PARTITIONING TRIPLES AND PARTIALLY ORDERED SETS

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ABSTRACT. We prove that if P is a partial order and $P \rightarrow (\omega)_\omega^1$, then

(a) $P \rightarrow (\omega + \omega + 1, 4)^3$, and

(b) $P \rightarrow (\omega + m, n)^3$ for each $m, n < \omega$.

Together these results represent the best progress known to us on the following question of P. Erdős and others. If $P \rightarrow (\omega)_\omega^1$, then does $P \rightarrow (\alpha, n)^3$ for each $\alpha < \omega_1$ and each $n < \omega$?

1. INTRODUCTION

The results presented here represent only the most recent leg of the journey begun in 1956 by P. Erdős and R. Rado when they proved the following.

Theorem 1 (P. Erdős and R. Rado, [2]). *If L is a real order, then $L \rightarrow (\omega + m, 4)^3$ for each $m < \omega$.*

(A *real order* is an uncountable linear order all of whose wellordered and anti-wellordered subsets are countable.) This result for triples stood alone for over thirty years, until 1987, when E. C. Milner and K. Prikry extended it from real orders to non-special linear orders.

Theorem 2 (E. C. Milner and K. Prikry, [7]). *If L is a non-special linear order, then $L \rightarrow (\omega + m, 4)^3$ for each $m < \omega$.*

(A partial order P is *non-special* if it cannot be decomposed into countably many anti-wellfounded subsets.) In particular, this implies that $\omega_1 \rightarrow (\omega + m, 4)^3$ for all $m < \omega$, a previously unknown fact. Six years later, Milner and Prikry improved on this more specific result.

Theorem 3 (E. C. Milner and K. Prikry, [8]). $\omega_1 \rightarrow (\omega + \omega + 1, 4)^3$.

In 1999, we were able to increase the four to an arbitrary finite integer, but at the cost of dropping back below $\omega + \omega$ in the other color.

Theorem 4 (A. L. Jones, [4]). $\omega_1 \rightarrow (\omega + m, n)^3$ for all $m, n < \omega$.

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The results presented below extend the domain of the results referenced above from that of the real orders, the non-special linear orders, or just ω_1 to that of *all* non-special partial orders. Even so, they fall almost unbelievably short of resolving the following.

Conjecture (P. Erdős, E. C. Milner, K. Prikry, A. L. Jones, *et al.*). *If P is a partial order and $P \rightarrow (\omega)_\omega^1$, then $P \rightarrow (\alpha, n)^3$ for all $\alpha < \omega_1$ and all $n < \omega$.*

We remark that this conjecture is sharp in the sense that if $P \rightarrow (\omega + 1, 4)^3$, then necessarily $P \rightarrow (\omega)_\omega^1$. The interested reader is kindly referred to [5] for more details.

2. NOTATION

Our set theoretic notation is essentially that of [3].

If P and Q are partial orders, then let $[P]^Q = \{X \subseteq P \mid X \cong Q\}$ be the set of all subsets of P which are order-isomorphic to Q . Thus, for any ordinal α , $[P]^\alpha$ is the set of wellordered chains of P of length α . For any ordinal α , let $[P]^{<\alpha} = \bigcup \{[P]^\beta \mid \beta < \alpha\}$, *etc.* Also, let $[P]^{n.s.} = \{X \subseteq P \mid X \rightarrow (\omega)_\omega^1\}$ be the set of non-special subsets of P . A partial order P is *non-special* if $P \rightarrow (\omega)_\omega^1$.

If A and B are subsets of P , then by $A < B$ we mean that each element of A is less than each element of B . If A is a subset of P and b is an element of P , then by $A < b$ we mean that each element of A is less than b . If a is an element of P and B is a subset of P , then by $a < B$ we mean that a is less than each element of B .

