ADMISSIBLE METRICS IN THE σ_k -YAMABE EQUATION

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ABSTRACT. In most previous works on the existence of solutions to the σ_k -Yamabe problem, one assumes that the initial metric g_0 is k -admissible. This is a pointwise condition. In this paper we prove that this condition can be replaced by a weaker integral condition.

INTRODUCTION

Let (M, g_0) be a compact Riemannian manifold of dimension $n \geq 3$ and $[g_0]$ the set of metrics conformal to g_0 . Denote

$$A_g^t = \frac{1}{n-2} \left(Ric_g - \frac{t}{2(n-1)} R_g \cdot g \right),$$

where Ric_g and R_g are the Ricci tensor and scalar curvature of g , respectively, $t \in \mathcal{R}$. When $t = 1$, A_g^t is the Schouten tensor [2]; when $t = n - 1$, it is the Einstein tensor [23]. Denote $\lambda(A_g^t) = (\lambda_1, \dots, \lambda_n)$ as the eigenvalues of A_g^t with respect to g . As usual denote

$$\sigma_k(\lambda_1, \dots, \lambda_n) = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

as the k -th elementary symmetric functions and

$$\Gamma_k = \{\lambda = (\lambda_1, \dots, \lambda_n) \in R^n \mid \sigma_j(\lambda) > 0 \text{ for } j = 1, \dots, k\}$$

as the corresponding open convex cone in R^n . A conformal metric $g \in [g_0]$ is called k -admissible if $\lambda(A_g^1) \in \Gamma_k$. Denote

$$[g_0]_k = \{g \in [g_0] \mid \lambda(A_g^1) \in \Gamma_k\}$$

as the set of all k -admissible metrics. The σ_k -Yamabe problem concerns the existence of a conformal metric $g \in [g_0]$ such that

$$\sigma_k(\lambda(A_g)) = 1.$$

The σ_k -Yamabe problem has been studied by many authors; see [1], [20], [24] for $k = 1$ and [27], [5], [6], [8], [12], [3], [18], [15], [21], [26] for $k \geq 2$. In most works the existence results were obtained under the assumption $[g_0]_k \neq \emptyset$, or equivalently, the initial metric g_0 is k -admissible.

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The assumption $[g_0]_k \neq \emptyset$ is a pointwise condition. An interesting problem is whether it can be replaced by integral conditions, namely by assumptions on conformal invariants. For the Yamabe problem ($k = 1$), it is known that if the Yamabe constant Y_1 is positive, then there exists a conformal metric such that the scalar curvature is a positive constant. In the case $n = 4$ and $k = 2$, this problem was investigated in [5]. The authors proved that $[g_0]_k \neq \emptyset$ if the Yamabe constants Y_1 and Y_2 are positive. Here the Yamabe constants Y_k are defined as

$$Y_k([g_0]) = \inf_{g \in [g_0]_k} F_k(g),$$

where

$$F_k(g) = \text{vol}(g)^{-\frac{n-2k}{n}} \int_M \sigma_k(g) \, d\text{vol}_g.$$

This result (again for the case $k = 2$, $n = 4$) was reproved in [14] by a different method.

For locally conformally flat manifolds, the above result was extended to general $2 \leq k \leq \frac{n}{2}$ in [10]. They defined the quantity

$$\widehat{Y}_k([g_0]) = \begin{cases} \inf_{g \in [g_0]_{k-1}} F_k(g), & \text{if } [g_0]_{k-1} \neq \emptyset, \\ -\infty, & \text{if } [g_0]_{k-1} = \emptyset, \end{cases}$$

and proved that $[g_0]_k \neq \emptyset$ if $\widehat{Y}_k([g_0]) > 0$ for $2 \leq k \leq \frac{n}{2}$. We remark that when $k = 2$, $n = 4$, $\widehat{Y}_k([g_0]) > 0$ is equivalent to $Y_1, Y_2 > 0$.

