INFINITE INDEX SUBALGEBRAS OF DEPTH TWO

LARS KADISON

(Communicated by Martin Lorenz)

ABSTRACT. An algebra extension $A|B$ is right depth two in this paper if its tensor-square is A - B -isomorphic to a direct summand of any (not necessarily finite) direct sum of A with itself. For example, normal subgroups of infinite groups, infinitely generated Hopf-Galois extensions and infinite-dimensional algebras are depth two in this extended sense. The added generality loses some duality results obtained in the finite theory (Kadison and Szlachányi, 2003) but extends the main theorem of depth two theory, as for example in (Kadison and Nikshych, 2001). That is, a right depth two extension has right bialgebroid $T = (A \otimes_B A)^B$ over its centralizer $R = C_A(B)$. The main theorem: An extension $A|B$ is right depth two and right balanced if and only if $A|B$ is T -Galois with respect to left projective, right R -bialgebroid T .

1. INTRODUCTION

Bialgebroids arise as the endomorphisms of fiber functors from certain tensor categories to a bimodule category over a base algebra. For example, bialgebras are bialgebroids over a one-dimensional base algebra, while weak bialgebras are bialgebroids over a separable base algebra. Hopf algebroids are bialgebroids with antipodes: various twisted Hopf algebras are also Hopf algebroids over a one-dimensional base algebra.

Like bialgebras and their actions/coactions, bialgebroids also act and coact on noncommutative algebras in a more general setting suitable to mathematical physics [1]. Initially appearing in the analytic theory of subfactors, the notion of depth two has been widened to Frobenius extensions in [5] and to arbitrary subalgebras in [6]. As shown in [6] and later papers, depth two is a *Galois* theory of actions and coactions for bialgebroids. In this paper, we widen the definition of a depth two algebra extension in [6] to include Hopf H -Galois extensions where H is an infinite-dimensional Hopf algebra, such as the universal enveloping algebra of a Lie algebra or an infinite-dimensional group algebra. Although we lose the dual theory of finite projective, left and right bialgebroids over the centralizer in [6], we retain the right bialgebroid T and its role in coaction in [4]. We then obtain the main theorem of depth two Galois theory with no finiteness conditions (Theorem 4.1): an algebra extension $A|B$ is right depth two with A_B a balanced module if and only if $A|B$ is T -Galois with respect to a left projective right R -bialgebroid T , for some base ring R which commutes within A with the subring of coinvariants B .

Received by the editors December 1, 2006.

2000 *Mathematics Subject Classification.* Primary 16W30; Secondary 46L37, 81R50.

1.1. Depth two preliminaries. By algebra we mean a unital associative algebra over a commutative ring k , and by algebra extension $A|B$, we mean any identity-preserving algebra homomorphism $B \rightarrow A$, proper if $B \rightarrow A$ is monic. In either case, the natural bimodule ${}_BA_B$ and its properties define the properties of the extension from this point of view. For example, we say $A|B$ is right faithfully flat if A_B is faithfully flat, in which case one notes the extension $A|B$ is proper.

An algebra extension $A|B$ is *left depth two (D2)* if its tensor-square $A \otimes_B A$ as a natural B - A -bimodule is isomorphic to a direct summand of a direct sum of the natural B - A -bimodule A : equivalently, for some set I , we have

$$(1) \quad A \otimes_B A \oplus * \cong A^{(I)},$$

where $A^{(I)}$ denotes the coproduct (weak direct product, direct sum $\sum_{i \in I} A_i$, each $A_i = A$) of A with itself indexed by I and consists of elements $(a_i)_{i \in I}$ where $a_i \in A$ and $a_i = 0$ for all but finitely many indices (almost everywhere, a.e.). An extension $A|B$ is *right D2* if (1) holds instead as natural A - B -bimodules. An algebra extension is of course D2 if it is both left D2 and right D2.

For example, if $A|B$ is a projective algebra (so B is commutative, maps into the center of A and the module A_B is projective), then $A|B$ is D2, since $A_B \oplus * \cong B^{(I)}$ for index set I , so we may tensor this by $-\otimes_B A_A$ to obtain (1).

As another example, suppose H is a Hopf algebra of finite or infinite dimension over a field, and A is a right H -comodule algebra with B equal to the subalgebra of coinvariants. If $A|B$ is an H -Galois extension, then $A|B$ is right D2 since $A \otimes_B A \cong A \otimes H$ via the Galois A - B -isomorphism, $x \otimes y \mapsto xy_{(0)} \otimes y_{(1)}$, where $y_{(0)} \otimes y_{(1)}$ denote finite sums of elements equal to the value in $A \otimes H$ of the coaction on $y \in A$. Let I be in one-to-one correspondence with a basis for H . Then $A \otimes_B A \cong A^{(I)}$. If H has a bijective antipode, use the equivalent Galois B - A -bimodule isomorphism given by $x \otimes y \mapsto x_{(0)}y \otimes x_{(1)}$ to conclude that $A|B$ is left D2.