If $P_1, \dots, P_n$ and $Q_1, \dots, Q_n$ are partial orders, then

$$[P_1, \dots, P_n]^{Q_1, \dots, Q_n} = \{X_1 \cup \dots \cup X_n \mid X_1 \in [P_1]^{Q_1}, \dots, X_n \in [P_n]^{Q_n}\}.$$

This notation will be most useful to us when n is small and each Q_i is a small ordinal. For example, if A and B are subsets of P with $A < B$, then $[A, B]^{2,1}$ is the set of three-element chains of P with two elements in A and one element in B . Similarly, $[A, B, C]^{1,1,1}$ might represent triples with one element in each of the sets A , B , and C , *etc.*

If $x, y \in [P]^{<\omega}$, then put $x \sqsubset y$ if $x \neq y$ and there are $u, v \in [P]^{<\omega}$ with $u < v$, $x = u$, and $y = u \cup v$. Put $x \ll y$ if $x \neq y$ and there are $u, v_0, v_1 \in [P]^{<\omega}$ with $u < v_0 < v_1$, $x = u \cup v_0$, and $y = u \cup v_1$.

If P , Q_0 , and Q_1 are partial orders and $m < \omega$, then the partition relation $P \rightarrow (Q_0, Q_1)^m$ holds if for every partition $[P]^m = K_0 \cup K_1$ either there is $X_0 \in [P]^{Q_0}$ with $[X_0]^m \subseteq K_0$ or there is $X_1 \in [P]^{Q_1}$ with $[X_1]^m \subseteq K_1$.

Similarly, if P and Q are partial orders, κ is a cardinal, and $m < \omega$, then the partition relation $P \rightarrow (Q)_\kappa^m$ holds if for every partition $f : [P]^m \rightarrow \kappa$ there is $X \in [P]^Q$ on which f is constant (*i.e.*, there is $i < \kappa$ with $f''[X]^m \subseteq \{i\}$).

The study of these relations (and much of the notation defined above) was introduced by P. Erdős and R. Rado in [2].

3. RESULTS

This section is devoted to our proof of the following.

Proposition. *If P is a partial order and $P \rightarrow (\omega)_\omega^1$, then*

- (a) $P \rightarrow (\omega + \omega + 1, 4)^3$, and

(b) $P \rightarrow (\omega + m, n)^3$ for all $m, n < \omega$.¹

By a result of S. Todorćević it is enough to prove this proposition for non-special trees of cardinality less than that of the continuum, under the additional assumption that $\mathfrak{p} = \mathfrak{c}$. We do so below, and refer the reader to either [10], [1], or [6] for a detailed explanation of this (now somewhat standard) metamathematical reduction.

Our kind reader should be familiar with several fundamental facts.

Theorem 5 (F. P. Ramsey, [9]). $\omega \rightarrow (\omega)_n^m$ for all $m, n < \omega$.

A family $\mathcal{F} \subseteq [\omega]^\omega$ is a *filter base* if the intersection of any finitely many of its elements is infinite. In particular, any subfamily of a non-principal ultrafilter over ω is a filter base. A set $X \in [\omega]^\omega$ is a *pseudo-intersection* of $\mathcal{F} \subseteq [\omega]^\omega$ if $X \subseteq^* F$ for every $F \in \mathcal{F}$. Here, $X \subseteq^* F$ if there is $N < \omega$ with $X \setminus N \subseteq F$.

The cardinal $\mathfrak{p}$ is the *pseudo-intersection number*, the minimal cardinality of a filter base for which there is no pseudo-intersection. Most importantly, every filter base of cardinality less than $\mathfrak{p}$ must have a pseudo-intersection. As usual, the cardinal $\mathfrak{c}$ is the cardinality of the continuum. Note that $\omega_1 \leq \mathfrak{p} \leq \mathfrak{c}$.

An ultrafilter $\mathcal{U}$ over ω is a *Ramsey ultrafilter* if $\omega \rightarrow (\mathcal{U})_n^m$ for all $m, n < \omega$. If we assume either CH or Martin's Axiom (or more generally that $\mathfrak{p} = \mathfrak{c}$), then such filters are easily constructed.