In this paper, we extend the result to its full generality. We prove that for any compact manifolds and any $2 \leq k \leq n$, $\widehat{Y}_k([g_0]) > 0$ implies that $[g_0]_k \neq \emptyset$. Note that once $[g_0]_k \neq \emptyset$, the existence results for the σ_k -Yamabe problem in the above mentioned papers applies.

Theorem 0.1. *Let (M, g_0) be a closed n -dimensional Riemannian manifold, $n \geq 3$. Suppose $\widehat{Y}_k([g_0]) > 0$ for some $k \geq 2$. Then there exists a conformal metric $g = e^{-2u}g_0$ with $R_g > 0$ and $\sigma_k(A_g) > 0$, and the Ricci tensor of g satisfies*

$$-\left(\frac{n-4}{2n}\right)Rg < Ric_g < Rg.$$

For proof of the theorem, we consider a family of equations involving a parameter t ,

$$(1) \quad \sigma_k^{1/k}(g_0^{-1}A_g^t) = f(x)e^{2u},$$

where $g = e^{-2u}g_0$, $f(x) > 0$ and $t \leq 1$. It is easy to compute that ([13, 19])

$$A_g^t = A_{g_0}^t + \nabla^2 u + \frac{1-t}{n-2}(\Delta u)g_0 + du \otimes du - \frac{2-t}{2}|\nabla u|^2 g_0.$$

Hence equation (1) is equivalent to

$$(2) \quad \begin{aligned} & \sigma_k^{1/k} \left(g_0^{-1} \left(A_{g_0}^t + \nabla^2 u + \frac{1-t}{n-2}(\Delta u)g_0 + du \otimes du - \frac{2-t}{2}|\nabla u|^2 g_0 \right) \right) \\ & = f(x)e^{2u}. \end{aligned}$$

The idea of introducing the family of equations (2) was used in [14]. This is a very useful technique, since equation (2) is uniformly elliptic when $t < 1$ [16, 17], and has been used in some other situations [25, 22].

It is easy to show that the linearized operator of this equation is invertible, since the exponent on the right hand side of (2) is a positive multiple of u . By choosing an appropriate function $f(x)$, we will establish the a priori estimates for solutions of (1). By the gradient and second derivative estimates in recent works [7, 28], it suffices to establish a uniform lower bound for the solutions. Our estimate here is different from those in [10]. The existence of solutions of (1) for $t = 1$ then follows by the continuity method.

1. PROOF OF THEOREM 0.1

By definition, if $\widehat{Y}_k([g_0]) > 0$, then $[g_0]_{k-1} \neq \emptyset$. For simplicity let us assume that $g_0 \in [g_0]_{k-1}$, i.e. $g_0^{-1}A_{g_0} \in \Gamma_{k-1}$, so that the scalar curvature $R_{g_0} = 2(n-1)\sigma_1(g_0^{-1}A_{g_0}) > 0$. Denote $g = e^{-2u}g_0$ and A_g^t as above. We first show that the linearized operator corresponding to equation (1) is invertible.

Lemma 1.1. *Let $u \in C^2(M)$ be a solution of (1) for some $t \leq 1$ with $A_g^t \in \Gamma_k$. Then the linearized operator at u , $L_t : C^{2,\alpha}(M) \rightarrow C^\alpha(M)$, is invertible ($0 < \alpha < 1$).*

