A REMARK ON THE IKEDA LIFT AND LOCAL SINGULAR SERIES POLYNOMIALS

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ABSTRACT. We relate a certain number-theoretic function occurring in a linear version of Ikeda’s lifting map to local singular series polynomials.

1. INTRODUCTION

Let k and n be positive integers with $k \equiv n \pmod{2}$. In his famous paper [1], T. Ikeda associated to a cuspidal Hecke eigenform f of weight $2k$ on $\Gamma_1 := SL_2(\mathbf{Z})$ a cuspidal Hecke eigenform F of weight $k+n$ on the Siegel modular group $\Gamma_{2n} := Sp_{2n}(\mathbf{Z})$ of genus $2n$. If $n = 1$, Ikeda’s lifting coincides with the classical Saito-Kurokawa lift.

Let T be a positive definite symmetric half-integral matrix of size $2n$ and let $D_T := (-1)^n \det(2T)$ be its discriminant. Then $D_T \equiv 0, 1 \pmod{4}$ and we shall write $D_T = D_{T,0} f_T^2$ with $D_{T,0}$ a fundamental discriminant and $f_T \in \mathbf{N}$.

Recall that the T -th Fourier coefficient of F —apart from the $|D_{T,0}|$ -th Fourier coefficient of a half-integral weight modular form—involves a product

$$\prod_{p|f_T} \tilde{F}_p(T; \alpha_p),$$

where $\tilde{F}_p(T; X)$ is a certain Laurent polynomial which essentially describes the “non-trivial” part of the local singular series polynomial of T at p . Moreover, α_p is the Satake parameter of the local component π_p of the automorphic cuspidal representation $\pi = \bigotimes_p \pi_p$ associated to f .

According to an important result of Katsurada [2], the polynomial $\tilde{F}_p(T; X)$ is symmetric, i.e., is invariant under $X \mapsto X^{-1}$, and of degree $\ell_p := \text{ord}_p f_T$ (i.e. $X^{\ell_p} \tilde{F}_p(T; X)$ is a polynomial of degree $2\ell_p$). Moreover, there are certain recursion formulas that allow us to compute $\tilde{F}_p(T; X)$ (knowing all the local p -invariants of T) in an explicit, though not quite simple way [2, sect. 4].

In [5], a linear version of Ikeda’s lifting map was given as a map from the “plus” space $S_{k+1/2}^+$ of cusp forms of weight $k+1/2$ and level 4 [4] to the space $S_{k+n}(\Gamma_{2n})$ of cusp forms of weight $k+n$ on Γ_{2n} . Here in the expression for the T -th Fourier coefficient of a lifted form, a certain somewhat mysterious, multiplicative integer-valued function $\phi(a; T)$ enters, defined for $a|f_T$. Its definition is complicated and involves a certain sum over the finitely many classes of matrices $T[G^{-1}]$, where

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G runs through matrices in $M_{2n}(\mathbf{Z}) \cap GL_{2n}(\mathbf{Q})$ such that $T[G^{-1}]$ is half-integral, modulo the action of $GL_{2n}(\mathbf{Z})$ by left-multiplication. Here we use the notation $A[B] := B'AB$ for matrices A and B of appropriate sizes. It seems quite difficult to compute $\phi(a; T)$ explicitly from its definition, in general, and so far no more explicit and simpler expressions for $\phi(a; T)$ (except in the case $n = 1$) have been given in the literature.

The purpose of this paper is to relate $\phi(p^j; T)$ ($0 \leq j \leq \ell_p$) in an explicit and simple way to $\tilde{F}_p(T; X)$. In fact, we will show that the latter are essentially the coefficients of $\tilde{F}_p(T; X)$ in a special basis of the space of symmetric Laurent polynomials of degree $\leq \ell_p$. In combination with the results of [2], this may lead to explicit expressions for $\phi(p^j; T)$ ($0 \leq j \leq \ell_p$).

In the proof of our result several calculations and arguments given in [5], especially certain formulas for the Fourier coefficients of Siegel-Eisenstein series, play an important role.

2. LOCAL SINGULAR SERIES

Let T be a positive definite symmetric half-integral matrix of size $2n$. If p is a prime, then the singular series attached to T and p is defined as

$$b_p(T; s) := \sum_R \nu_p(R)^{-s} e_p(\text{tr}(TR)) \quad (s \in \mathbf{C}),$$

where R runs over all symmetric $(2n, 2n)$ -matrices with entries in $\mathbf{Q}_p/\mathbf{Z}_p$ and $\nu_p(R)$ is a power of p equal to the product of denominators of elementary divisors of R . Furthermore, for $x \in \mathbf{Q}_p$ we have put $e_p(x) := e^{2\pi i x'}$ where x' denotes the fractional part of x .