Theorem 6 (S. Todorćević, [10]). *Non-special trees are much like ω_1 in that*

- (i) *if T is a non-special tree and $f : T \rightarrow T$ is regressive (that is, $f(t) < t$ for all $t \in T$), then there must be $S \in [T]^{n.s.}$ on which f is constant,*
- (ii) *$n.s. \text{ tree} \rightarrow (n.s. \text{ tree}, \omega + 1)^2$, and*
- (iii) *$n.s. \text{ tree} \rightarrow (\alpha)_n^2$ for all $\alpha < \omega_1$ and $n < \omega$.*

Each part of the above theorem is a generalization to non-special trees of an important result about ω_1 :

- (i) Fodor's (Pressing Down) Lemma,
- (ii) The Erdős–Dushnik–Miller Theorem, and
- (iii) The Baumgartner–Hajnal Theorem.

We refer the interested reader to [3], [2], [1], and [10] for more information on these and other related results.

If T is a non-special tree and P is a partial order, then a function $f : T \rightarrow P$ is said to be *almost always non-increasing* if for every $s \in T$ the set

$$I_f(s) = \{t \in T \mid s < t \wedge f(s) < f(t)\}$$

is special.

Corollary 7. *If T is a non-special tree, W is a wellordering, and $f : T \rightarrow W$ is almost always non-increasing, then there is $S \in [T]^{n.s.}$ on which f is constant.*

Proof. Let $I = \{t \in T \mid \exists s < t [t \in I_f(s)]\}$ be the diagonal union of $\{I_f(s) \mid s \in T\}$. Let $\bar{T} = T \setminus I$. By (i), $\bar{T}$ is non-special. Define $[\bar{T}]^2 = K_0 \cup K_1$ as follows. For $s, t \in \bar{T}$ with $s < t$, put $\{s, t\} \in K_0$ if $f(s) = f(t)$ and put $\{s, t\} \in K_1$ if $f(s) > f(t)$. Because W is wellordered there can be no $X_1 \in [\bar{T}]^\omega$ with $[X]_1^2 \subseteq K_1$. By (ii), there must be $X_0 \in [\bar{T}]^{n.s.}$ with $[X_0]_1^2 \subseteq K_0$. Choose $S \in [X_0]^{n.s.}$ with a unique minimal element. Evidently f is constant on S . $\square$

¹Part (b) of the proposition first appeared in [4], though the proof given there was much more difficult to follow and was marred by a few typographical errors.

By (i) we can (and do henceforth) assume that each non-special tree T is *well-pruned* in that $T(b) = \{t \in T \mid b < t\}$ is non-special for each $t \in T$. In particular, if $A < b$ for $A \subseteq T$ and $b \in T$, then there must be $B \in [T]^{n.s.}$ (namely $B = T(b)$) with $A < B$.

We now proceed to our proof of the proposition. Part (a) of the proposition will follow from Lemma A and Lemma C directly, while part (b) of the proposition will follow from Lemma B and Lemma C via a straightforward inductive argument.

Lemma A. *Let T be a non-special tree with $[T]^3 = K_0 \cup K_1$. If there are $A \in [T]^\omega$ and $B \in [T]^{n.s.}$ with $A < B$ and $[A, B]^{2,1} \subseteq K_0$, then either*

- (i) *there is $X \in [T]^{\omega+\omega+1}$ with $[X]^3 \subseteq K_0$, or*
- (ii) *there is $Y \in [T]^4$ with $[Y]^3 \subseteq K_1$.*

Proof. Let $\mathcal{U}$ be a non-principal ultrafilter over A . Thus, for each pair $\{r, s\} \in [B]^2$, there are $i_{r,s} \in \{0, 1\}$ and $A_{r,s} \in \mathcal{U}$ with $\{a, r, s\} \in K_{i_{r,s}}$ for each $a \in A_{r,s}$. For each $t \in B$, call the pair $\langle x, y \rangle$ *good* for t if

- (1) $x \in [A]^{<\omega}$, $y \in [B]^{<\omega}$, and $y < t$,
- (2) $[x, y]^{1,2} \cup [x, y, \{t\}]^{1,1,1} \cup [y, \{t\}]^{2,1} \subseteq K_0$,
- (3) $i_{r,s} = 0$ for each pair $\{r, s\} \in [y \cup \{t\}]^2$.