Proof. Denote

$$F_t(x, u, \nabla u, \nabla^2 u) = \sigma_k(g_0^{-1}A_u^t) - f(x)^k e^{2ku}.$$

Suppose that $u \in C^{2,\alpha}(M)$ satisfies $F_t(x, u, \nabla u, \nabla^2 u) = 0$ and $A_u^t \in \Gamma_k$, where $A_u^t = A_g^t$ and $g = e^{-2u}g_0$. Let $u_\tau = u + \tau\phi$; then

$$\begin{aligned} L_t(\phi) &= \frac{d}{d\tau} F_t(x, u_\tau, \nabla u_\tau, \nabla^2 u_\tau) |_{\tau=0} \\ &= [T_{k-1}(g_0^{-1}A_u^t)]_{ij} \frac{d}{d\tau} (g_0^{-1}A_{u_\tau}^t)_{ij} |_{\tau=0} - 2k\phi f(x)^k e^{2ku} \\ &= [T_{k-1}(g_0^{-1}A_u^t)]_{ij} \left\{ g_0^{-1}(\nabla^2 \phi + \frac{1-t}{n-2}(\Delta \phi)g_0) \right. \\ &\quad \left. + 2du \otimes d\phi - (2-t)\langle \nabla u, \nabla \phi \rangle g_0 \right\}_{ij} - 2k\phi f(x)^k e^{2ku} \end{aligned}$$

where $[T_{k-1}(g_0^{-1}A_u^t)]_{ij} = \partial F_t / \partial r_{ij}$ is the $(k-1)$ -th Newton transformation associated with $g_0^{-1}A_u^t$ (see [19] for instance). Since $A_u^t \in \Gamma_k$, $[T_{k-1}(g_0^{-1}A_u^t)]_{ij}$ is positive definite, so L_t is uniformly elliptic. Note that the coefficient of ϕ in the zeroth-order term is strictly negative. Hence L_t is invertible in the Hölder space $C^{2,\alpha}(M)$ (see [9]). $\square$

Since $R_{g_0} > 0$, there exists $a > -\infty$ so that $A_{g_0}^a$ is positive definite. For $t \in [a, 1]$, we consider the deformation equations

$$(3) \quad \sigma_k^{1/k}(g_0^{-1}A_{u_t}^t) = f(x) e^{2u_t},$$

where $f(x) = \sigma_k^{1/k}(g_0^{-1}A_{u_a}^a) > 0$ and $u \equiv 0$ is a solution of (3) for $t = a$.

Lemma 1.2. *Let $u_t \in C^2(M)$ be a solution of (3) for some $a \leq t \leq 1$. Then $u_t \leq C$, where C depends only on g_0 .*

Proof. By Newton's inequality $\sigma_k^{1/k}(g_0^{-1}A_{u_t}^t) \leq C_1 \sigma_1(g_0^{-1}A_{u_t}^t)$, where C_1 is a positive constant depending only on n and k , we have from (3)

$$f(x) e^{2u_t} \leq C_1 \sigma_1(g_0^{-1}A_{u_t}^t).$$

Let x_0 be a maximum point of u_t ; then $\nabla u_t(x_0) = 0$ and $\nabla^2 u_t(x_0) \leq 0$. It follows that

$$f(x_0) e^{2u_t(x_0)} \leq C_1 \sigma_1(g_0^{-1} A_{g_0}^t).$$

Hence $u_t \leq C$ for $t \geq a$. $\square$

Next we need the C^1 and C^2 estimates for solutions of (3). When $t = 1$, the a priori estimates were first established in [11]. For general $t \leq 1$, they were proved in [28] and [7] by different methods.

Lemma 1.3. *Let $u_t(x)$ be a C^4 solution of (3) for $a \leq t \leq 1$ with $u_t \leq C$. Then there exists a positive constant $C_2 = C_2(n, k, \max_M u_t, a, \|g_0\|_{C^3})$ but independent of $\min_M u_t$, such that*

$$|\nabla^2 u_t| + |\nabla u|^2 \leq C_2.$$

From Lemma 1.3, we only need to give a lower bound for the solution u_t to equation (3).

Lemma 1.4. *Assume that for some $a \leq t \leq 1$,*

$$(4) \quad \hat{Y}_k([g_0]) + \frac{n!}{(n-k)!k!} \left(\frac{1-t}{n-2} \right)^k \frac{1}{2^k (n-1)^k} (Y_1[g_0])^k = \lambda_t > 0.$$

If $u_t \in C^2(M)$ is a solution of (3) satisfying $\|\nabla u_t\|_{L^\infty} \leq C_3$ and $g_t = e^{-2u_t} g_0 \in [g_0]_{k-1}$, then $u_t > C$, where C depends only on g_0, C_3 .

