DEGREE OF THE CANONICAL MAP OF A GORENSTEIN 3-FOLD OF GENERAL TYPE

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ABSTRACT. We prove that, for a complex projective Gorenstein 3-fold X of general type with locally factorial terminal singularities, if $p_g(X) > 105411$ and the canonical map ϕ_X of X is generically finite, then $\deg \phi_X \leq 72$.

1. INTRODUCTION

Let X be a complex projective Gorenstein 3-fold of general type with locally factorial terminal singularities. In [Ha], C. Hacon proved that, if the canonical map ϕ_X of X is generically finite, then $\deg \phi_X \leq 576$. Recall that for a smooth surface S of general type, if the canonical map ϕ_S of S is generically finite, then $\deg \phi_S \leq 36$, and $\deg \phi_S \leq 9$ provided $p_g(S) \geq 30$ [Be]. By analogy with the case of surfaces, an interesting problem is to give the best bound for the degree of the canonical map of X when $p_g(X)$ is sufficiently large. The aim of this note is to prove the following.

Theorem 1.1. *Let X be a complex projective Gorenstein 3-fold of general type with locally factorial terminal singularities. Assume that $p_g(X) > 105411$, and that the canonical map ϕ_X of X is generically finite. Then $\deg \phi_X \leq 72$.*

Remark 1.2. (i) The assumption that X is Gorenstein is indispensable for Theorem 1.1 by Hacon's example [Ha, Example 2.1], and also that $p_g(X) > 105411$ is technical.

(ii) The bound in Theorem 1.1 seems to be the best possible. See Remark 3.5 below for a discussion.

Theorem 1.1 is proved in Section 3. Thanks to Hacon's argument in [Ha], we reduce the problem to the case when the Albanese map of X is a fibration $f : X \rightarrow C$ over a curve C , with $g(C)$ being sufficiently large and $p_g(F)$ being small, where F is a general fiber of f . In this case, we have $\deg \phi_X \leq \deg \phi_{X|F} \cdot \deg \phi_{\mathcal{L}}$ for any invertible subsheaf $\mathcal{L} \subset f_*\omega_X$ with $h^0(\mathcal{L}) \geq 2$, where ω_X is the dualizing sheaf of X and $\phi_{\mathcal{L}}$ is the rational map associated to $\mathcal{L}$. So the main point is to show that there exists an invertible subsheaf $\mathcal{L} \subset f_*\omega_X$ with higher degree compared with $g(C)$, which is done in Section 2.

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2. TECHNICAL PREPARATIONS

2.1. Let X be a smooth complex projective variety of general type, $f: X \rightarrow C$ a fiber space over a smooth projective curve C , and F a general fiber of f . Assume that the canonical map ϕ_X of X is generically finite. Then we have that $p_g(F) \geq \dim X$. (If $p_g(F) < \dim X$, then the dimension of the image of the restriction of ϕ_X to F is at most $\dim X - 2$, and hence $\dim \operatorname{Im} \phi_X \leq \dim X - 1$, which contradicts the assumption that ϕ_X is generically finite.)

Let $\Sigma \subset \mathbb{P}^{p_g(X)-1}$ be the image of ϕ_X . Desingularizing Σ and blowing-up X if necessary, we can assume that Σ is smooth and ϕ_X is a morphism. Let H be a very ample divisor on C , and $H_i \in |H|$ ($i = 1, 2$) such that $\operatorname{supp} H_1 \cap \operatorname{supp} H_2 = \emptyset$. Let $D_i = \phi_{X*}(f^*H_i)$. We have that D_1 is linearly equivalent to D_2 . Let $\alpha': \Sigma \rightarrow \mathbb{P}^1$ be the rational map associated to the pencil generated by D_1 and D_2 . Blowing-up Σ and X if necessary, we can assume that α' and ϕ_X are morphisms. Let $\alpha: \Sigma \rightarrow B$ be the Stein factorization of α' . By construction, we have that for each point $p \in B$, $(\alpha \circ \phi_X)^*p$ is contained in some union of fibers of f . So there is a morphism $\pi: C \rightarrow B$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi_X} & \Sigma \\ f \downarrow & & \alpha \downarrow \\ C & \xrightarrow{\pi} & B \end{array}$$

is commutative. We have

$$\deg \phi_X = \deg \phi_{X|F} \deg \pi.$$

Lemma 2.2. *Let $n \geq 2$ be an integer, and let $p_1, p_2, \dots, p_n$ be different points of C with f^*p_i smooth for all i . Assume that $h^1(f_*\omega_X) = 0$ and*

$$h^1(f_*\omega_X \otimes \mathcal{O}_C(-\sum_{i=1}^n p_i)) \leq (n-1)p_g(F) - 1.$$

Then $\pi(p_i) \neq \pi(p_j)$ for some $1 \leq i, j \leq n$ (in other words, ϕ_X separates f^*p_i and f^*p_j).