If $\langle x_0, y_0 \rangle$ and $\langle x_1, y_1 \rangle$ are both good pairs for t , then put $\langle x_0, y_0 \rangle \sqsubset \langle x_1, y_1 \rangle$ if both $x_0 \sqsubset x_1$ and $y_0 \sqsubset y_1$. Note that $\langle \emptyset, \emptyset \rangle$ is a good pair for each $t \in B$.

Claim. If for some $t \in B$ there is an infinite increasing sequence $\langle x_0, y_0 \rangle \sqsubset \langle x_1, y_1 \rangle \sqsubset \langle x_2, y_2 \rangle \sqsubset \cdots$ of pairs that are good for t , then either (i) or (ii) holds.

Proof. Let $X = \bigcup \{x_n \mid n < \omega\}$ and $Y = \bigcup \{y_n \mid n < \omega\}$. Then $X, Y \in [T]^\omega$ because $\langle x_n, y_n \rangle \sqsubseteq \langle x_{n+1}, y_{n+1} \rangle$ for each $n < \omega$. Note that $X < Y$ and

$$[X, Y \cup \{t\}]^{2,1} \subseteq K_0$$

because $X \subseteq A$ and $Y \cup \{t\} \subseteq B$. Also,

$$[X, Y]^{1,2} \cup [X, Y, \{t\}]^{1,1,1} \cup [Y, \{t\}]^{2,1} \subseteq K_0$$

because each $\langle x_n, y_n \rangle$ is a good pair for t . This is almost enough to make $[X \cup Y \cup \{t\}]^3 \subseteq K_0$; all that is lacking is that $[X]^3 \subseteq K_0$ and $[Y]^3 \subseteq K_0$. But because $X \rightarrow (\omega, 4)^3$, either there are $X_0 \in [X]^\omega$ and $Y_0 \in [Y]^\omega$ with $[X_0]^3 \subseteq K_0$ and $[Y_0]^3 \subseteq K_0$, in which case $X_0 \cup Y_0 \cup \{t\} \in [T]^{\omega+\omega+1}$ and $[X_0 \cup Y_0 \cup \{t\}]^3 \subseteq K_0$ (and hence (i) holds), or there is either $Z \in [X]^4$ or $Z \in [Y]^4$ with $[Z]^3 \subseteq K_1$ (and hence (ii) holds). $\square$

Without loss of generality, we may therefore assume that for each $t \in B$ there is a $\sqsubset$ -maximal good pair $\langle x_t, y_t \rangle$. Moreover, we may assume (by pressing down in B , if necessary) that $\langle x_t, y_t \rangle$ is the same pair $\langle x, y \rangle$ for all $t \in B$. Note that this implies that for each pair $\{s, t\} \in [B]^2$, either

- (1) there is $r \in x \cup y$ with $\{r, s, t\} \in K_1$, or
- (2) $i_{s,t} = 1$.

Otherwise, any a in $\bigcap \{A_{p,q} \mid \{p, q\} \in [y \cup \{s\} \cup \{t\}]^2\}$ would make $\langle x \cup \{a\}, y \cup \{s\} \rangle$ a good pair for t , contradicting the supposed maximality of $\langle x, y \rangle$ for t .