If the index set I is finite, then the algebra extension $A|B$ is right or left D2 in the earlier sense of [5, 6, 3, 4]. The lemma below notes that the earlier definition is recovered for any finitely generated (f.g.) extension.

Lemma 1.1. *If $A|B$ is right or left D2 and either of the natural modules ${}_BA$ or A_B is finitely generated, then I in (1) may be chosen to be finite.*

Proof. Suppose $A|B$ is right D2. If either ${}_BA$ or A_B is f.g., then ${}_AA \otimes_B A_B$ is f.g. It follows that ${}_AA \otimes_B A_B$ is isomorphic to a direct summand of a finite direct sum $A^n \subseteq A^{(I)}$. The argument is entirely similar starting with a left D2, left or right f.g. extension.

More explicitly using the A - B -epi f and A - B -monic g defined below, if $A \otimes_B A = Aw_1B + \cdots + Aw_NB$ for N elements $w_j \in A \otimes_B A$, then $g(w_j) = (a_{ij})_{i \in I}$ has finite support on $I_j \subset I$. Then $I' = I_1 \cup \cdots \cup I_N$ is finite, g corestricts, and f restricts to $A^{I'}$ so that $f \circ g = \text{id}_{A \otimes_B A}$. $\square$

In analogy with projective bases for projective modules, we similarly develop D2 quasibases for depth two extensions.

Proposition 1.2. *An algebra extension is right D2 if and only if there is an index set I and sets of elements $u_i = u_i^1 \otimes_B u_i^2 \in (A \otimes_B A)^B$, $\gamma_i \in \text{End } {}_BA_B$, both indexed*

by I , such that for each $a \in A$, $\gamma_i(a) = 0$ a.e. on I , and

$$(2) \quad x \otimes_B y = \sum_{i \in I} x \gamma_i(y) u_i^1 \otimes_B u_i^2$$

for all $x, y \in A$.

Proof. Let $\pi_i : A^{(I)} \rightarrow A$ and $\iota_i : A \rightarrow A^{(I)}$ be the usual projection and inclusion mappings of a coproduct, so that $\pi_j \circ \iota_i = \delta_{ij} \text{id}_A$ and $\sum_{i \in I} \iota_i \circ \pi_i = \text{id}$ on $A^{(I)}$.

Given a right D2 extension $A \mid B$, there is an A - B -split epimorphism $f : A^{(I)} \rightarrow A \otimes_B A$, say with section $g : A \otimes_B A \rightarrow A^{(I)}$. Then $f \circ g = \text{id}_{A \otimes_B A}$. Define $f_i = f \circ \iota_i \in \text{Hom}(A, A \otimes_B A)$ and define $g_i = \pi_i \circ g \in \text{Hom}(A \otimes_B A, A)$, both hom-groups of the natural A - B -bimodules. Then $\sum_{i \in I} f_i \circ g_i = \text{id}_{A \otimes_B A}$. But

$$(3) \quad \text{Hom}(A, A \otimes_B A) \cong (A \otimes_B A)^B$$

via $f \mapsto f(1_A)$, and

$$(4) \quad \text{Hom}(A \otimes_B A, A) \cong \text{End}_B A_B$$

via $F \mapsto F(1_A \otimes -)$ with inverse $\alpha \mapsto (x \otimes y \mapsto x\alpha(y))$. In this case, there are $\gamma_i \in \text{End}_B A_B$ such that $\gamma_i(a) = g_i(1 \otimes a)$, all $a \in A$, and $u_i \in (A \otimes_B A)^B$ such that $f_i(1_A) = u_i$, for each $i \in I$. Note that $\gamma_i(a) = 0$ a.e. on I , since $g_i(1 \otimes a) = \pi_i(g(1 \otimes a))$ is zero a.e. on I . It follows from $\text{id}_{A \otimes_B A} = \sum_{i \in I} f_i \circ g_i$ that

$$x \otimes y = \sum_{i \in I} x \gamma_i(y) u_i.$$

Conversely, given right D2 quasibases $\{\gamma_i\}_{i \in I}$, $\{u_i\}_{i \in I}$ as above, define an epimorphism $\pi : A^{(I)} \rightarrow A \otimes_B A$ of natural A - B -bimodules by

$$(5) \quad \pi[(a_i)] = \sum_{i \in I} a_i u_i$$

with A - B -bimodule section $\iota : A \otimes_B A \hookrightarrow A^{(I)}$, given by

$$(6) \quad \iota(x \otimes y) = (x \gamma_i(y))_{i \in I},$$

well defined in $A^{(I)}$ since for all $a \in A$, $\gamma_i(a) = 0$ a.e. on I . □

A similar proposition holds for a left D2 extension $A \mid B$ and left D2 quasibase $t_i \in (A \otimes_B A)^B$ and $\beta_i \in \text{End}_B A_B$ for each $i \in I$. In this case,

$$(7) \quad \sum_{i \in I} t_i^1 \otimes_B t_i^2 \beta_i(x) y = x \otimes_B y,$$

for all $x, y \in A$, which is equivalently expressed as $a \otimes 1 = \sum_i t_i \beta_i(a)$ for all $a \in A$, where again $\beta_i(a) = 0$ a.e. on the index set I . We fix our notation for right and left D2 quasibases throughout the paper. In addition, we denote $T = (A \otimes_B A)^B$ and (less importantly) $S = \text{End}_B A_B$.

