SPACES BETWEEN H^1 AND L^1

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ABSTRACT. In this paper we consider the spaces X_s that lie between $H^1(R^n)$ and $L^1(R^n)$. We discuss their interpolation properties and the behavior of maximal functions and singular integrals acting on them.

Because of their similarities, but mainly because of their differences, it is a matter of interest to determine the relationship between the Hardy space $H^1(R^n)$ and the space of integrable functions $L^1(R^n)$. The purpose of this paper is to gain a better understanding of the gap that separates them. We thus continue with the study of the spaces $X_s(R^n)$ that lie between H^1 and L^1 , and discuss their interpolation properties and the behavior of maximal functions and singular integrals acting on them.

The spaces X_s were introduced by Sweezy; see [10]. They form a nested family that starts at $H^1 = X_1$ and approaches L^1_0 , the subspace of L^1 functions with vanishing integral, as $s \rightarrow \infty$. Here we consider the whole range of X_s spaces. First, $X_s = H^1$ for $0 < s \leq 1$; also, $X_\infty = L^1_0$; see [1]. Further we show that, for $f \in X_s$,

$$K(t, f; H^1, L^1) \leq \min(t, t^{1/s'}) \|f\|_{X_s}.$$

This estimate allows us to interpolate between H^1 and L^1 . In particular, it gives the fact that X_s is continuously embedded in the Hardy-Lorentz space $H^{1,r}$ consisting of those distributions with non-tangential maximal function in the Lorentz space $L^{1,r}$, for $1 < s < r \leq \infty$. Therefore, the spaces $H^{1,s} \cap L^1$, $1 \leq s < \infty$, also form a nested family of subspaces of L^1 that increase from H^1 to L^1 . As for Calderón-Zygmund singular integral operators, they map X_s into $L^{1,r}$ for $1 < s < r \leq \infty$. We conclude the paper by introducing the closely related family of X^s spaces, $0 < s \leq \infty$, that increases towards L^1 , and then showing how X_s and X^s atoms can be used to build other spaces, including analogues of the local spaces considered in [7], that lie between H^1 and L^1 .

1. ATOMIC DECOMPOSITIONS IN BANACH SPACES

Our first result is of a general nature and will ensure that the various atomic spaces considered below are indeed Banach spaces.

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Let $\mathcal{A}$ be a non-empty subset of the unit ball of a Banach space $(B, \|\cdot\|)$. The atomic space A spanned by $\mathcal{A}$ consists of all those $\varphi \in B$ of the form

$$\varphi = \sum \lambda_j a_j, \quad \sum |\lambda_j| < \infty, a_j \in \mathcal{A}.$$

It is readily seen that, endowed with the atomic norm $\|\varphi\|_A = \inf \{ \sum_1^\infty |\lambda_j| : \varphi = \sum_1^\infty \lambda_j a_j \}$, A becomes a normed space. In fact, it is also complete.

Lemma 1.1. *The atomic space $(A, \|\cdot\|_A)$ is a Banach space.*

Proof. Since for $\varphi = \sum_1^\infty \lambda_j a_j$ we have $\|\varphi\| \leq \sum_1^\infty |\lambda_j|$, it readily follows that $\|\varphi\| \leq \|\varphi\|_A$, and A is continuously embedded in B .

To verify that $(A, \|\cdot\|_A)$ is complete, it suffices to show that if $\{\varphi_n\}$ is a sequence of elements in A such that $\sum_1^\infty \|\varphi_n\|_A < \infty$, then the sum converges in A , i.e., for some $\varphi \in A$, $\lim_{N \rightarrow \infty} \|\varphi - \sum_{n=1}^N \varphi_n\|_A = 0$. First observe that, since $\sum_1^\infty \|\varphi_n\| < \infty$, also the sum converges to some φ in B . We will show that $\varphi \in A$, and that the sum also converges to φ in A .