As is well-known, $b_p(T; s)$ is a product of two polynomials in p^{-s} with coefficients in $\mathbf{Z}$. More precisely, put

$$\gamma_p(T; X) := (1 - X)(1 - p^n(\frac{D_{T,0}}{p})X)^{-1} \cdot \prod_{j=1}^n (1 - p^{2j}X^2).$$

Then

$$b_p(T; s) = \gamma_p(T; p^{-s}) F_p(T; p^{-s}),$$

where $F_p(T; X)$ is a certain integral polynomial with constant term equal to 1.

If we put

$$\tilde{F}_p(T; X) := X^{-\ell_p} F_p(T; p^{-n-1/2}X),$$

then $\tilde{F}_p(T; X)$ is a symmetric Laurent polynomial of degree ℓ_p [2].

3. IKEDA'S LIFTING MAP

Let f be a normalized Hecke eigenform of weight $2k$ on Γ_1 . For a prime p we denote by $\{\alpha_p, \alpha_p^{-1}\}$ “the” Satake p -parameter of f , so

$$1 - \lambda(p)X + p^{2k-1}X^2 = (1 - p^{k-1/2}\alpha_p X)(1 - p^{k-1/2}\alpha_p^{-1}X),$$

where $\lambda(p)$ is the p -th Hecke eigenvalue of f (normalized in the usual way).

Denote by $c(m)$ ($m \geq 1$; $(-1)^k m \equiv 0, 1 \pmod{4}$) the Fourier coefficients of a Hecke eigenform of weight $k + 1/2$ and level 4 in the “plus” space $S_{k+1/2}^+$ corresponding to f under the Shimura correspondence [4, 6].

Then Ikeda's main result [1] is that

$$F(Z) := \sum_{T>0} \left(c(|D_{T,0}|) f_T^{k-1/2} \prod_{p|f_T} \tilde{F}_p(T; \alpha_p) \right) e^{2\pi i \operatorname{tr}(TZ)}$$

is a Hecke eigenform in $S_{k+n}(\Gamma_{2n})$. Here Z is the Siegel upper half-space of genus $2n$ and T runs over all positive definite symmetric half-integral matrices of size $2n$.

We define the function $\phi(a; T) (a|f_T)$ as follows [5]. First, for a prime p and an integer $j \geq 0$ define $\rho_T(p^j)$ as the coefficient of X^j in $(1 - X^2)H_{n,p}(T; X)$ where $H_{n,p}(T; X)$ is the "Kitaoka's polynomial" [3], a certain integral polynomial depending on the local p -invariants of T (for a precise definition we refer, e.g., to (6), sect. 3 in [5]).

We extend the function ρ_T multiplicatively to the whole of $\mathbf{N}$ by defining

$$\sum_{a \geq 1} \rho_T(a) a^{-s} := \prod_{p|f_T} ((1 - p^{-2s}) H_{n,p}(T; p^{-s})).$$

Finally, for $a \in \mathbf{N}$ with $a|f_T$ we set

$$\phi(a; T) := a^{1/2} \sum_{d^2|a} \sum_{G \in \mathcal{D}(T), |\det(G)|=d} \rho_{T[G^{-1}]} \left(\frac{a}{d^2} \right).$$

Here

$$\mathcal{D}(T) := GL_{2n}(\mathbf{Z}) \backslash \{G \in M_{2n}(\mathbf{Z}) \cap GL_{2n}(\mathbf{Q}) \mid T[G^{-1}] \text{ half-integral} \},$$

where $GL_{2n}(\mathbf{Z})$ operates by left-multiplication. It follows from the definitions that $\phi(a; T)$ is an integer and $a \mapsto \phi(a; T)$ is multiplicative (cf., e.g., Remarks (ii)) after the Corollary to Theorem 1 in [5].

The main result of [5] then states that the T -th Fourier coefficient of the Ikeda lift F of f is given by

$$(1) \quad \sum_{a|f_T} a^{k-1} \phi(a; T) c(|D_{T,0}| \left(\frac{f_T}{a} \right)^2).$$

As a consequence, (1) defines a linear map from $S_{k+1/2}^+$ to $S_{k+n}(\Gamma_{2n})$ which on Hecke eigenforms coincides with Ikeda's lift.

4. MAIN RESULT AND PROOF

For a non-negative integer ℓ we denote by V_ℓ the $(\ell + 1)$ -dimensional complex vector space of symmetric Laurent polynomials

$$\sum_{\nu=0}^{\ell} c_\nu (X^\nu + X^{-\nu}) \quad (c_\nu \in \mathbf{C})$$

of degree $\leq \ell$.

For $j \in \{0, 1, \dots, \ell + 1\}$ we set

$$\psi_j(X) := \frac{X^j - X^{-j}}{X - X^{-1}}.$$

Lemma 1. *Let $\epsilon \in \{0, \pm 1\}$. Then the Laurent polynomials*

$$\psi_j(X) - \epsilon p^{-1/2} \psi_{j-1}(X) \quad (j = 1, \dots, \ell + 1)$$

form a basis of V_ℓ .