Proof. Since

$$A^t = A^1 + \frac{1-t}{n-2} \sigma_1(A^1) g,$$

where $\sigma_1(A^1) = \frac{1}{2(n-1)} Rg$, we have

$$\sigma_k(A^t) = \sum_{i=0}^k \binom{n-i}{n-k} \left(\frac{1-t}{n-2} \right)^{k-i} \sigma_i(A^1) (\sigma_1(A^1))^{k-i}.$$

Let $g_t = e^{-2u_t} g_0$. By (3)

$$\begin{aligned} e^{2ku_t} f(x)^k &= \sigma_k(g_0^{-1} A_{u_t}^t) \\ &= \sum_{i=0}^k \binom{n-i}{n-k} \left(\frac{1-t}{n-2} \right)^{k-i} \sigma_i(g_0^{-1} A_{u_t}^1) (\sigma_1(g_0^{-1} A_{u_t}^1))^{k-i} \\ (5) \quad &= e^{-2ku_t} \sum_{i=0}^k \binom{n-i}{n-k} \left(\frac{1-t}{n-2} \right)^{k-i} \sigma_i(g_t^{-1} A_{u_t}^1) (\sigma_1(g_t^{-1} A_{u_t}^1))^{k-i}. \end{aligned}$$

Since $g_t^{-1} A_{u_t}^1 \in \Gamma_{k-1}$ and $a \leq t \leq 1$, we have

$$\begin{aligned} e^{(4k-n)u_t} f(x)^k &\geq e^{-nu_t} \left[\sigma_k(g_t^{-1} A_{u_t}^1) + \frac{n!}{(n-k)!k!} \left(\frac{1-t}{n-2} \right)^k (\sigma_1(g_t^{-1} A_{u_t}^1))^k \right] \\ &= e^{-nu_t} \left[\sigma_k(g_t^{-1} A_{u_t}^1) + \frac{n!}{(n-k)!k!} \left(\frac{1-t}{n-2} \right)^k \frac{1}{2^k (n-1)^k} (R_{g_t})^k \right]. \end{aligned}$$

Integrating the inequality, we get

$$\begin{aligned} C_4 \int_M e^{(4k-n)u_t} d\text{vol}_{g_0} &\geq \int_M e^{(4k-n)u_t} f(x)^k d\text{vol}_{g_0} \\ &\geq \int_M \sigma_k(g_t^{-1} A_{u_t}^1) d\text{vol}_{g_t} \\ &\quad + \frac{n!}{(n-k)!k!} \left(\frac{1-t}{n-2} \right)^k \frac{1}{2^k (n-1)^k} \int_M (R_{g_t})^k d\text{vol}_{g_t}, \end{aligned}$$

where $C_4 > 0$ is chosen so that $f(x)^k \leq C_4$. In the above inequality,

$$\begin{aligned} &\{\text{vol}(g_t)\}^{-\frac{n-2k}{n}} \int_M \sigma_k(g_t^{-1} A_{g_t}^1) d\text{vol}_{g_t} \\ &\geq \inf_{g \in [g_0]_{k-1}} \text{vol}(g)^{-\frac{n-2k}{n}} \int_M \sigma_k(g) d\text{Vol}(g) = \widehat{Y}_k([g_0]). \end{aligned}$$