Let $\varphi_n = \sum_{j=1}^\infty \lambda_{j,n} a_{j,n}$, where the $\lambda_{j,n}$'s satisfy $\sum_{j=1}^\infty |\lambda_{j,n}| \leq 2\|\varphi_n\|_A$, $n = 1, 2, \dots$. Having fixed these decompositions, we may restrict our attention to the countable set of atoms $\{a_{j,n}\}$. So, we rename these elements $\{a_j\}$ and, by adding zeroes to the original $\lambda_{j,n}$'s as needed, we have $\varphi_n = \sum_{j=1}^\infty \lambda_{j,n} a_j$ for all n . Clearly $\sum_{n=1}^\infty \sum_{j=1}^\infty |\lambda_{j,n}| < \infty$. Moreover, if $\mu_j = \sum_{n=1}^\infty \lambda_{j,n}$, then $\sum_j |\mu_j| \leq \sum_{j=1}^\infty \sum_{n=1}^\infty |\lambda_{j,n}| < \infty$. Now, since $\varphi = \sum \varphi_n = \sum_{j=1}^\infty \mu_j a_j$, $\varphi \in A$. Finally, given $\varepsilon > 0$, let N_0 be such that $\sum_{n=N}^\infty \sum_{j=1}^\infty |\lambda_{j,n}| \leq \varepsilon$, for $N \geq N_0$. Then, for $N \geq N_0$,

$$\begin{aligned} \left\| \varphi - \sum_{j=1}^\infty \left(\sum_{n=1}^{N-1} \lambda_{j,n} \right) a_j \right\|_A &= \left\| \sum_{j=1}^\infty \left(\sum_{n=N}^\infty \lambda_{j,n} \right) a_j \right\|_A \\ &\leq \sum_{n=N}^\infty \sum_{j=1}^\infty |\lambda_{j,n}| \leq \varepsilon, \end{aligned}$$

and we have finished. $\square$

2. THE SPACES X_s

For $1 < q \leq 2$ with conjugate $2 \leq p < \infty$, $1/p + 1/q = 1$, and $0 < s \leq \infty$, we say that a compactly supported function a with vanishing integral is a (q, s) atom with defining cube Q if

$$\text{supp}(a) \subseteq Q, \quad \int_Q a(x) dx = 0, \quad p^{1/s} |Q| \left(\frac{1}{|Q|} \int_Q |a(x)|^q dx \right)^{1/q} \leq 1.$$

When $s = \infty$, a is a usual L^q 1-atom.

We denote by $\mathcal{A}_s$ the collection of (q, s) atoms, $0 < s \leq \infty$. Since for $a \in \mathcal{A}_s$ we have

$$\int_Q |a| \leq |Q| \left(\frac{1}{|Q|} \int_Q |a(x)|^q dx \right)^{1/q} \leq p^{-1/s} \leq 1,$$

$\mathcal{A}_s$ is contained in the unit ball of L^1 , and Lemma 1.1 applies. The resulting Banach space is $X_s(R^n) = X_s$, the space introduced by Sweezy, who also showed that $X_1 = H^1$; see [10].

As a first step in determining the relationship between the X_s 's for the different values of s we have

Lemma 2.1. *Suppose a is a (q, s) atom. Then,*

$$\|a\|_{X_u} \leq p^{1/u-1/s}, \quad 0 < u \leq \infty.$$

Proof. Since for a (q, s) atom a we have

$$p^{1/u}|Q| \left(\frac{1}{|Q|} \int_Q \left[\frac{|a(x)|}{p^{1/u-1/s}} \right]^q dx \right)^{1/q} \leq 1,$$

the conclusion follows readily. $\square$

From Lemma 2.1 we get $\|f\|_{X_s} \leq \|f\|_{X_r}$, $0 < r \leq s \leq \infty$, and the X_s 's are nested. Also, the norm in X_s reduces to the atomic H^1 norm for $0 < s < 1$.

Proposition 2.1. *Suppose $0 < s < 1$. Then $X_s = H^1$, with equivalent norms.*

Proof. We have already noted that $\|f\|_{H^1} \leq \|f\|_{X_s}$. Now let $f \in H^1$ have an atomic decomposition $f = \sum_j \lambda_j a_j$ in terms of L^∞ 1-atoms a_j . Then,

$$2^{1/s}|Q| \left(\frac{1}{|Q|} \int_Q \left[\frac{|a_j(x)|}{2^{1/s}} \right]^2 dx \right)^{1/2} \leq 1,$$

$a_j(x)/2^{1/s}$ is a $(2, s)$ atom, and $\|f\|_{X_s} \leq 2^{1/s} \sum_j |\lambda_j|$. Taking the infimum over all possible decompositions of f it follows that $\|f\|_{X_s} \leq 2^{1/s} \|f\|_{H^1}$, and we have finished. $\square$

The situation is different for $s > 1$. As the q_j 's approach 1, the p_j 's tend to ∞ , the sums escape H^1 , and $X_s \neq H^1$. In fact, X_s contains strictly X_r for $1 \leq r < s$, and, although X_r is densely embedded in X_s for $r < s$, it is of first category in $(X_s, \|\cdot\|_{X_s})$.