Because $B \rightarrow (\omega + \omega + 1)_{|x|+|y|+1}^2$, either there is $r \in x \cup y$ and $X \in [B]^{\omega+\omega+1}$ with $\{r, s, t\} \in K_1$ for each pair $\{s, t\} \in [X]^2$, or there is $X \in [B]^{\omega+\omega+1}$ with $i_{s,t} = 1$ for all $\{s, t\} \in [X]^2$. In either case, either $[X]^3 \subseteq K_0$ (and hence (i) holds) or there

are $Y \in [X]^3$ and $r \in \bigcap \{A_{s,t} \mid \{s,t\} \in [Y]^2\}$ with $[\{r\} \cup Y]^3 \subseteq K_1$ (and hence (ii) holds). $\square$

Lemma B. *Suppose that $m, n < \omega$ and that n.s. tree $\rightarrow (\omega + m, n)^3$. Let T be a non-special tree with $[T]^3 = K_0 \cup K_1$. If there are $A \in [T]^\omega$ and $B \in [T]^{n.s.}$ with $A < B$ and $[A, B]^{2,1} \subseteq K_0$, then either*

- (i) *there is $X \in [T]^{\omega+m}$ with $[X]^3 \subseteq K_0$, or*
- (ii) *there is $Y \in [T]^{n+1}$ with $[Y]^3 \subseteq K_1$.*

Proof. Let $\mathcal{U}$ be a non-principal ultrafilter over A . Thus, for each pair $\{r, s\} \in [B]^2$, there are $i_{r,s} \in \{0, 1\}$ and $A_{r,s} \in \mathcal{U}$ with $\{a, r, s\} \in K_{i_{r,s}}$ for each $a \in A_{r,s}$. Since $B \rightarrow (\text{n.s. tree}, \omega)^2$, either

- (1) *there is $C \in [B]^{n.s.}$ with $i_{r,s} = 1$ for each pair $\{r, s\} \in [C]^2$, or*
- (2) *there is $D \in [B]^\omega$ with $i_{r,s} = 0$ for each pair $\{r, s\} \in [D]^2$.*

If (1) holds, then because $C \rightarrow (\omega + m, n)^3$, either there is $E \in [C]^{\omega+m}$ with $[E]^3 \subseteq K_0$ (and hence (i) holds), or there is $F \in [C]^n$ with $[F]^3 \subseteq K_1$. In the latter case, let a be any element of $\bigcap \{A_{r,s} \mid \{r, s\} \in [F]^2\}$, and let $\bar{F} = \{a\} \cup F$. Clearly, $\bar{F} \in [T]^{n+1}$ and $[\bar{F}]^3 \subseteq K_1$ (and hence (ii) holds).

But if (2) holds, then since $D \rightarrow (\omega, n + 1)^3$, either there is $\bar{D} \in [D]^\omega$ with $[\bar{D}]^3 \subseteq K_1$, or there is $F \in [D]^{n+1}$ with $[F]^3 \subseteq K_1$ (and hence (ii) holds). In the former case, choose any $E \in [\bar{D}]^m$ and let $\bar{A} = \bigcap \{A_{r,s} \mid \{r, s\} \in [E]^2\}$. Since $\bar{A} \rightarrow (\omega, n + 1)^3$, either there is $\bar{\bar{A}} \in [\bar{A}]^\omega$ with $[\bar{\bar{A}}]^3 \subseteq K_0$, or there is $\bar{F} \in [\bar{A}]^{n+1}$ with $[\bar{F}]^3 \subseteq K_1$ (and hence (ii) holds). In the former case, let $G = \bar{\bar{A}} \cup E$. Clearly, $G \in [T]^{\omega+m}$ and $[G]^3 \subseteq K_0$ (and hence (i) holds). $\square$

Lemma C. *Let T be a non-special tree with $|T| < \mathfrak{p} = \mathfrak{c}$ and $[T]^3 = K_0 \cup K_1$. Then either*

- (i) *there are $A \in [T]^\omega$ and $B \in [T]^{n.s.}$ with $A < B$ and $[A, B]^{2,1} \subseteq K_0$, or*
- (ii) *for each $n < \omega$ there is $C \in [T]^n$ with $[C]^3 \subseteq K_1$.*

Informally, n.s. tree $\rightarrow ((\omega : \text{n.s. tree})^{2,1}, n)^3$ for each $n < \omega$.