From Hölder's inequality,

$$\int_M R_g d\text{vol}_g \leq \left\{ \int_M (R_g)^k d\text{vol}_g \right\}^{1/k} \{\text{vol}(g)\}^{(k-1)/k},$$

we obtain

$$\begin{aligned} Y_1[g_0] &\leq \{\text{vol}(g_t)\}^{-(n-2)/n} \int_M R_{g_t} d\text{vol}_{g_t} \\ &= \{\text{vol}(g_t)\}^{-\frac{1}{k} + \frac{2}{n}} \{\text{vol}(g_t)\}^{-\frac{k-1}{k}} \int_M R_{g_t} d\text{vol}_{g_t} \\ &\leq \{\text{vol}(g_t)\}^{-\frac{1}{k} + \frac{2}{n}} \left\{ \int_M (R_g)^k d\text{vol}_g \right\}^{1/k}. \end{aligned}$$

It follows that

$$\{\text{vol}(g_t)\}^{-1 + \frac{2k}{n}} \int_M (R_g)^k d\text{vol}_g \geq (Y_1[g_0])^k.$$

Therefore we obtain

$$\begin{aligned} &\{\text{vol}(g_t)\}^{-\frac{n-2k}{n}} C_4 \int_M e^{(4k-n)u_t} d\text{vol}_{g_0} \\ &\geq \widehat{Y}_k([g_0]) + \frac{n!}{(n-k)!k!} \left(\frac{1-t}{n-2} \right)^k \frac{1}{2^k (n-1)^k} (Y_1[g_0])^k. \end{aligned}$$

We compute

$$\begin{aligned} &\{\text{vol}(g_t)\}^{-\frac{n-2k}{n}} C_4 \int_M e^{(4k-n)u_t} d\text{vol}_{g_0} \\ &= C_4 \{\text{vol}(g_t)\}^{-\frac{n-2k}{n}} \int_M e^{4ku_t} d\text{vol}_{g_t} \\ &\leq C_4 e^{4k \max_M u_t} \{\text{vol}(g_t)\}^{2k/n} \\ &= C_4 e^{4k \max_M u_t} \left\{ \int_M e^{-nu_t} d\text{vol}_{g_0} \right\}^{2k/n} \\ &\leq C_4 e^{4k \max_M u_t} \{e^{-n \min_M u_t} \text{vol}(g_0)\}^{2k/n} \\ &\leq C_4 \exp \left\{ 2k \max_M u_t + 2k \left(\max_M u_t - \min_M u_t \right) \right\} (\text{vol}(g_0))^{2k/n}. \end{aligned}$$

By (4)

$$\begin{aligned} & C_4 \exp \left\{ 2k \max_M u_t + 2k \left(\max_M u_t - \min_M u_t \right) \right\} (\text{vol}(g_0))^{2k/n} \\ & \geq \widehat{Y}_k([g_0]) + \frac{n!}{(n-k)!k!} \left(\frac{1-t}{n-2} \right)^k \frac{1}{2^k (n-1)^k} (Y_1[g_0])^k \\ & \geq \lambda_t > 0. \end{aligned}$$

We obtain

$$\max_M u_t + \left(\max_M u_t - \min_M u_t \right) \geq \frac{1}{2k} \log \lambda_t - C(g_0).$$

Note that the estimate $|\nabla u_t| < C_3$ implies the Harnack inequality

$$\max_M u_t - \min_M u_t \leq C(C_3, g_0).$$

Hence

$$\max_M u_t \geq \frac{1}{2k} \log \lambda_t - C(g_0) - C(C_3, g_0).$$

Substituting this estimate into the Harnack inequality, we get

$$\min_M u_t \geq \frac{1}{2k} \log \lambda_t - C. \quad \square$$

Now we use the continuity method to prove

Theorem 1.5. *Let (M, g_0) be a closed n -dimensional Riemannian manifold with positive scalar curvature, $n \geq 3$. If*

$$\widehat{Y}_k([g_0]) + \frac{n!}{(n-k)!k!} \left(\frac{1-t_0}{n-2} \right)^k \frac{1}{2^k (n-1)^k} (Y_1[g_0])^k = \lambda_{t_0} > 0$$

for some $t_0 \leq 1$, then there exists a conformal metric $g = e^{-2u} g_0$ with $R_g > 0$ and $\sigma_k(A_g^{t_0}) > 0$, where $k \geq 2$. This implies the pointwise inequalities

$$- \left(\frac{n-4}{2n} + \frac{1-t_0}{2} \right) Rg < Ric_g < \frac{2-t_0}{2} Rg.$$