Concerning s large, let $X = \bigcup_{s < \infty} X_s$. We introduce a topology in X that is easier to deal with than the inductive topology there. For $f \in X$, let $\|f\|_X = \lim_{s \rightarrow \infty} \|f\|_{X_s}$. It is not hard to see that $\|\cdot\|_X$ is a norm. The homogeneity and triangle inequality follow easily. Moreover, if $\|f\|_X = 0$, $\|f\|_1 = 0$, and $f = 0$ a.e. In fact, for each s , the inclusion mapping is continuous from X_s to $(X, \|\cdot\|_X)$, and, consequently, $\varinjlim X_s$ is also continuously included in X . Similarly, $(X, \|\cdot\|_X)$ is continuously embedded in $(X_\infty, \|\cdot\|_{X_\infty})$, but X and X_∞ are not the same space. To see this note that X_r is of first category in X_∞ for $r < \infty$, and the same is true for X : if $\mathcal{U}$ is open in X_∞ , it cannot be open in any X_r , and hence it is not open in X . Finally, $X_\infty = L^1_0$; see [1].

2.1. X_s as an intermediate space between X_{s_1} and X_{s_2} . Recall that the K functional of $f \in X_{s_1} + X_{s_2}$ at $t > 0$ is defined by

$$K(t, f; X_{s_1}, X_{s_2}) = \inf_{f=f_1+f_2} \|f_1\|_{X_{s_1}} + t \|f_2\|_{X_{s_2}},$$

where $f = f_1 + f_2$, $f_1 \in X_{s_1}$ and $f_2 \in X_{s_2}$. We begin by estimating the K functional for $f \in X_s$, $1 \leq s_1 < s < s_2 \leq \infty$. The reader will have no difficulty in verifying that a similar result holds with X in place of X_∞ .

Lemma 2.2. *Given $1 \leq s_1 < s < s_2 \leq \infty$, let $0 < \eta < 1$ be given by $1/s = (1 - \eta)/s_1 + \eta/s_2$. Then, for $f \in X_s$,*

$$K(t, f; X_{s_1}, X_{s_2}) \leq \min(t, t^\eta) \|f\|_{X_s}.$$

Proof. Since $X_s \hookrightarrow X_{s_2}$, $K(t, f; X_{s_1}, X_{s_2}) \leq t \|f\|_{X_{s_2}} \leq t \|f\|_{X_s}$. This estimate suffices for t small.

Suppose now that t is large, $t > 1$, say, and let $\alpha > 0$ be given by $1/\alpha = 1/s_1 - 1/s_2$. Let $f \in X_s$ have an atomic decomposition $f = \sum_j \lambda_j a_j$ in terms of (q_j, s) atoms a_j , and let p_j denote the conjugate exponent to q_j . Finally, put $\mathcal{J}_1 = \{j : p_j \leq t^\alpha\}$ and $\mathcal{J}_2 = \{j : p_j > t^\alpha\}$. By Lemma 2.1 we have

$$\begin{aligned} \left\| \sum_{j \in \mathcal{J}_1} \lambda_j a_j \right\|_{X_{s_1}} &\leq \sum_{j \in \mathcal{J}_1} |\lambda_j| \|a_j\|_{X_{s_1}} \\ &\leq \sum_{j \in \mathcal{J}_1} |\lambda_j| p_j^{1/s_1 - 1/s} \leq t^{\alpha(1/s_1 - 1/s)} \sum_{j \in \mathcal{J}_1} |\lambda_j|, \\ \left\| \sum_{j \in \mathcal{J}_2} \lambda_j a_j \right\|_{X_{s_2}} &\leq \sum_{j \in \mathcal{J}_2} |\lambda_j| \|a_j\|_{X_{s_2}} \\ &\leq \sum_{j \in \mathcal{J}_2} |\lambda_j| p_j^{1/s_2 - 1/s} \leq t^{\alpha(1/s_2 - 1/s)} \sum_{j \in \mathcal{J}_2} |\lambda_j|. \end{aligned}$$