For example, left and right D2 quasibases are obtained as follows for group algebras $A = k[G] \supseteq B = k[N]$ where G is a group, possibly of infinite order, N is a normal subgroup of possibly infinite index, and k is a commutative ring. Let $\{g_i\}_{i \in I}$ be a transversal of N in G . Define straightforwardly a projection onto the i th coset by $\gamma_i(a) = \sum_{j \in J} \lambda_{ij} g_i n_j$ where $a \in A$ and therefore of the form $a = \sum_{i \in I} \sum_{j \in J} \lambda_{ij} g_i n_j$, where J is an indexing set in one-to-one correspondence with N and the k -coefficients $\lambda_{ij} = 0$ a.e. on $I \times J$. In this case for any basis element $g \in G \hookrightarrow A$ all but one of the projections γ_i vanish on g : if g is in the

coset Ng_j , then $\gamma_j(g) = g$. Of course, the γ_i are B - B -bimodule projections since $gN = Ng$ for all $g \in G$. It is then easy to see that

$$(8) \quad 1 \otimes_B g = \sum_{i \in I} \gamma_i(g) g_i^{-1} \otimes_B g_i,$$

whence (2) follows by choosing $u_i = g_i^{-1} \otimes_B g_i$. Note that $u_i \in (A \otimes_B A)^B$ since $ng_i^{-1} \otimes_B g_i = g_i^{-1} \otimes_B g_i n$ for $n \in N$.

Similarly a left D2 quasibase is given by $\{\gamma_i\}$ and $\{g_i \otimes_B g_i^{-1}\}$ since $g_i N = Ng_i$.

We end this section with a proposition collecting various necessary conditions on a right depth two algebra extension.

Proposition 1.3. *Suppose an algebra extension $A | B$ is right D2 with centralizer R . Then the following is true:*

- (1) *for each two-sided ideal in A , $(I \cap R)A \subseteq A(I \cap R)$;*
- (2) *${}_B A$ is projective if $A | B$ is moreover a split extension;*
- (3) *for some indexing set I , $\text{End } {}_B A \oplus * \cong A^I$ as natural B - A -bimodules;*
- (4) *for each H -separable extension $B | C$, or equivalently an extension satisfying*

$$(9) \quad B \otimes_C B \oplus * \cong B^{(J)} \text{ natural } B\text{-}B\text{-bimodules}$$

for any index set J , the composite algebra extension $A | C$ is right D2.

Proof. The proof of each statement follows in the order above.

- (1) Given $x \in I \cap R$ and $a \in A$, apply (2) and a right D2 quasibase: $xa = \sum_i \gamma_i(a) u_i^1 x u_i^2$. Note that for each i , $u_i^1 x u_i^2 \in I \cap R$.
- (2) Given a B - B -bimodule projection $p : A \rightarrow B$, apply $p \otimes_B \text{id}_A$ to (2) with $x = 1$, obtaining $y = \sum_{i \in I} p(\gamma_i(y) u_i^1) u_i^2$ for all $y \in A$, which shows ${}_B A$ has dual bases.
- (3) Note that $\text{Hom}({}_A A \otimes_B A, {}_A A) \cong \text{End } {}_B A$ as A - A -bimodules via $F \mapsto F(1 \otimes -)$. Apply $\text{Hom}({}_A -, {}_A A)$ to ${}_A A \otimes_B A_B \oplus * \cong {}_A A_B^{(I)}$, noting that $\text{Hom}(A^{(I)}, A) \cong A^I$ (the direct product) as B - A -bimodules.
- (4) Apply the functor $A \otimes_B - \otimes_B A$ from B - B -bimodules into A - B -bimodules to the isomorphism (9). Then ${}_A A \otimes_C A_B \oplus * \cong {}_A A \otimes_B A_B^{(J)}$. Clearly $A \otimes_B A^{(J)} \oplus * \cong (A^{(I)})^{(J)} \cong A^{(I \times J)}$ as A - B -bimodules, whence the composite extension $A | C$ satisfies the right D2 condition

$$(10) \quad A \otimes_C A \oplus * \cong A^{(I \times J)}.$$

Finally, J in (9) may be replaced by the finite support of the image of $1 \otimes 1$ in $A^{(J)}$, under a split A - A -monomorphism $A \otimes_B A \rightarrow A^{(J)}$, whence an algebra extension satisfying (9) is H -separable [3].