Proof. For any subset $I \subset N := \{1, 2, \dots, n\}$, consider the exact sequence of sheaves

$$0 \rightarrow f_*\omega_X \otimes \mathcal{O}_C(-\sum_{i \in I} p_i) \rightarrow f_*\omega_X \rightarrow \bigoplus_{i \in I} (f_*\omega_X \otimes \mathcal{O}_{p_i}) \rightarrow 0.$$

Taking cohomology, we get the long exact sequence

$$\begin{aligned} 0 &\rightarrow H^0(f_*\omega_X \otimes \mathcal{O}_C(-\sum_{i \in I} p_i)) \rightarrow H^0(f_*\omega_X) \xrightarrow{r_I} \bigoplus_{i \in I} H^0(f_*\omega_X \otimes \mathcal{O}_{p_i}) \\ &\rightarrow H^1(f_*\omega_X \otimes \mathcal{O}_C(-\sum_{i \in I} p_i)) \rightarrow H^1(f_*\omega_X) \rightarrow \dots \end{aligned}$$

The assumptions imply that

$$(2.2.1) \quad \dim \operatorname{Im} r_N \geq p_g(F) + 1.$$

For any subsets $I, J \subset N$, if $I \subset J$, we have $q_{J,I} \circ r_J = r_I$, where

$$q_{J,I}: \bigoplus_{j \in J} H^0(f_*\omega_X \otimes \mathcal{O}_{p_j}) \rightarrow \bigoplus_{i \in I} H^0(f_*\omega_X \otimes \mathcal{O}_{p_i})$$

is the projection. For convenience, if $I = \{i\}$, we set $r_i = r_I$.

Note that for any non-zero section $s \in H^0(f_*\omega_X)$, let $D \in |K_X|$ be the divisor corresponding to s ; we have $r_I(s) = 0$ if and only if $\sum_{i \in I} f^*p_i < D$.

We proceed by induction on n . If $n = 2$, by (2.2.1), we have that for $i = 1$ and 2 , there exists a non-zero section s_i with $r_{\{1,2\}}(s_i) \neq 0$ and $r_i(s_i) = 0$. Let $D_i \in |K_X|$ be the divisor corresponding to s_i . Then we have

$$f^*p_1 < D_1, \quad f^*p_1 \not< D_2, \quad f^*p_2 \not< D_1, \quad f^*p_2 < D_2.$$

Let h be a rational function on X such that $\operatorname{div}(h) = D_1 - D_2$, and let $\phi_h : X \dashrightarrow \mathbb{P}^1$ be the rational map associated to h . Then for a general point $x_i \in f^*p_i$ ($i = 1, 2$), we have $h(x_1) = 0$ and $h(x_2) = \infty$. This implies $\pi(p_1) \neq \pi(p_2)$. Indeed, if $\pi(p_1) = \pi(p_2)$, then ϕ_X maps both f^*p_1 and f^*p_2 onto α^*b , where $b = \pi(p_1)$ (and α is as in subsection 2.1). For any point $x'_i \in f^*p_i \cap \phi_X^{-1}(z)$ ($i = 1, 2$), where $z \in \alpha^*b$ is a given general point, we have $\phi_X(x'_1) = \phi_X(x'_2)$, but $h(x'_1) = 0$ and $h(x'_2) = \infty$. This is a contradiction.

If $n > 2$, by induction, we may assume that $\dim \operatorname{Im} r_{N \setminus \{i\}} \leq p_g(F)$ for all $i \in N$. By (2.2.1), we have that $\operatorname{Ker} q_{N, N \setminus \{i\}} \cap \operatorname{Im} r_N \neq 0$ for all $i \in N$. This implies that for each $i \in N$, there exists a non-zero section s_i with $r_i(s_i) \neq 0$ and $r_j(s_i) = 0$ for all $j \neq i$. Now by the same argument as in case $n = 2$, we get the result. $\square$

2.3. We assume that $h^1(f_*\omega_X) = 0$ and $\deg \pi \geq 2$. Let $2 \leq u \leq \deg \pi$ be an integer. Let $b \in B$ be a general point, and $p_1, p_2, \dots, p_{\deg \pi} \in C$ the inverse image of b under π . For any locally free sheaf $\mathcal{E}$ on C , we denote by $\mathcal{E}^\vee$ the dual of $\mathcal{E}$. By Lemma 2.2,

$$h^0((f_*\omega_{X/C})^\vee \otimes \mathcal{O}_C(\sum_{i=1}^u p_i)) = h^1(f_*\omega_X \otimes \mathcal{O}_C(-\sum_{i=1}^u p_i)) \geq (u-1)p_g(F),$$

where $f_*\omega_{X/C} = f_*\omega_X \otimes \omega_C^{-1}$. Let $\mathcal{F}'$ be the subsheaf of $(f_*\omega_{X/C})^\vee \otimes \mathcal{O}_C(\sum_{i=1}^u p_i)$ generated by its global sections, and $\mathcal{F}$ the saturation of $\mathcal{F}'$ in $(f_*\omega_{X/C})^\vee \otimes \mathcal{O}_C(\sum_{i=1}^u p_i)$. We have $h^0(\mathcal{F}) \geq (u-1)p_g(F)$, and there is an exact sequence of locally free sheaves