Proof. For any fixed t_0 with $a \leq t_0 \leq 1$, we consider the 1-parameter family of equations (3), for $t \in [a, t_0]$, with $f(x) = \sigma_k^{1/k} \left(g_0^{-1} A_{g(a)}^a \right) > 0$, and a is chosen so that $A_{g_a}^a$ is positive definite. We define

$$I = \{ t \in [a, t_0] \mid \exists \text{ a solution } u_t \in C^{2,\alpha}(M) \text{ of (3) with } A_{g_t}^t \in \Gamma_k \}.$$

Since $A_{g_a}^a$ is positive definite, I is nonempty. For any given $t \in I$, let u_t be a solution. By Lemma 1.1, the linear operator at u_t , $L_t : C^{2,\alpha}(M) \rightarrow C^\alpha(M)$, is invertible. The implicit function theorem implies that I is open. Since $f \in C^\infty(M)$, $u_t \in C^\infty(M)$. The a priori estimates in Lemmas 1.2, 1.3, and 1.4 also imply that I is closed. Therefore $I = [a, t_0]$. The metric $g = e^{-2u_{t_0}} g_0$ then satisfies $\sigma_k(A_g^{t_0}) > 0$ and $A_g^{t_0} \in \Gamma_k$ for $k \geq 2$. Therefore $R_g > 0$. The last inequalities are obtained for $k = 2$, and therefore for any $k \geq 2$. By Proposition 5.2 of [14], if $A_g^{t_0} \in \Gamma_2$, then

$$- \frac{n-2}{n} \sigma_1(A_g^{t_0}) g < A_g^{t_0} < \sigma_1(A_g^{t_0}) g.$$

Since

$$\begin{aligned} A^{t_0} &= A^1 + \frac{1-t_0}{n-2} \sigma_1(A^1) g, \\ \sigma_1(A^{t_0}) &= \frac{n-2+n(1-t_0)}{n-2} \sigma_1(A^1), \\ \sigma_1(A^1) &= \frac{1}{2(n-1)} R, \end{aligned}$$

we have

$$\begin{aligned} & - \left(\frac{n-2+n(1-t_0)}{n} + \frac{1-t_0}{n-2} \right) \sigma_1(A^1) g \\ & < A^1 < \left(\frac{n-2+(n-1)(1-t_0)}{n-2} \right) \sigma_1(A^1) g. \end{aligned}$$

Substituting $A^1 = \frac{1}{n-2} \left(Ric - \frac{1}{2(n-1)} Rg \right)$ into the above inequality, we get

$$- \left(\frac{n-4}{2n} + \frac{1-t_0}{2} \right) Rg < Ric < \frac{2-t_0}{2} Rg. \quad \square$$

Note that Theorem 0.1 follows from Theorem 1.5 by taking $t_0 = 1$.