□

Similar statements hold for a left D2 extension, one of which results in

Corollary 1.4. *If $A | B$ is D2, then the centralizer R is a normal subalgebra; i.e., for each two-sided ideal I in A , the contraction of I to R is A -invariant:*

$$(11) \quad A(I \cap R) = (I \cap R)A.$$

For example, any trivial extension $A | A$ is D2, in which case R is the center of A , which is of course a normal subalgebra.

2. THE BIALGEBROID T FOR A DEPTH TWO EXTENSION

In this section we establish that if $A \mid B$ is a right or left D2 algebra extension, then the construct $T = (A \otimes_B A)^B$, whose acquaintance we made in the last section, is a right bialgebroid over the centralizer $C_A(B) = R$. Moreover, T is right or left projective as a module over R according to which depth two condition, left or right, respectively, we assume.

Lemma 2.1. *Let T be equipped with the natural R - R -bimodule structure given by*

$$(12) \quad r \cdot t \cdot s = rt^1 \otimes_B t^2 s$$

for each $r, s \in R$ and $t \in T$. If $A \mid B$ is left D2 (right D2), then T is a projective right (left, resp.) R -module.

Proof. This follows from (7) by restricting to elements of $T \subseteq A \otimes_B A$. We obtain $t = \sum_i t_i \beta_i(t^1) t^2$ where $t_i \in T$. But $\beta_i(t^1) t^2 \in R$, so define elements $f_i \in \text{Hom}(T_R, R_R)$, indexed by I , by $f_i(t) = \beta_i(t^1) t^2$. Substitution yields $t = \sum_i t_i f_i(t)$, where $f_i(t) = 0$ a.e. on I , whence T_R is projective with dual basis $\{t_i\}, \{f_i\}$.

The proof that $A \mid B$ is right D2 implies ${}_R T$ is projective follows similarly from (2). $\square$

The next theorem may be viewed as a generalization of the first statement in [6, theorem 5.2].

Theorem 2.2. *If $A \mid B$ is right D2 or left D2, then T is a right bialgebroid over the centralizer R .*

Proof. The algebra structure on T comes from the isomorphism $T \cong \text{End}_A A \otimes_B A_A$ via

$$t \longmapsto (x \otimes_B y \mapsto xt^1 \otimes_B t^2 y)$$

with inverse $F \mapsto F(1 \otimes 1)$. The endomorphism algebra structure on T becomes

$$(13) \quad tu = u^1 t^1 \otimes_B t^2 u^2, \quad 1_T = 1_A \otimes_B 1_A.$$

It follows from this that there is an algebra homomorphism $s_R : R \rightarrow T$ and an algebra anti-homomorphism $t_R : R \rightarrow T$, satisfying a commutativity condition and inducing an R - R -bimodule from the right of T , given by $(r, s \in R, t \in T)$

$$(14) \quad s_R(r) = 1_A \otimes_B r,$$

$$(15) \quad t_R(s) = s \otimes_B 1_A,$$

$$(16) \quad s_R(r) t_R(s) = t_R(s) s_R(r) = r \otimes_B s,$$

$$(17) \quad t t_R(r) s_R(s) = r t^1 \otimes_B t^2 s.$$

Henceforth, the bimodule ${}_R T_R$ referred to is the one above, which is the same as the bimodule in (12).

An R -coring structure $(T, R, \Delta, \varepsilon)$ with comultiplication $\Delta : T \rightarrow T \otimes_R T$ and counit $\varepsilon : T \rightarrow R$ is given by

$$(18) \quad \Delta(t) = \sum_{i \in I} (t^1 \otimes_B \gamma_i(t^2)) \otimes_R u_i,$$

$$(19) \quad \varepsilon(t) = \sum t^1 t^2;$$

i.e., ε is the restriction of $\mu : A \otimes_B A \rightarrow A$ defined by $\mu(x \otimes y) = xy$ to $T \subseteq A \otimes_B A$. The coproduct Δ is well defined since for any given $t \in T$, there are only finitely

many nonzero terms on the right. It is immediate that Δ is left R -linear, ε is left and right R -linear, and

$$(\varepsilon \otimes_R \text{id}_T) \circ \Delta = \text{id}_T = (\text{id}_T \otimes_R \varepsilon) \circ \Delta$$

follows from variants of (2). We postpone the proof of coassociativity of Δ for one paragraph.

Additionally, note that the coproduct and counit are unit-preserving, $\varepsilon(1_T) = 1_A = 1_R$ and $\Delta(1_T) = 1_T \otimes_R 1_T$, since $\gamma_i(1_A) \in R$.