Proof. Let $\prec$ be a fixed wellordering of $H(\mathfrak{c}^+)$, the collection of all sets of hereditary cardinality not greater than the continuum.

For each $n < \omega$ and $x, y \in [\omega]^n$, note that $x \ll y$ if and only if there are $m < n$, $u \in [\omega]^m$, and $v_0, v_1 \in [\omega]^{n-m}$ with $u < v_0 < v_1$, $x = u \cup v_0$, and $y = u \cup v_1$. For each $n < \omega$ and $A \in [T]^\omega$ there is a unique order isomorphism between $\langle [\omega]^n, \ll_{\text{lex}} \rangle$ and $\langle A, <_T \rangle$. For each $x \in [\omega]^n$, let $A(x)$ be the element of A identified with x via this isomorphism.

Fix a Ramsey ultrafilter $\mathcal{U}$ over ω . For $n < \omega$, $m < n$, $A \in [T]^{\omega^m}$, and $b \in T$ with $A < b$, define the partition $f_{A,m}^b : [\omega]^{2n-m} \rightarrow \{0, 1\}$ as follows. Decompose each $x \in [\omega]^{2n-m}$ as $x = u \cup v_0 \cup v_1$ with $u < v_0 < v_1$, $|u| = m$, and $|v_0| = |v_1| = n - m$. Let

$$f_{A,m}^b(x) = i \quad \text{only if} \quad \{A(u \cup v_0), A(u \cup v_1), b\} \in K_i.$$

Because $\mathcal{U}$ is a Ramsey ultrafilter, there must be $i_{A,m}^b \in \{0, 1\}$ and $X_{A,m}^b \in \mathcal{U}$ with $f_{A,m}^b \restriction [X_{A,m}^b]^{2n-m} = \{i_{A,m}^b\}$. Let X_A be the $\prec$ -least $X \in [\omega]^\omega$ with $X \subseteq^* X_{A,m}^b$ for all $m < n$ and all $b \in T$ with $A < b$. Let $N_{A,m}^b$ be the least $N < \omega$ with $X_A \setminus N \subseteq X_{A,m}^b$.

For $n < \omega$, $m < n$, and $A \in [T]^{\omega^n}$ let

$$B_{A,m} = \{b \in T \mid A < b \wedge i_{A,m}^b = 0\}.$$

For each $n < \omega$ let $\mathcal{B}_n$ be the collection of $B \in [T]^{n.s.}$ for which there exist $A \in [T]^{\omega^n}$ and $m < n$ with $B \subseteq B_{A,m}$.

Claim. If (i) fails, then $\mathcal{B}_n$ is empty for all $n < \omega$. (Or contrapositively, if $\mathcal{B}_n$ is non-empty for some $n < \omega$, then (i) holds.)

Proof. Fix $n < \omega$ and suppose $B \in \mathcal{B}_n$. Choose $A \in [T]^{\omega^n}$ and $m < n$ with $B \subseteq B_{A,m}$. Because $B \rightarrow (n.s. \text{ tree})_\omega^1$ there must be $\bar{B} \in [B]^{n.s.}$ and $\bar{N} < \omega$ with $\bar{N} = \bar{N}_{A,m}^b$ for each $b \in \bar{B}$. Choose $u \in [X_A]^m$ and $\{v_i \mid i < \omega\} \subseteq [X_A]^{n-m}$ with

$$\bar{N} < u < v_0 < v_1 < v_2 < \dots$$

Let $\bar{A} = \{A(u \cup v_i) \mid i < \omega\}$. Note that $\bar{A} \in [T]^\omega$, $\bar{B} \in [T]^{n.s.}$, $\bar{A} < \bar{B}$, and $[\bar{A}, \bar{B}]^{2,1} \subseteq K_0$. Thus (i) holds. $\square$