Now, since, as is readily seen, $\alpha(1/s_1 - 1/s) = \eta$ and $\alpha(1/s_2 - 1/s) = \eta - 1$, we get

$$K(t, f; X_{s_1}, X_{s_2}) \leq \left\| \sum_{j \in \mathcal{J}_1} \lambda_j a_j \right\|_{X_{s_1}} + t \left\| \sum_{j \in \mathcal{J}_2} \lambda_j a_j \right\|_{X_{s_2}} \leq t^\eta \sum_j |\lambda_j|.$$

Thus, taking the infimum over the decompositions of f in X_s , it follows that

$$K(t, f; X_{s_1}, X_{s_2}) \leq t^\eta \|f\|_{X_s}.$$

The conclusion now obtains by combining the estimates for t small and t large. $\square$

Corollary 2.1.1. *Let $1 < s < \infty$. Then, for $f \in X_s$,*

$$K(t, f; H^1, L^1) \leq \min(t, t^{1/s'}) \|f\|_{X_s}.$$

Proposition 2.2. *Let $f \in X_s$, $1 < s < \infty$. Then f is in the Hardy-Lorentz space $H^{1,r}$, $r > s$, and $\|f\|_{H^{1,r}} \leq c_r \|f\|_{X_s}$, $c_r = O(c/(1/s - 1/r))$.*

Proof. Let $f \in X_s$. We will show that the non-tangential maximal function Nf is in $L^{1,r}$ for $r > s$. Since the non-tangential maximal function of a function in H^1 is in L^1 , and that of a function in L^1 is in $L^{1,\infty}$, by elementary interpolation considerations and Corollary 2.1.1 it follows that

$$K(t, Nf; L^1, L^{1,\infty}) \leq c K(t, f; H^1, L^1) \leq c \min(t, t^{1/s'}) \|f\|_{X_s}.$$

Given $1 < s < r < \infty$, let $1/s' < \theta < 1$ be chosen so that $1/r = 1 - \theta$. Integrating the above inequality we get

$$\begin{aligned} \|Nf\|_{(L^1, L^{1,\infty})_{\theta,s}}^s &\leq c \left(\int_0^\infty \left(\frac{\min(t, t^{1/s'})}{t^\theta} \right)^s \frac{dt}{t} \right) \|f\|_{X_s}^s \\ &\leq c_r \|f\|_{X_s}^s. \end{aligned}$$

Clearly, $c_r \leq c/(1/s - 1/r)$. Furthermore, since $s < r$, we also have (see [2])

$$\|Nf\|_{1,r} \sim \|Nf\|_{(L^1, L^{1,\infty})_{\theta,r}} \leq c \|Nf\|_{(L^1, L^{1,\infty})_{\theta,s}},$$

and the conclusion follows by combining the two estimates. $\square$

Lemma 2.2 applies to Calderón-Zygmund singular integrals, and other operators, such as the Marcinkiewicz integral (see [4]) that map H^1 into L^1 , and L^1 into weak L^1 . Thus, these operators also map X_s into the Lorentz space $L^{1,r}$, for $1 < s < r \leq \infty$. It also applies to some Calderón-Zygmund singular integral operators with rough kernels that are known to be of weak-type $(1, 1)$ (see [8]) and to map H^1 into $L^{1,2}$ (see [9]). Lemma 2.2 then gives that they also map X_s into $L^{1,r}$ for $r > 2s$.