$$0 \longrightarrow \mathcal{F} \otimes \mathcal{O}_C(-\sum_{i=1}^u p_i) \longrightarrow (f_*\omega_{X/C})^\vee \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where $\mathcal{Q}$ is the quotient sheaf $(f_*\omega_{X/C})^\vee / (\mathcal{F} \otimes \mathcal{O}_C(-\sum_{i=1}^u p_i))$. Taking dual, we get an exact sequence of locally free sheaves

$$(2.3.1) \quad 0 \longrightarrow \mathcal{Q}^\vee \longrightarrow f_*\omega_{X/C} \longrightarrow \mathcal{F}^\vee \otimes \mathcal{O}_C(\sum_{i=1}^u p_i) \longrightarrow 0.$$

Here we insert a lemma that will be used in the sequel.

Lemma 2.4. *Let $\mathcal{E}$ be a locally free sheaf of rank r on a projective smooth curve C of positive genus.*

- (i) *If $\mathcal{E}$ is generically generated by global sections, then $\deg \mathcal{E} \geq h^0(\mathcal{E}) - r$.*
- (ii) *If $\mathcal{E}^\vee \otimes \mathcal{H}$ is semipositive, where $\mathcal{H}$ is an invertible sheaf of degree $m \geq 1$, then $h^0(\mathcal{E}) \leq mr$.*

Proof. (i) The assumption on $\mathcal{E}$ implies $h^0(\mathcal{E}) \geq r$, $\deg \mathcal{E} \geq 0$, and $\deg \mathcal{E} = 0$ holds only when $\mathcal{E} \simeq \mathcal{O}_C^{\oplus r}$. If $r = 1$, by the Riemann-Roch theorem, we have $h^0(\mathcal{E}) - 1 - \deg \mathcal{E} = h^0(\mathcal{E}^\vee \otimes \omega_C) - g(C) \leq 0$. Now we assume that $r \geq 2$. A non-zero section s in $H^0(\mathcal{E})$ induces an exact sequence of locally free sheaves on C

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{E} \rightarrow \mathcal{F} := \mathcal{E}/\mathcal{L} \rightarrow 0$$

with $h^0(\mathcal{L}) > 0$. Note that $\mathcal{F}$ is also generically generated by global sections. By induction on rank $\mathcal{E}$, we have $\deg \mathcal{F} \geq h^0(\mathcal{F}) - (r - 1)$. So

$$\deg \mathcal{E} = \deg \mathcal{F} + \deg \mathcal{L} \geq h^0(\mathcal{F}) - (r - 1) + h^0(\mathcal{L}) - 1 \geq h^0(\mathcal{E}) - r.$$

(ii) First we assume that $r = 1$. We can assume that $h^0(\mathcal{E}) > 0$. We have $\deg \mathcal{E} \geq 0$, and the equality holds only when $\mathcal{E} \simeq \mathcal{O}_C$. If $\deg \mathcal{E} = 0$, the lemma is trivially true.

If $\deg \mathcal{E} > 0$, then $h^0(\mathcal{E}^\vee \otimes \omega_C) \leq g(C) - 1$. By the Riemann-Roch theorem, we have

$$(2.4.1) \quad h^0(\mathcal{E}) - \deg \mathcal{E} = h^0(\mathcal{E}^\vee \otimes \omega_C) + 1 - g(C) \leq 0.$$

Since $\mathcal{E}^\vee \otimes \mathcal{H}$ is semipositive, we have $\deg \mathcal{E}^\vee \otimes \mathcal{H} \geq 0$. So $\deg \mathcal{E} \leq m$. Adding the above two inequalities we get $h^0(\mathcal{E}) \leq m$.

Now we assume that $r > 1$. We will show the following

Claim. Set $\mathcal{E}_0 = \mathcal{E}$. Suppose that $h^0(\mathcal{E}_0) > mr$. Then for $j = 1, \dots, r - 1$, there exist exact sequences of locally free sheaves on C ,

$$0 \rightarrow \mathcal{L}_j \rightarrow \mathcal{E}_{j-1} \xrightarrow{\alpha_j} \mathcal{E}_j \rightarrow 0,$$

with the following properties:

- (a) $\deg \mathcal{L}_j > 0$ for all j ;
- (b) $\sum_{i=1}^j \deg \mathcal{L}_i \leq jm$ for $j = 1, \dots, r - 1$;
- (c) $h^0(\mathcal{E}_j) > m(r - j)$ for $j = 1, \dots, r - 1$.