We employ the usual Sweedler notation $\Delta(t) = t_{(1)} \otimes_R t_{(2)}$. In order to show the bialgebroid identities

$$(20) \quad \Delta(tr) = t_{(1)} \otimes_R t_{(2)} r,$$

$$(21) \quad s_R(r) t_{(1)} \otimes_R t_{(2)} = t_{(1)} \otimes_R t_R(r) t_{(2)},$$

$$(22) \quad \Delta(tu) = t_{(1)} u_{(1)} \otimes_R t_{(2)} u_{(2)},$$

it will be useful to know that

$$(23) \quad T \otimes_R T \xrightarrow{\cong} (A \otimes_B A \otimes_B A)^B, \quad t \otimes_R u \mapsto t^1 \otimes t^2 u^1 \otimes u^2$$

with inverse

$$v \mapsto \sum_i (v^1 \otimes_B v^2 \gamma_i(v^3)) \otimes_R u_i.$$

Note that the LHS and RHS of (20) are the expressions $\sum_i (t^1 \otimes \gamma_i(t^2 r)) \otimes u_i$ and $\sum_i (t^1 \otimes \gamma_i(t^2)) \otimes u_i r$, which both map bijectively into $t^1 \otimes 1_A \otimes t^2 r$ in $(A \otimes_B A \otimes_B A)^B$, whence LHS = RHS indeed. Similarly, the LHS of (21) is $\sum_i (t^1 \otimes r \gamma_i(t^2)) \otimes_R u_i$ while the RHS is $\sum_i (t^1 \otimes \gamma_i(t^2)) \otimes_R (u_i^1 r \otimes u_i^2)$, both mapping into the same element, $t^1 \otimes_B r \otimes_B t^2$. Hence, this equation holds, giving meaning to the next equation for all $t, u \in T$ (the tensor product algebra over noncommutative rings ordinarily makes no sense, cf. [2]). Equation (22) holds because both expressions map isomorphically into the element $u^1 t^1 \otimes_B 1_A \otimes_B t^2 u^2$.

Finally the coproduct is coassociative, $(\Delta \otimes_R \text{id}_T) \circ \Delta = (\text{id}_T \otimes_R \Delta) \circ \Delta$, since we first note that

$$\begin{aligned} T \otimes_R T \otimes_R T &\xrightarrow{\cong} (A \otimes_B A \otimes_B A \otimes_B A)^B \\ t \otimes u \otimes v &\mapsto t^1 \otimes t^2 u^1 \otimes u^2 v^1 \otimes v^2. \end{aligned}$$

Secondly, $\sum_i \Delta(t^1 \otimes \gamma_i(t^2)) \otimes_R u_i$ maps into $t^1 \otimes 1 \otimes 1 \otimes t^2$, as does $\sum_i (t^1 \otimes \gamma_i(t^2)) \otimes_R \Delta(u_i)$, which establishes this, the last of the axioms of a right bialgebroid.

The proof that T is a right bialgebroid using a left D2 quasibase instead is very similar. $\square$

If A and B are commutative algebras where A is B -projective, then the bialgebroid T is just the tensor algebra $A \otimes_B A$, $R = A$, with Sweedler A -coring $\Delta(x \otimes y) = x \otimes 1 \otimes y$ and $\varepsilon = \mu$. This particular bialgebroid has an antipode $\tau : T \rightarrow T$ given by $\tau(x \otimes y) = y \otimes x$ (cf. [7, 9, 6]).

P. Xu [9] defines bialgebroid using an anchor map $T \rightarrow \text{End } R$ instead of the counit $\varepsilon : T \rightarrow R$. The anchor map is a right T -module algebra structure on R given by

$$(24) \quad r \triangleleft t = t^1 r t^2,$$

for $r \in R$, $t \in T$. We will study this and an extended right T -module algebra structure on $\text{End}_B A$ in the next section. The counit is the anchor map evaluated at 1_R , which is indeed the case above.

Remark 2.3. If I is a finite set, a D2 extension $A \mid B$ has a left bialgebroid structure on $S = \text{End}_B A_B$ such that A is a left S -module algebra, the left or right endomorphism algebras are smash products of A with S , and T is the R -dual bialgebroid of S [6]. In the proofs of these facts, most of the formulas in [6] do not make sense if I is an infinite set.

3. A RIGHT T -MODULE ENDOMORPHISM ALGEBRA

We continue in this section with a right depth two extension $A \mid B$ and our notation for $T = (A \otimes_B A)^B$, $R = C_A(B)$, left and right D2 quasibases $t_i, u_i \in T$, $\beta_i, \gamma_i \in S$ where $i \in I$, respectively, in an index set I of possibly infinite cardinality. Given any right R -bialgebroid T , recall that a right T -module algebra A is an algebra in the tensor category of right T -modules [2, 6].