3. CONCLUDING REMARKS

In order to reach L^1 from H^1 , one more atom, this one with non-vanishing integral, needs to be added to the families $\mathcal{A}_s$; the characteristic function of Q_1 , the cube of sidelength 1 centered at the origin, will do. Let X^s denote the Banach space spanned by $\mathcal{A}_s \cup \chi_{Q_1}$, $1 \leq s \leq \infty$, and note that if $f \in X^s$ has an atomic decomposition of the form $f = \sum_j \lambda_j a_j + \lambda \chi_{Q_1}$, λ is uniquely determined and is equal to $\int_{\mathbb{R}^n} f$. Thus, for f in X^s we have $\|f\|_{X^s} = \inf \sum_j |\lambda_j| + |\lambda|$, where the infimum is taken over the atomic decompositions of f . In other words, $X^s = X_s + \text{sp}(\chi_{Q_1})$, in the sense of the sum of Banach spaces. Moreover, the family X^s is nested and reaches L^1 . Other than this important property, the family X^s behaves very much like X_s , and, consequently, we only give the description of its dual. It is the space BMO^s consisting of those functions $\varphi(x)$ such that

$$A(\varphi) = \sup_{p>1} \frac{1}{p^{1/s}} \sup_Q \left(\frac{1}{|Q|} \int_Q |\varphi(x) - \varphi_Q|^p dx \right)^{1/p} < \infty,$$

$$B(\varphi) = \left| \int_{Q_1} \varphi(x) dx \right| < \infty,$$

normed with $\|\varphi\|_{\text{BMO}^s} = \max(A(\varphi), B(\varphi))$, $1 \leq s < \infty$. We are grateful to the referee for pointing out that the Chang-Wilson-Wolff inequality (see [3]) provides a sufficient condition for a function to be in BMO^2 . Of course, the dual of X^∞ is L^∞ ; see [1].

One can also define spaces analogous to the local version of H^1 spaces at the origin introduced in [7], with the H^1 atoms there replaced by X_s atoms. Let Q_δ denote the cube of sidelength δ centered at the origin. The family $\mathcal{C}_s$ of central (q, s) atoms consists of those $a \in \mathcal{A}_s$ with defining cube Q_δ for some $\delta > 0$. The atomic space generated by $\mathcal{C}_s$ is denoted HX_s , for $1 \leq s \leq \infty$.

HX_1 is a dense subset of H^1 which is embedded continuously in H^1 and, as we will see below, it is not H^1 . The HX_s 's form a nested family of subspaces of L_0^1 and, for each s , HX_s is a dense subset of X_s continuously embedded in X_s .

The dual of HX_s is the space $\text{CMO}(s)$ which consists of those functions φ such that

$$\|\varphi\|_{\text{CMO}(s)} = \sup_{p>1} \frac{1}{p^{1/s}} \sup_{\delta>0} \left(\frac{1}{|Q_\delta|} \int_{Q_\delta} |\varphi(x) - \varphi_{Q_\delta}|^p dx \right)^{1/p} < \infty, \quad s < \infty,$$

$$\|\varphi\|_{\text{CMO}(\infty)} = \sup_{p>1} \sup_{\delta>0} \left(\frac{1}{|Q_\delta|} \int_{Q_\delta} |\varphi(x) - \varphi_{Q_\delta}|^p dx \right)^{1/p} < \infty, \quad s = \infty.$$

Now, $\text{CMO}(1)$ strictly contains BMO . To see this consider, for $n = 1$,

$$\varphi(x) = \begin{cases} \ln|x+2|, & -\infty < x < -2, \\ 0, & -2 \leq x \leq -1, \\ \ln|x+1|, & -1 < x < \infty. \end{cases}$$

φ is in $\text{CMO}(1)$ but not in BMO , which shows that, unlike X_1 , HX_1 is not H^1 . The space $\text{CMO}(\infty)$ coincides with $L^{\infty,*}$, the dual of L_0^1 , as a set of functions, but with a weaker norm.

Finally, let HX^s denote the Banach space spanned by $\mathcal{C}_s \cup \chi_{Q_1}$. The HX^s 's form an increasing family of subspaces of L^1 , and HX^s contains strictly HX_s . HX^∞ is close to L^1 , so the HX^s 's cover the gap left by the HX_s 's. The dual of HX^s is CMO^s , which consists of all functions φ in $\text{CMO}(s)$, but normed with $\|\varphi\|_{\text{CMO}^s} = \|\varphi\|_{\text{CMO}(s)} + \left| \int_{Q_1} \varphi(x) dx \right|$, $1 \leq s < \infty$.

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