Admitting the Claim for the moment, we have

$$\begin{aligned} \deg \mathcal{E}_{i-1} &= \deg \mathcal{E}_i + \deg \mathcal{L}_i, \\ h^0(\mathcal{E}_i) + h^0(\mathcal{L}_i) &\geq h^0(\mathcal{E}_{i-1}) \end{aligned}$$

for $i = 1, \dots, r - 1$. Adding these equalities and inequalities, we get

$$\deg \mathcal{E}_0 + h^0(\mathcal{E}_{r-1}) + \sum_{i=1}^{r-1} h^0(\mathcal{L}_i) \geq \deg \mathcal{E}_{r-1} + \sum_{i=1}^{r-1} \deg \mathcal{L}_i + h^0(\mathcal{E}_0).$$

Note that for any invertible sheaf $\mathcal{L}$ of positive degree, we have $h^0(\mathcal{L}) \leq \deg \mathcal{L}$ (see (2.4.1)). Hence

$$\deg \mathcal{E}_0 \geq (\deg \mathcal{E}_{r-1} - h^0(\mathcal{E}_{r-1})) + \sum_{i=1}^{r-1} (\deg \mathcal{L}_i - h^0(\mathcal{L}_i)) + h^0(\mathcal{E}_0) \geq h^0(\mathcal{E}_0) > mr,$$

which contradicts the semi-positive property of $\mathcal{E}_0^\vee \otimes \mathcal{H}$.

It remains to prove the Claim. We proceed by induction on j . Let $p \in C$ be a point. Consider the exact sequence of sheaves

$$0 \rightarrow \mathcal{E}_0 \otimes \mathcal{O}_C(-p) \rightarrow \mathcal{E}_0 \rightarrow \mathcal{E}_0 \otimes \mathcal{O}_p \rightarrow 0.$$

Taking cohomology, we get $h^0(\mathcal{E}_0 \otimes \mathcal{O}_C(-p)) > (m - 1)r \geq 0$. So there is a non-zero section s in $H^0(\mathcal{E}_0 \otimes \mathcal{O}_C(-p))$. Then s induces an exact sequence of locally free sheaves on C ,

$$0 \rightarrow \mathcal{L}_1 \rightarrow \mathcal{E}_0 \xrightarrow{\alpha_1} \mathcal{E}_1 := \mathcal{E}_0 / \mathcal{L}_1 \rightarrow 0,$$

with $\deg \mathcal{L}_1 > 0$. Since $\mathcal{E}_0^\vee \otimes \mathcal{H}$ is semipositive, we have $\deg(\mathcal{L}_1^\vee \otimes \mathcal{H}) \geq 0$. So $\deg \mathcal{L}_1 \leq m$, and hence

$$h^0(\mathcal{E}_1) \geq h^0(\mathcal{E}_0) - h^0(\mathcal{L}_1) > mr - \deg \mathcal{L}_1 \geq m(r - 1).$$

Thus the Claim is true for $j = 1$.

Inductively, assume that for $j \leq k$, there exist exact sequences of locally free sheaves on C ,

$$0 \rightarrow \mathcal{L}_j \rightarrow \mathcal{E}_{j-1} \xrightarrow{\alpha_j} \mathcal{E}_j \rightarrow 0,$$

with the above properties (a)–(c). Since $h^0(\mathcal{E}_k) > m(r - k)$ (by (c)), by the same argument as above, there exists an exact sequence of locally free sheaves on C ,

$$0 \rightarrow \mathcal{L}_{k+1} \rightarrow \mathcal{E}_k \xrightarrow{\alpha_k} \mathcal{E}_{k+1} := \mathcal{E}_k / \mathcal{L}_{k+1} \rightarrow 0,$$

with $\deg \mathcal{L}_{k+1} > 0$. Set $\mathcal{F}_1 = \mathcal{L}_1$ and $\mathcal{F}_i = (\alpha_{i-1} \circ \cdots \circ \alpha_1)^{-1}(\mathcal{L}_i)$ for $2 \leq i \leq k+1$. Since $\mathcal{L}_i \subset \alpha_i^{-1}(\mathcal{L}_{i+1})$, we have exact sequences of locally free sheaves

$$0 \rightarrow \mathcal{F}_{i-1} \rightarrow \mathcal{F}_i \xrightarrow{\alpha_i \circ \cdots \circ \alpha_1|_{\mathcal{F}_i}} \mathcal{L}_i \rightarrow 0$$

for $i = 2, \dots, k+1$. So

$$\sum_{i=1}^{k+1} \deg \mathcal{L}_i = \sum_{i=2}^{k+1} (\deg \mathcal{F}_i - \deg \mathcal{F}_{i-1}) + \deg \mathcal{L}_1 = \deg \mathcal{F}_{k+1} \leq (k+1)m,$$

where the last inequality holds since $\mathcal{F}_{k+1}^\vee \otimes \mathcal{H}$ is a quotient of $\mathcal{E}^\vee \otimes \mathcal{H}$, which is semi-positive. We have $h^0(\mathcal{E}_i) \geq h^0(\mathcal{E}_{i-1}) - h^0(\mathcal{L}_i) \geq h^0(\mathcal{E}_{i-1}) - \deg(\mathcal{L}_i)$ for $i = 1, \dots, k+1$. Adding these inequalities, we get

$$h^0(\mathcal{E}_{k+1}) \geq h^0(\mathcal{E}_0) - \sum_{i=1}^{k+1} \deg \mathcal{L}_i > mr - m(k+1) = m(r - (k+1)).$$