Suppose ${}_A M$ is a left A -module. Let $\mathcal{E}$ denote its endomorphism ring as a module restricted to a B -module: $\mathcal{E} = \text{End}_B M$. There is a right action of T on $\mathcal{E}$ given by $f \triangleleft t = t^1 f(t^2 -)$ for $f \in \mathcal{E}$. This is a measuring action and $\mathcal{E}$ is a right T -module algebra (as defined in [6, 2]), since

$$(f \triangleleft t_{(1)}) \circ (g \triangleleft t_{(2)}) = \sum_i t^1 f(\gamma_i(t^2) u_i^1 g(u_i^2 -)) = (f \circ g) \triangleleft t,$$

and $1_{\mathcal{E}} \in \mathcal{E}^T$ is a T -invariant, since $\text{id}_M \triangleleft t = \text{id}_M \triangleleft s_R(\varepsilon(t))$. The subring of invariants $\mathcal{E}^T$ in $\mathcal{E}$ is $\text{End}_A M$ since $\text{End}_A M \subseteq \mathcal{E}^T$ is obvious, and $\phi \in \mathcal{E}^T$ satisfies for $m \in M, a \in A$:

$$\phi(am) = \sum_i \gamma_i(a)(\phi \triangleleft u_i)(m) = \sum_i \gamma_i(a) \varepsilon_T(u_i) \phi(m) = a \phi(m).$$

With similar arguments for a left D2 quasibase, we have established:

Theorem 3.1. *If $B \rightarrow A$ is right or left D2 and ${}_A M$ is a module, then $\text{End}_B M$ is a right T -module algebra with invariant subalgebra $\text{End}_A M$.*

By specializing $M = A$, we obtain

Corollary 3.2. *If $A \mid B$ is D2, then $\text{End}_B A$ is a right T -module algebra with invariant subalgebra $\rho(A)$ and right T -module subalgebra $\lambda(R)$.*

Proof. For any $r \in R$, we have $\lambda(r) \triangleleft t = \lambda(r \triangleleft t)$ w.r.t. the right action of T on R in (24) in the previous section. Of course, $\text{Hom}({}_A A, {}_A A) \cong \rho(A)$ where we fix the notation for right multiplication, $\rho(a)(x) = xa$ (all $a, x \in A$). $\square$

The right T -module $\text{End}_B A$ is identifiable with composition of endomorphism and homomorphism under the ring isomorphism $T \cong \text{End}_A A \otimes_B A_A$ and the A - A -bimodule isomorphism $\text{End}_B A \cong \text{Hom}({}_A A \otimes_B A, {}_A A)$ via $f \mapsto (x \otimes y \mapsto xf(y))$. We leave this remark as an exercise.

4. MAIN THEOREM CHARACTERIZING GALOIS EXTENSION

Given any right R -bialgebroid T , recall that a right T -comodule algebra A is an algebra in the category of right T -comodules [2]. If B denotes its subalgebra of coinvariants $A^{\text{co } T}$, which are the elements $\delta : x \mapsto x \otimes_R 1_T$ under the coaction, we say $A \mid B$ is right T -Galois if the canonical mapping $\beta : A \otimes_B A \rightarrow A \otimes_R T$ given by $\beta(x \otimes y) = xy_{(0)} \otimes y_{(1)}$ is bijective. Note that any $r \in R$ and $b \in B$ necessarily commute in A , since the coaction is monic and

$$\delta(rb) = \delta(r)\delta(b) = b \otimes_R s_R(r) = \delta(br).$$

Among other things, we show in the theorem that if $A \mid B$ is right depth two, then A is a right T -comodule algebra and the isomorphism $A \otimes_B A \cong A \otimes_R T$ projects to the Galois mapping via $A \otimes_B A \rightarrow A \otimes_{A^{\text{co } T}} A$. If moreover the natural module A_B is faithfully flat (apply to (28) below) or balanced, i.e., the map $\rho : B \rightarrow \text{End}_E A$ is surjective where $E = \text{End } A_B$, then $B = A^{\text{co } T}$.

Theorem 4.1. *An algebra extension $A \mid B$ is right D2 and right balanced if and only if $A \mid B$ is a right T -Galois extension for some left projective right bialgebroid T over some algebra R .*

Proof. ($\Leftarrow$) Since ${}_R T$ is projective, ${}_R T \oplus * \cong {}_R R^{(I)}$ for some set I . Then $A \otimes_R T \oplus * \cong A^{(I)}$ as A - B -bimodules, since R and B commute in A and $A \otimes_R R^{(I)} \cong (A \otimes_R R)^{(I)}$. But the Galois isomorphism $A \otimes_B A \xrightarrow{\cong} A \otimes_R T$, $\beta(x \otimes y) = xy_{(0)} \otimes y_{(1)}$ is an A - B -bimodule isomorphism; hence $A \mid B$ is right D2.

To see that the natural map $\rho : B \rightarrow \text{End}_E A$ is surjective, we let $F \in \text{End}_E A$. Then for each $a \in A$, left multiplication $\lambda_a \in E$, whence $F \circ \lambda_a = \lambda_a \circ F$. It follows that $F = \rho_x$ where $x = F(1)$. Since $B = A^{\text{co } T}$, it suffices to show that $x_{(0)} \otimes x_{(1)} = x \otimes 1$ under the coaction. For this we pause for a lemma.