Suppose $A, B \subseteq T$. Then A is *bounded* in B if there is $b \in B$ with $A < b$. For each $n < \omega$ let $\mathcal{A}_n(B)$ be the collection of all $A \in [B]^{\omega^n}$ with A bounded in B and $\{A(x), A(y), A(z)\} \in K_1$ for all $x, y, z \in [\omega]^n$ with $x \ll y \ll z$ and $|x \cap z| < |x \cap y|$. Let $\mathcal{A}_n = \mathcal{A}_n(T)$.

Claim. If $\mathcal{B}_n$ is empty for all $n < \omega$, then $\mathcal{A}_n$ is non-empty for all $n < \omega$.

Proof. Suppose that $\mathcal{B}_n$ is empty for all $n < \omega$. Then for each $A \in [T]^{\omega^n}$ the set

$$B_A = \bigcap \{B_{A,m} \mid m < n\} = \{b \in T \mid A < b \wedge \exists m < n [i_{A,m}^b = 0]\}$$

is special. Consequently, for every $B \in [T]^{n.s.}$ each $A \in \mathcal{A}_n(B)$ is *nicely bounded* in B , in that there is $b \in B$ with $A < b$ and $i_{A,m}^b = 1$ for all $m < n$.

We will prove (by induction on n) that $\mathcal{A}_n(B)$ is non-empty for every $B \in [T]^{n.s.}$. Note that $\mathcal{A}_0(B)$ and $\mathcal{A}_1(B)$ are certainly non-empty for every $B \in [T]^{n.s.}$, being vacuously equal to $[B]^1$ and $\{A \in [B]^\omega \mid \exists b \in B [A < b]\}$, respectively.

Fix a non-zero $n < \omega$. Suppose that $\mathcal{A}_n(B)$ is non-empty for every $B \in [T]^{n.s.}$. Fix $B \in [T]^{n.s.}$. We will prove that $\mathcal{A}_1^{n+1}(B)$ is non-empty. For each $b \in B$, we first try to construct a sequence $\langle A_k^b \mid k < \omega \rangle \subseteq \mathcal{A}_n(B)$. Fix $k < \omega$. Suppose that A_j^b has been defined for each $j < k$. If there is $A \in \mathcal{A}_n(B)$ with

- (1) $A_j^b < A < b$ for all $j < k$,
- (2) $i_{A_j^b, m}^a = i_{A_j^b, m}^b = i_{A, m}^b = 1$ for all $j < k$, $m < n$, and $a \in A$, and
- (3) $N_{A_j^b, m}^a = N_{A_j^b, m}^b$ for all $j < k$, $m < n$, and $a \in A$,

then let A_k^b be the $\prec$ -least such $A \in \mathcal{A}_n(B)$. Otherwise, stop the construction and let $k^b = k$.

We claim that there must be $b \in B$ for which the construction of $\langle A_k^b \mid k < \omega \rangle$ succeeds. If not, then by repeated application of Theorem 6 and its corollary, there must be $\bar{B} \in [B]^{n.s.}$, $\bar{k} < \omega$, $\langle \bar{A}_j \mid j < \bar{k} \rangle \subseteq \mathcal{A}_n(B)$, and $\langle \bar{N}_{j, m} \mid j < \bar{k}, m < n \rangle \subseteq \omega$ with

- (1) $\bar{k} = k^b$ for all $b \in \bar{B}$,
- (2) $\bar{A}_j = A_j^b$ for all $j < \bar{k}$ and $b \in \bar{B}$,
- (3) $\bar{N}_{j, m} = N_{A_j^b, m}^b$ for all $j < \bar{k}$, $m < n$, and $b \in \bar{B}$.