So the Claim is true for $j = k+1$. This completes the proof of the Claim. $\square$

Lemma 2.5. *Notation as in subsection 2.3. We have $\text{rank } \mathcal{Q}^\vee \leq \frac{p_g(F)}{u}$.*

Proof. Set $s = \text{rank } \mathcal{F}$. By Lemma 2.4 (ii), we have

$$su \geq h^0(\mathcal{F}) \geq (u-1)p_g(F).$$

So we have $s \geq \frac{(u-1)p_g(F)}{u}$, and hence $\text{rank } \mathcal{Q}^\vee = p_g(F) - s \leq \frac{p_g(F)}{u}$. $\square$

Lemma 2.6. *Let $2 \leq u \leq \deg \pi$ be an integer. There exists an invertible subsheaf $\mathcal{L} \subset f_*\omega_X$ of degree at least*

$$\min\left\{\left(1 + \frac{u}{p_g(F)}\right)g(C) + u(u-1) - \frac{(p_g(F)-2)(u+1)}{2} - 2, 2g(C) - 2 - (p_g(F)-1)u\right\}.$$

Proof. We can assume that $h^1(f_*\omega_X) = 0$; indeed, if $h^1(f_*\omega_X) \neq 0$, by [Fu, Theorem 3.1], $\mathcal{O}_C$ is a direct summand of $f_*\omega_{X/C}$. So $\omega_C \subset f_*\omega_X$, and the lemma is trivially true in this case.

Set $t = \text{rank } \mathcal{Q}^\vee$. We distinguish two cases according to whether $t \geq 2$ or not.

Case 1. $t \geq 2$. Let $\mathcal{L}' \subset \mathcal{Q}^\vee$ be an invertible subsheaf of maximal degree. We have

$$\begin{aligned}
 \deg \mathcal{L}' &\geq \frac{1}{t} \deg \mathcal{Q}^\vee - \frac{t-1}{t} g(C) \quad (\text{by [Mu-Sa, Theorem]}) \\
 &= \frac{1}{t} (\deg f_* \omega_{X/C} - \deg \mathcal{F}^\vee \otimes \mathcal{O}_C(\sum_{i=1}^u p_i)) - \frac{t-1}{t} g(C) \quad (\text{by (2.3.1)}) \\
 &\geq \frac{1}{t} (\deg \mathcal{F} - (p_g(F) - t)u) - \frac{t-1}{t} g(C) \quad (f_* \omega_{X/C} \text{ is semipositive [Fu]}) \\
 &\geq \frac{1}{t} ((u-1)p_g(F) - (p_g(F) - t)(u+1)) - \frac{t-1}{t} g(C) \quad (\text{by Lemma 2.4 (i)}) \\
 &= \frac{1}{t} ((u-1)p_g(F) + g(C)) - \frac{(p_g(F) - t)}{t} (u+1) - g(C) \\
 &\geq \frac{u}{p_g(F)} ((u-1)p_g(F) + g(C)) - \frac{(p_g(F) - 2)}{2} (u+1) - g(C) \quad (\text{by (2.5)}).
 \end{aligned}$$

So $\mathcal{L} := \mathcal{L}' \otimes \omega_C$ is an invertible subsheaf of $f_* \omega_X$ of degree

$$\deg \mathcal{L} \geq (1 + \frac{u}{p_g(F)})g(C) + u(u-1) - \frac{(p_g(F) - 2)(u+1)}{2} - 2.$$

Case 2. $t \leq 1$. In this case, $\text{rank } \mathcal{F} \geq p_g(F) - 1$. So we can choose $p_g(F) - 1$ algebraically independent global sections of $(f_* \omega_{X/C})^\vee \otimes \mathcal{O}_C(\sum_{i=1}^u p_i)$, and let $\widetilde{\mathcal{F}}'$ be the subsheaf of $(f_* \omega_{X/C})^\vee \otimes \mathcal{O}_C(\sum_{i=1}^u p_i)$ generated by these sections, and $\widetilde{\mathcal{F}}$ the saturation of $\widetilde{\mathcal{F}}'$ in $(f_* \omega_{X/C})^\vee \otimes \mathcal{O}_C(\sum_{i=1}^u p_i)$. Then we have an exact sequence of locally free sheaves

$$0 \longrightarrow \widetilde{\mathcal{L}}' \longrightarrow f_* \omega_{X/C} \longrightarrow \widetilde{\mathcal{F}}^\vee \otimes \mathcal{O}_C(\sum_{i=1}^u p_i) \longrightarrow 0,$$

where $\widetilde{\mathcal{L}}'$ is an invertible sheaf by the choice of $\widetilde{\mathcal{F}}$. We have

$$\begin{aligned}
 \deg \widetilde{\mathcal{L}}' &= \deg f_* \omega_{X/C} - \deg \widetilde{\mathcal{F}}^\vee \otimes \mathcal{O}_C(\sum_{i=1}^u p_i) \\
 &= \deg f_* \omega_{X/C} + \deg \widetilde{\mathcal{F}} - (p_g(F) - 1)u \\
 &\geq -(p_g(F) - 1)u.
 \end{aligned}$$