Lemma. *Let R be an algebra with modules M_R and ${}_R V$ where V is projective with dual bases $w_i \in V$, $f_i \in \text{Hom}({}_R V, {}_R R) = {}^*V$ for some possibly infinite cardinality index set $i \in I$. If for some $m_j \in M$, $v_j \in V$ and finite index set J , we have $\sum_{j \in J} m_j \phi(v_j) = 0$ for each $\phi \in {}^*V$, then $\sum_{j \in J} m_j \otimes_R v_j = 0$. This statement follows of course by substitution of $\sum_{i \in I} f_i(v_j)w_i$ for each $j \in J$.*

To see that $x \otimes 1 - x_{(0)} \otimes x_{(1)} = 0$, we define for each $\nu \in \text{Hom}({}_R T, {}_R R)$, the right B -endomorphism $\bar{\nu} \in E$ by $\bar{\nu}(a) = a_{(0)}\nu(a_{(1)})$. Since also $\rho_r \in E$ for $r = \nu(1_T) \in R$, we compute:

$$x\nu(1_T) = \rho_{\nu(1_T)}F(1_A) = F(\bar{\nu}(1_A)) = \bar{\nu}(F(1_A)) = x_{(0)}\nu(x_{(1)}).$$

By the lemma then $x \in B$, so that A_B is a balanced module.

($\Rightarrow$) If $A \mid B$ is right D2, we have explicit formulas in the previous section for $T = (A \otimes_B A)^B$ as a left R -projective right bialgebroid over $R = C_A(B)$. Define a coaction $\delta : A \rightarrow A \otimes_R T$ by

$$(25) \quad \delta(a) = \sum_{i \in I} \gamma_i(a) \otimes_R u_i.$$

We claim that A is a right T -comodule algebra, an argument similar to [4, 5.1] but with infinite index set, and postpone a sketch of the proof for two paragraphs.

It is clear that $B \subseteq A^{\text{co } T}$, since $\gamma_i(b) = b\gamma_i(1)$ and $\sum_i \gamma(1)u_i = 1_T$. For the reverse inclusion, suppose $x \in A$ such that $x_{(0)} \otimes x_{(1)} = x \otimes 1_T$. Note the

isomorphism

$$(26) \quad A \otimes_R T \xrightarrow{\cong} A \otimes_B A$$

given by $a \otimes t \mapsto at^1 \otimes t^2$, with inverse

$$(27) \quad \beta(x \otimes y) = \sum_i x \gamma_i(y) \otimes_R u_i, \quad \beta : A \otimes_B A \xrightarrow{\cong} A \otimes_R T.$$

Since $\sum_i \gamma_i(x) \otimes u_i = x \otimes 1_T$, the image under β^{-1} is

$$(28) \quad 1_A \otimes_B x = x \otimes_B 1_A.$$

Given any $f \in E$, apply $\mu \circ (f \circ \lambda_a \otimes \text{id}_A)$ to this equation obtaining $f(a)x = f(ax)$ for each $a \in A$. Then $\rho_x \circ f = f \circ \rho_x$, whence $\rho_x \in \text{End}_E A$. Then $\rho_x \in \rho(B)$ since A_B is balanced. Hence $x \in B$.

The Galois condition on the algebra extension $A | B$ follows immediately from the fact that β in (27) is an isomorphism. Indeed using the isomorphism β^{-1} as an identification between $A \otimes_R T$ and $A \otimes_B A$ is the easiest way to show δ defines a right T -comodule structure on A .

The conditions that A must meet to be a right T -comodule algebra are

- (1) an algebra homomorphism $R \rightarrow A$;
- (2) a right T -comodule structure (A, δ) : δ is right R -linear, $a_{(0)}\varepsilon(a_{(1)}) = a$ for all $a \in A$, $(\text{id}_A \otimes \Delta) \circ \delta = (\delta \otimes \text{id}_T) \circ \delta$;
- (3) $\delta(1_A) = 1_A \otimes 1_T$;
- (4) for all $r \in R, a \in A$, $ra_{(0)} \otimes_R a_{(1)} = a_{(0)} \otimes_R t_R(r)a_{(1)}$;
- (5) $\delta(xy) = x_{(0)}y_{(0)} \otimes_R x_{(1)}y_{(1)}$ for all $x, y \in A$.

The following is a sketch of the proof, the details being left to the reader. For $R \rightarrow A$ we take the inclusion $C_A(B) \hookrightarrow A$. Note that $\delta(ar) = a_{(0)} \otimes_R a_{(1)}s_R(t)$ since both expressions map into $1 \otimes_B ar$ under $\beta^{-1} : A \otimes_R T \xrightarrow{\cong} A \otimes_B A$. Next we note that A is counital since $\sum_i \gamma_i(a)u_i^1 u_i^2 = a$. The coaction is coassociative on any $a \in A$ since both expressions map into $1_A \otimes 1_A \otimes a$ under the isomorphism

$$(29) \quad A \otimes_R T \otimes_R T \xrightarrow{\cong} A \otimes_B A \otimes_B A, \quad a \otimes t \otimes u \mapsto at^1 \otimes t^2 u^1 \otimes u^2.$$

The expressions in the last two items map bijectively via β^{-1} into $r \otimes a$ and $1 \otimes xy$ in $A \otimes_B A$, respectively, so the equalities hold. $\square$

The following by-product of the proof above is a characterization of right (similarly left) depth two in terms of T .

