z. 178, (1981), 443-447. MR638810 (83a:57048)

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