So $\mathcal{L} := \widetilde{\mathcal{L}}' \otimes \omega_C$ is an invertible subsheaf of $f_* \omega_X$ of degree

$$\deg \mathcal{L} \geq 2g(C) - 2 - (p_g(F) - 1)u. \quad \square$$

Lemma 2.7. *Let C be a smooth projective curve of genus $g \geq 2$, and let $\mathcal{L}$ be an invertible sheaf of positive degree on C . If*

$$\deg \mathcal{L} > (1 + \frac{1}{a})(g - h^1(\mathcal{L}))$$

for some positive integer a , then $\deg \phi_{\mathcal{L}} \leq a$, where $\phi_{\mathcal{L}}$ is the rational map associated to $\mathcal{L}$.

Proof. Since $\deg \mathcal{L} > 0$, we have $h^1(\mathcal{L}) < g$. By the Riemann-Roch theorem and the assumption,

$$(2.7.1) \quad h^0(\mathcal{L}) - 1 = \deg \mathcal{L} - (g - h^1(\mathcal{L})) > \frac{1}{a+1} \deg \mathcal{L}.$$

Hence $h^0(\mathcal{L}) \geq 2$. Let d be the degree of the rational map $\phi_{\mathcal{L}} : C \rightarrow \mathbb{P}^{h^0(\mathcal{L})-1}$. We have

$$d(h^0(\mathcal{L}) - 1) = d(\operatorname{codim} \operatorname{Im} \phi_{\mathcal{L}} + 1) \leq d \deg \operatorname{Im} \phi_{\mathcal{L}} \leq \deg \mathcal{L}.$$

So

$$d \leq \frac{\deg \mathcal{L}}{h^0(\mathcal{L}) - 1} < a + 1,$$

where the last inequality follows by (2.7.1). $\square$

Proposition 2.8. *Notation as in subsection 2.1. Let u, a be integers with $2 \leq u \leq a$ and $ua > p_g(F)$. Assume that $g(C) > M(u, a, p_g(F))$, where we set $M(u, a, p_g(F))$ to be*

$$\max\left\{\frac{ap_g(F)}{ua - p_g(F)}\left(2 + \frac{(p_g(F) - 2)(u + 1)}{2} - u(u - 1)\right), \frac{a}{a - 1}(2 + (p_g(F) - 1)u)\right\}.$$

Then $\deg \pi \leq a$.

Proof. We can assume that $\deg \pi \geq u$; otherwise, the proposition is trivially true. By Lemma 2.6, there exists an invertible subsheaf $\mathcal{L} \subset f_*\omega_X$ of degree at least

$$\begin{aligned} \min\left\{\left(1 + \frac{u}{p_g(F)}\right)g(C) + u(u - 1) - \frac{(p_g(F) - 2)(u + 1)}{2} - 2, 2g(C) - 2 - (p_g(F) - 1)u\right\} \\ > \left(1 + \frac{1}{a}\right)g(C), \text{ provided } g(C) > M(u, a, p_g(F)). \end{aligned}$$

By Lemma 2.7, we get $\deg \phi_{\mathcal{L}} \leq a$.

On the other hand, since $f^*|\mathcal{L}| + (\text{some effective divisors})$ is a sublinear system of $|K_X|$, $\phi_{\mathcal{L}}$ factors through π , where π is as in subsection 2.1. This implies $\deg \pi \leq \deg \phi_{\mathcal{L}}$, and the result follows. $\square$

3. PROOF OF THEOREM 1.1

3.1. Notation as in Theorem 1.1. Let $d = \deg \phi_X$. By the Miyaoka-Yau inequality [M], we have

$$d(p_g(X) - 3) \leq d \deg \operatorname{Im} \phi_X \leq K_X^3 \leq 72\chi(\omega_X),$$

where the first inequality holds since $\operatorname{Im} \phi_X$ is a non-degenerate 3-fold in $\mathbb{P}^{p_g(X)-1}$. So

$$(3.1.1) \quad d \leq 72 \frac{\chi(\omega_X)}{p_g(X) - 3}.$$

For convenience, we set the right side of (3.1.1) by d_X .

If $q(X) \leq 1462$, then $\chi(\omega_X) \leq p_g(X) + 1461$. So $d_X < 73$ if $p_g(X) > 105411$ and the result follows in this case.

3.2. Now we can assume hereafter that $q(X) \geq 1463$. Consider the Albanese map $\operatorname{alb}_X : X \rightarrow \operatorname{Alb} X$. Let $f : X \rightarrow Y$ be the Stein factorization of alb_X . By Hacon's argument (see the proof of [Ha, Theorem 1.1]), one has

$$(3.2.1) \quad \chi(\omega_X) \leq p_g(X), \text{ if } \dim Y \geq 2;$$

$$(3.2.2) \quad \chi(\omega_X) \leq p_g(X) + \chi(\omega_Y) \text{ and } \chi(\omega_Y)p_g(F) \leq p_g(X), \text{ if } \dim Y = 1.$$

Hence if $\dim Y \geq 2$, by (3.2.1), $d_X < 73$ if $p_g(X) > 219$.