Corollary 4.2. *Let $A | B$ be an algebra extension with $T = (A \otimes_B A)^B$ and $R = C_A(B)$. The extension $A | B$ is right D2 if and only if $A \otimes_R T \cong A \otimes_B A$ via $a \otimes_R t \mapsto at^1 \otimes_B t^2$ and the module ${}_R T$ is projective.*

The main theorem is most interesting for subalgebras with small centralizers. An example of what can happen for large centralizers: the theorem shows that any field extension $K \supseteq F$ is T -Galois, since the underlying vector space of the F -algebra K is free, therefore balanced, and any algebra over a field is depth two. The bialgebroid T in this case is described after Theorem 2.2.

The paper [4] sketches how the main theorem in this paper would extend the main theorem in [5] for extensions with trivial centralizer as follows. We call an algebra extension $A | B$ semisimple-Hopf-Galois if H is a semisimple Hopf algebra, A is an H -comodule algebra with coinvariants B and the Galois mapping $A \otimes_B A \rightarrow A \otimes H$

is bijective [8]. Recall that an algebra extension $A \mid B$ is a Frobenius extension if A_B is f.g. projective and $A \cong \operatorname{Hom}(A_B, B_B)$ as natural B - A -bimodules. Left and right depth two are equivalent conditions on a Frobenius extension [6]. Recall too that an algebra extension $A \mid B$ is separable if the multiplication $\mu : A \otimes_B A \rightarrow A$ is a split A - A -epi.

Corollary 4.3. *Suppose $A \mid B$ is a Frobenius extension of k -algebras with trivial centralizer $R = 1_A \cdot k$ and k a field of characteristic zero. Then $A \mid B$ is semisimple-Hopf-Galois if and only if $A \mid B$ is a separable and depth two extension.*

Also, various pseudo-Galois and almost-Galois extensions over groups, Hopf algebras or weak Hopf algebras are depth two, balanced extensions, and so Galois extensions with respect to bialgebroids. The following corollary is an example using Hopf algebras, although the corollary may be stated more generally for bialgebroids by using the proof of $\Leftarrow$ above, which stays valid if the Galois mapping $\beta : A \otimes_B A \rightarrow A \otimes_R T$ is weakened from isomorphism to split A - B -monomorphism.

Corollary 4.4. *Suppose H is a Hopf algebra and $A \mid B$ is a right H -extension. If the Galois mapping β is a split A - B -monomorphism, then $A \mid B$ is a right $(A \otimes_B A)^B$ -Galois extension, where $(A \otimes_B A)^B$ is the bialgebroid T over $C_A(B)$ studied in Section 2.*

This corollary fits in with the current study of weak Hopf-Galois extensions in case the centralizer $C_A(B)$ is a separable algebra over a field, whence the bialgebroid T is a weak bialgebra [6, Prop. 7.4].

REFERENCES

- [1] G. Böhm and T. Brzeziński, Strong connections and the relative Chern-Galois character for corings, *Int. Math. Res. Not.* (2005), 2579–2625. MR2182707 (2006i:58011)
- [2] T. Brzeziński and R. Wisbauer, *Corings and Comodules*, London Math. Soc. **309**, Cambridge University Press, 2003. MR2012570 (2004k:16093)
- [3] L. Kadison, Hopf algebroids and H -separable extensions, *Proc. A.M.S.* **131** (2003), 2993–3002. MR1993204 (2004f:16068)
- [4] L. Kadison, Depth two and the Galois coring, *Contemp. Math* **391** (2005), A.M.S., 149–156. MR2184019 (2006h:16057)
- [5] L. Kadison and D. Nikshych, Hopf algebra actions on strongly separable extensions of depth two, *Adv. in Math.* **163** (2001), 258–286. MR1864835 (2003h:46098)
- [6] L. Kadison and K. Szlachányi, Bialgebroid actions on depth two extensions and duality, *Adv. in Math.* **179** (2003), 75–121. MR2004729 (2004i:16055)
- [7] J.-H. Lu, Hopf algebroids and quantum groupoids, *Int. J. Math.* **7** (1996), 47–70. MR1369905 (97a:16073)
- [8] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, CBMS Regional Conf. Series in Math. Vol. 82, AMS, Providence, 1993. MR1243637 (94i:16019)
- [9] Ping Xu, Quantum groupoids, *Commun. Math. Physics* **216** (2001), 539–581. MR1815717 (2002f:17033)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, 209 SOUTH 33RD STREET, PHILADELPHIA, PENNSYLVANIA 19104-6395

E-mail address: lkadison@math.upenn.edu