But $\mathcal{A}_n(\bar{B})$ is non-empty. Choose $\bar{A} \in \mathcal{A}_n(\bar{B})$ and $\bar{b} \in \bar{B}$ with $\bar{A} < \bar{b}$ and $i_{\bar{A},m}^{\bar{b}} = 1$ for all $m < n$. This A is a candidate for $A_k^{\bar{b}}$, contradicting our assumption that the construction stopped at step $\bar{k}$ for $\bar{b}$.

Choose $b \in B$ for which the construction of $\langle A_k^b \mid k < \omega \rangle$ succeeds. For each $k < \omega$, let $N_k = \max\{N_{A_k^b, m}^b \mid m < n\}$. Let $A = \{A_k^b(x) \mid x \in [X_{A_k^b} \setminus N_k]^n\}$. It is now easily verified that $A \in \mathcal{A}_{n+1}(B)$. $\square$

Claim. If $\mathcal{A}_n$ is non-empty for all $n < \omega$, then (ii) holds. (More specifically, for each $n < \omega$, if $\mathcal{A}_n$ is non-empty, then there is $Y \in [T]^{n+1}$ with $[Y]^3 \subseteq K_1$.)

Proof. Fix $n < \omega$. Suppose $A \in \mathcal{A}_n$. For each $k \leq n$ let

$$x_{n,k} = \{c \mid c < n - k\} \cup \{kn + c \mid c < k\}.$$

For all $i < j < k \leq n$ it is easily verified that $x_{n,i} \ll x_{n,j} \ll x_{n,k}$ and $|x_{n,i} \cap x_{n,k}| = n - k < n - j = |x_{n,i} \cap x_{n,j}|$. Let $Y = \{A(x_{n,k}) \mid k \leq n\}$. Then $Y \in [T]^{n+1}$ and $[Y]^3 \subseteq K_1$. $\square$

The lemma now follows directly from the three claims above. $\square$

Finally—with these three lemmas in hand—the proposition is easily demonstrated.

Proof of (a). Let T be a non-special tree with $[T]^3 = K_0 \cup K_1$. Suppose that there is no $Y \in [T]^4$ with $[Y]^3 \subseteq K_1$. By Lemma C there must then be $A \in [T]^\omega$ and $B \in [T]^{n.s.}$ with $A < B$ and $[A, B]^{2,1} \subseteq K_0$. It now follows from Lemma A that there must be $X \in [T]^{\omega+\omega+1}$ with $[X]^3 \subseteq K_0$. As T was arbitrary, we may conclude that n.s. tree $\rightarrow (\omega + \omega + 1, 4)^3$. $\square$

Proof of (b). Fix $m < \omega$ and proceed by induction on $n < \omega$. Assume that n.s. tree $\rightarrow (\omega + m, n)^3$. Let T be a non-special tree with $[T]^3 = K_0 \cup K_1$. Suppose that there is no $Y \in [T]^{n+1}$ with $[Y]^3 \subseteq K_1$. By Lemma C there must then be $A \in [T]^\omega$ and $B \in [T]^{n.s.}$ with $A < B$ and $[A, B]^{2,1} \subseteq K_0$. It now follows from Lemma B that there must be $X \in [T]^{\omega+m}$ with $[X]^3 \subseteq K_0$. As T was arbitrary, we may conclude that n.s. tree $\rightarrow (\omega + m, n + 1)^3$. $\square$

4. CONCLUSION

Finally, we remark that we have recently proven the following result, to be published elsewhere.

Proposition. $\omega_1 \rightarrow (\omega + \omega + 1, n)^3$ for all $n < \omega$.

Unfortunately, we have been unable to extend our proof to arbitrary non-special trees. This asymmetry leaves us with the following unbalanced pair of open problems.

Problem 1. Does $\omega_1 \rightarrow (\omega + \omega + 2, 4)^3$?

Problem 2. Does n.s. tree $\rightarrow (\omega + \omega, 5)^3$?

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