3.3. Now we assume that $C := Y$ is a curve (of genus $g(C) = q(X)$). Desingularizing X if necessary, we can assume that X is smooth. Let F be a general fiber of f . Then F is a surface of general type with $p_g(F) \geq 3$ (cf. subsection 2.1). By (3.2.2), we have

$$(3.3.1) \quad d_X \leq 72(1 + \frac{1}{p_g(F)}) \frac{p_g(X)}{p_g(X) - 3}.$$

If $p_g(F) \geq 73$, then $d_X \leq 72(1 + \frac{1}{73}) \frac{p_g(X)}{p_g(X) - 3} < 73$ if $p_g(X) > 15987$, and the result follows in this case. Hence we can assume that $p_g(F) \leq 72$.

Let r be the rank of the image of $H^0(f_*\omega_X) \otimes \mathcal{O}_C \rightarrow f_*\omega_X$. By [X, Proposition, p. 476], we have

$$(3.3.2) \quad (g(C) - 1)(p_g(F) - r) \leq r.$$

If $r < p_g(F)$, by (3.3.2) we have $g(C) \leq r + 1 \leq p_g(F) \leq 72$, which contradicts the assumption on $g(C)$ as in subsection 3.2. So we have $r = p_g(F)$. This implies that the restriction map $H^0(\omega_X) \rightarrow H^0(\omega_F)$ is surjective for the general fiber F of f . So the restriction of ϕ_X to F is just the canonical map ϕ_F of F .

Now let

$$\begin{array}{ccc} X & \xrightarrow{\phi_X} & \Sigma \\ f \downarrow & & \alpha \downarrow \\ C & \xrightarrow{\pi} & B \end{array}$$

be as in subsection 2.1. We have

$$(3.3.3) \quad d = \deg \phi_F \cdot \deg \pi.$$

Note that

$$(3.3.4) \quad \deg \phi_F \leq 9 + \frac{27}{p_g(F) - 2}$$

by the the Miyaoka-Yau inequality.

Let $M(u, a, p_g(F))$ be as in Proposition 2.8. We consider several cases according to $p_g(F)$.

If $3 \leq p_g(F) \leq 11$, by elementary calculations, we have

$$M(2, [\frac{p_g(F)}{2}] + 1, p_g(F)) \leq 297 < g(C),$$

where for rational number x , we define $[x]$ to be the largest integer $\leq x$. So the conditions of Proposition 2.8 are satisfied for $u = 2$, $a = [\frac{p_g(F)}{2}] + 1$ and $3 \leq p_g(F) \leq 11$. By Proposition 2.8 with $u = 2$ and $a = [\frac{p_g(F)}{2}] + 1$, we get

$$(3.3.5) \quad \deg \pi \leq [\frac{p_g(F)}{2}] + 1.$$

By (3.3.3), (3.3.4) and (3.3.5), we have $d \leq 72$.

Similarly, if $12 \leq p_g(F) \leq 23$ (resp. $24 \leq p_g(F) \leq 34$), by Proposition 2.8 with $u = 6$ and $a = [\frac{p_g(F)}{6}] + 4$ (resp. with $u = 7$ and $a = [\frac{p_g(F)}{7}] + 4$), we get $\deg \pi \leq [\frac{p_g(F)}{6}] + 4$ (resp. $\deg \pi \leq [\frac{p_g(F)}{7}] + 4$). Combining this with (3.3.3) and (3.3.4), we have $d \leq 72$.

Finally we assume that $35 \leq p_g(F) \leq 72$. In this case, we have

$$M(20, 36, p_g(F)) \leq M(20, 36, 72) = \frac{36 \cdot 1422}{35} < 1463.$$

Since $g(C) = q(X) \geq 1463$ by the assumption (3.2), the conditions of Proposition 2.8 are satisfied for $u = 20$, $a = 36$ and $35 \leq p_g(F) \leq 72$. By Proposition 2.8 with $u = 20$ and $a = 36$, we get

$$(3.3.6) \quad \deg \pi \leq 36.$$

By (3.3.1), we have $d_X \leq 72(1 + \frac{1}{35})\frac{p_g(X)}{p_g(X)-3} < 75$, provided that $p_g(X) > \frac{2625}{11}$. So $d \leq 74$. We show that $d = 74$ does not occur. Otherwise, by (3.3.3) and (3.3.4), we have either $\deg \phi_F = 1$ and $\deg \pi = 74$ or $\deg \phi_F = 2$ and $\deg \pi = 37$; in both cases we get a contradiction by (3.3.6). Similarly, $d = 73$ does not occur. This completes the proof of Theorem 1.1. $\square$

In the end of this section, we give some examples of 3-folds with high canonical degree.