Proof. It is clearly sufficient to show that the polynomials $\psi_j(X)$ ($j = 1, \dots, \ell + 1$) form a basis of V_ℓ , and this is obvious because of the formula

$$\psi_j(X) = X^{j-1} + X^{j-3} + \dots + X^{-j+3} + X^{-j+1}. \quad \square$$

Theorem. *Let T be a positive definite symmetric half-integral matrix of size $2n$. Let p be a prime with $p|f_T$ and let $\ell = \ell_p := \text{ord}_p f_T$. Then*

$$\tilde{F}_p(T; X) = \sum_{j=1}^{\ell+1} \phi(p^{\ell-j+1}; T) p^{-\frac{\ell-j+1}{2}} \left(\psi_j(X) - \left(\frac{D_{T,0}}{p} \right) p^{-1/2} \psi_{j-1}(X) \right).$$

Proof. Let us write $D_0 = D_{T,0}$ and $f = f_T$ (this will not lead to confusion with earlier notation). According to Siegel, the T -th Fourier coefficient of the Siegel-Eisenstein series of weight $k > 2n + 1$ on Γ_{2n} is given by

$$(2) \quad a_{k,2n}(T) = \frac{2^n}{\zeta(1-k) \prod_{j=1}^n \zeta(1-2k+2j)} \cdot L(1-k+n, \left(\frac{D_0}{\cdot}\right)) \cdot f^{k-n-\frac{1}{2}} \cdot \prod_{p|f} \tilde{F}_p(T; p^{n+\frac{1}{2}-k});$$

cf. (10) in sect. 3 of [5]. By Theorem 2 in [5], we also have

$$(3) \quad a_{k,2n}(T) = \frac{2^n}{\zeta(1-k) \prod_{j=1}^n \zeta(1-2k+2j)} \sum_{a|f} a^{k-n-1} \phi(a; T) H(k-n, |D_0| \left(\frac{f}{a}\right)^2),$$

where

$$H(k, |D_0| m^2) := L(1-k, \left(\frac{D_0}{\cdot}\right)) \sum_{d|m} \mu(d) \left(\frac{D_0}{d}\right) d^{k-1} \sigma_{2k-1}\left(\frac{m}{d}\right) \quad (m \in \mathbf{N})$$

is Cohen's class number function. (Recall that $\sigma_{2k-1}(a) := \sum_{d|a} d^{2k-1}$ ($a \in \mathbf{N}$).)

Combining (2) and (3), replacing k by $k + n$, and using multiplicativity, we find that for all positive even integers k large enough one has

$$(4) \quad \prod_{p|f} p^{\ell_p(k-\frac{1}{2})} \tilde{F}_p(T; p^{-k+\frac{1}{2}}) = \prod_{p|f} \left(\sum_{j=0}^{\ell_p} p^{j(k-\frac{1}{2})} \cdot p^{-\frac{j}{2}} \phi(p^j; T) (\sigma_{2k-1}(p^{\ell_p-j}) - \left(\frac{D_0}{p}\right) p^{k-\frac{1}{2}} \cdot p^{-\frac{1}{2}} \sigma_{2k-1}(p^{\ell_p-j-1})) \right).$$

We observe that

$$(5) \quad \sigma_{2k-1}(p^\nu) = \frac{(p^{k-\frac{1}{2}})^{2\nu+2} - 1}{(p^{k-\frac{1}{2}})^2 - 1} \quad (\nu \in \mathbf{N}_0)\ell$$

and enter (5) with $\nu = \ell_p - j$, $\ell_p - j - 1$ into (4).

Then by a well-known argument we deduce that identity (4) holds with $p^{k-\frac{1}{2}}$ replaced by a variable X_p , for each $p|f$, and then it follows that

$$(6) \quad X^\ell \tilde{F}_p(T; X) = \sum_{j=0}^{\ell} p^{-\frac{j}{2}} \phi(p^j; T) X^j \left(\frac{X^{2(\ell-j+1)} - 1}{X^2 - 1} - \left(\frac{D_0}{p}\right) p^{-\frac{1}{2}} X \frac{X^{2(\ell-j)} - 1}{X^2 - 1} \right),$$

for each $p|f$ individually (with the abbreviation $\ell = \ell_p$), since the polynomials on both sides of (6) have coefficients at the highest terms equal to 1.

Clearly, (6) can be rewritten as

$$\tilde{F}_p(T; X) = \sum_{j=0}^{\ell} p^{-\frac{j}{2}} \phi(p^j; T) \left(\frac{X^{\ell-j+1} - X^{-\ell+j-1}}{X - X^{-1}} - \left(\frac{D_0}{p}\right) p^{-\frac{1}{2}} \frac{X^{\ell-j} - X^{-\ell+j}}{X - X^{-1}} \right),$$

and therefore it is equivalent to the assertion of the theorem. $\square$

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