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REFERENCES

1. T. Aubin, *Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl., **55** (1976), 269-296. MR0431287 (55:4288)
2. Arthur L. Besse, *Einstein manifolds*, Springer-Verlag, Berlin, 1987. MR867684 (88f:53087)
3. S. Brendle and J. Viaclovsky, *A variational characterization for $\sigma_{n/2}$* , Calc. Var. PDE., **20** (2004), 399-402. MR2071927 (2005c:53081)
4. L. Caffarelli, L. Nirenberg, and J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian*, Acta Math., **155** (1985), 261-301. MR806416 (87f:35098)
5. A. Chang, M. Gursky, and P. Yang, *An equation of Monge-Ampère type in conformal geometry, and four-manifolds of positive Ricci curvature*, Ann. of Math. (2), **155** (2002), 709-787. MR1923964 (2003j:53048)
6. A. Chang, M. Gursky, and P. Yang, *An a priori estimate for a fully nonlinear equation on four-manifolds*, J. Anal. Math., **87** (2002), 151-186. MR1945280 (2003k:53036)
7. S. Chen, *Local estimates for some fully nonlinear elliptic equations*, arXiv: math.AP/0510652.
8. Y. Ge and G. Wang, *On a fully nonlinear Yamabe problem*, preprint. MR2290138
9. D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer, Berlin, 1983. MR737190 (86c:35035)
10. P. Guan, C.-S. Lin, and G. Wang, *Application of the method of moving planes to conformally invariant equations*, Math. Z., **247** (2004), 1-19. MR2054518 (2005a:53059)
11. P. Guan and G. Wang, *Local estimates for a class of conformal equation arising from conformal geometry*, Int. Math. Res. Not., **2003** (2003), 1413-1432. MR1976045 (2003m:53055)
12. P. Guan and G. Wang, *A fully nonlinear conformal flow on locally conformally flat manifolds*, J. Reine Angew. Math., **557** (2003), 219-238. MR1978409 (2004e:53101)
13. M. Gursky and J. Viaclovsky, *Fully nonlinear equations on Riemannian manifolds with negative curvature*, Indiana Univ. Math. J., **52** (2003), 399-419. MR1976082 (2004a:53039)

14. M. Gursky and J. Viaclovsky, *A fully nonlinear equation on four-manifolds with positive scalar curvature*, J. Differential Geom., **63** (2003), 131-154. MR2015262 (2004h:53052)
15. M. Gursky and J. Viaclovsky, *Prescribing symmetric functions of the eigenvalues of the Ricci tensor*, Ann. of Math., to appear.
16. N.V. Krylov, *Nonlinear elliptic and parabolic equations of second order*, Reidel, 1987.
17. N.V. Krylov, *On the general notion of fully nonlinear second-order elliptic equations*, Trans. Amer. Math. Soc., **347** (1995), 857-895. MR1284912 (95f:35075)
18. A. Li and Y.Y. Li, *On some conformally invariant fully nonlinear equations*, Comm. Pure Appl. Math., **56** (2003), 1416-1464. MR1988895 (2004e:35072)
19. J. Li and W.M. Sheng, *Deforming metrics with negative curvature by a fully nonlinear flow*, Calc. Var. PDE., **23** (2005), 33-50. MR2133660 (2005m:53121)
20. R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom., **20** (1984), 479-495. MR788292 (86i:58137)
21. W.M. Sheng, N. Trudinger and X.-J. Wang, *The Yamabe problem for higher order curvatures*, J. Differential Geom. **77** (2007), 515-553.
22. W.M. Sheng, J. Urbas, and X.-J. Wang, *Interior curvature bounds for a class of curvature equations*, Duke Math. J., **123** (2004), 235-264. MR2066938 (2005d:35087)
23. W.M. Sheng and Y. Zhang, *A class of fully nonlinear equations arising conformal geometry*, Math. Z., **255** (2007), 17-34. MR2262720
24. N. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math., **20** (1967), 721-747. MR0226198 (37:1788)
25. N.S. Trudinger, *The Dirichlet problem for the prescribed curvature equations*, Arch. Rational Mech. Anal., **111** (1990), 153-179. MR1057653 (91g:35118)
26. N. Trudinger and X.-J. Wang, *On Harnack inequalities and singularities of admissible metrics in the Yamabe problem*, arXiv:math.DG/0509341.
27. J. Viaclovsky, *Conformal geometry, contact geometry, and the calculus of variations*, Duke Math. J., **101** (2000), 283-316. MR1738176 (2001b:53038)
28. X.-J. Wang, *A priori estimates and existence for a class of fully nonlinear elliptic equations in conformal geometry*, Chin. Ann. of Math. (B), **27** (2006), 169-178. MR2243678 (2007e:53036)

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