Example 3.4. Let S be a smooth minimal surface with $\deg \phi_S = 16$ (see [Pe, Example 5.8] for such surfaces), and let C be a hyperelliptic curve of genus $g \geq 2$. Let $X = C \times S$. Then X is a smooth minimal 3-fold with $p_g(X) = 3g$ and $\deg \phi_X = 32$.

Remark 3.5. Let S be a smooth minimal surface of general type. If the canonical map ϕ_S of S is generically finite, then $\deg \phi_S \leq 36$, and the equality can only occur if $p_g(S) = 3$, $q(S) = 0$, $K_S^2 = 36$, and $|K_S|$ is base-point free (cf. (3.3.4)). A difficult problem is, does there exist a smooth surface with $p_g(S) = 3$, $q(S) = 0$, $K_S^2 = 36$, and $|K_S|$ being base-point free? An affirmative answer to this problem implies that there exist examples of 3-folds of arbitrary large p_g with $\deg \phi_X = 72$.

Example 3.6. Here are more examples of 3-folds of arbitrary large p_g with $\deg \phi_X = 32$.

Let $n \geq 3$ be an integer. Let C be a smooth curve of genus 3, and $\eta \in \text{Pic}^0 C$ a non-trivial 2-torsion point. Let $Y = C \times C \times \mathbb{P}^1$, and p_i ($i = 1, 2, 3$) the projections to its factors. Let $\delta = p_1^* \eta \otimes p_2^* \eta \otimes p_3^* \mathcal{O}_{\mathbb{P}^1}(n)$. Then the linear system $|2\delta| = |p_3^* \mathcal{O}_{\mathbb{P}^1}(2n)|$ is base-point free. Take $B \in |2\delta|$ be a smooth divisor. Let $\pi : X \rightarrow Y$ be the morphism corresponding to the double cover data (B, δ) . Then X is 3-fold with $p_g(X) = 4(n-1)$ and $K_X^3 = 192(n-2)$. We have $\pi_* \omega_X = \omega_Y \otimes \delta \oplus \omega_Y$. Since $p_g(Y) = 0$, $\phi_X = \phi_{\omega_Y \otimes \delta} \circ \pi$. Since $\omega_Y \otimes \delta = p_1^*(\omega_C \otimes \eta) \otimes p_2^*(\omega_C \otimes \eta) \otimes p_3^* \mathcal{O}_{\mathbb{P}^1}(n-2)$, we have that $\phi_{\omega_Y \otimes \delta}$ is the composition of the following morphisms:

$$\begin{array}{ccccc} C \times C \times \mathbb{P}^1 & \xrightarrow{\phi_{\omega_C \otimes \eta} \times \text{id}} & \mathbb{P}^1 \times C \times \mathbb{P}^1 & \xrightarrow{\text{id} \times \phi_{\omega_C \otimes \eta} \times \text{id}} & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\ & \searrow \text{id} \times \phi_{\mathcal{O}_{\mathbb{P}^1}(n-2)} & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\iota} & \mathbb{P}^{4n-5}, \end{array}$$

where ι is the Segre embedding. So $\deg \phi_{\omega_Y \otimes \delta} = 16$ and hence $\deg \phi_X = 32$.

Example 3.7 (3-folds of small p_g with $\deg \phi_X = 64$). Let S be a numerical Campedelli surface (i.e., a smooth minimal projective surface of general type with $p_g = q = 0$ and $K^2 = 2$) with $|2K_S|$ being base-point free (see e.g., [Pe] for such surfaces), and let C be a smooth curve of genus 3. Let $Y = C \times S$, and p, q the projections to C, S , respectively. Let $\delta = p^* \eta \otimes q^* \omega_S$, where $\eta \in \text{Pic}^0 C$ is a non-trivial 2-torsion point. Then the linear system $|2\delta| = |q^* \omega_S^{\otimes 2}|$ is base-point free. Take $B \in |2\delta|$ be a smooth divisor. Let $\pi : X \rightarrow Y$ be the morphism corresponding to the double cover data (B, δ) . Then X is 3-fold with $p_g(X) = 6$ and $K_X^3 = 192$. We have that $\pi_* \omega_X = \omega_Y \otimes \delta \oplus \omega_Y$. Since $p_g(Y) = 0$, $\phi_X = \phi_{\omega_Y \otimes \delta} \circ \pi$.

Since $\omega_Y \otimes \delta = p^*(\omega_C \otimes \eta) \otimes q^*\omega_S^{\otimes 2}$, we have that $\phi_{\omega_Y \otimes \delta}$ is the composition of the following morphisms:

$$C \times S \xrightarrow{\phi_{\omega_C \otimes \eta} \times \text{id}} \mathbb{P}^1 \times S \xrightarrow{\text{id} \times \phi_{\omega_S^{\otimes 2}}} \mathbb{P}^1 \times \mathbb{P}^2 \xrightarrow{\iota} \mathbb{P}^5,$$

where ι is the Segre embedding. So $\deg \phi_{\omega_Y \otimes \delta} = 32$ and hence $\deg \phi_X = 64$.

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