A COUNTEREXAMPLE FOR BOUNDEDNESS OF PSEUDO-DIFFERENTIAL OPERATORS ON MODULATION SPACES

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ABSTRACT. We prove that pseudo-differential operators with symbols in the class $S_{1,\delta}^0$ ($0 < \delta < 1$) are not always bounded on the modulation space $M^{p,q}$ ($q \neq 2$).

1. INTRODUCTION

In the 1980s, by Feichtinger [2, 3], the modulation spaces $M^{p,q}$ were introduced as a fundamental function space of time-frequency analysis, which originated in signal analysis or quantum mechanics. See also [4] or Triebel [12]. The exact definition will be given in the next section, but the main idea is to consider the decaying property of a function with respect to the space variable and the variable of its Fourier transform simultaneously. Based on the same idea, Sjöstrand [9] independently introduced a symbol class which assures the L^2 -boundedness of corresponding pseudo-differential operators. Since this pioneering work, the modulation spaces have also been recognized as a useful tool for the theory of pseudo-differential operators (see Gröchenig [6]).

In this paper, we investigate the boundedness property of pseudo-differential operators with symbols in $S_{\rho,\delta}^m$ on the modulation spaces $M^{p,q}$. There have already been several papers on this subject. For example, Gröchenig and Heil [7], Tachizawa [10], Toft [11] proved that pseudo-differential operators with symbols in $S_{0,0}^0$ are $M^{p,q}$ -bounded. On the other hand, Calderón and Vaillancourt [1] proved that pseudo-differential operators with symbols in $S_{\delta,\delta}^0$ with $0 < \delta < 1$ (hence $S_{1,\delta}^0$) are L^2 -bounded (hence $M^{2,2}$ -bounded) by reducing them to the case of $S_{0,0}^0$. In view of these results, the class $S_{1,\delta}^0$ with $0 < \delta < 1$ appears to induce the $M^{p,q}$ -boundedness as well. The objective of this paper is to show that this is not true:

Theorem 1.1. *Let $1 < p, q < \infty$, $m \in \mathbb{R}$ and $0 < \delta < 1$. If $m > -|1/q - 1/2|\delta n$, then there exists a symbol $\sigma \in S_{1,\delta}^m$ such that $\sigma(X, D)$ is not bounded on $M^{p,q}(\mathbb{R}^n)$.*

In particular, Theorem 1.1 actually says that symbols in the class $S_{1,\delta}^0$ ($0 < \delta < 1$) do not always induce the $M^{p,q}$ -boundedness in the case $q \neq 2$. We will prove this fact by constructing a counterexample.

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We remark that we can admit the endpoints for p and q in the statement of Theorem 1.1. For example, if we assume that symbols in the class $S_{1,\delta}^m$ with $m > -\delta n/2$ induce the $M^{1,1}$ -boundedness, then those with $m > -|1/q - 1/2|\delta n$ induce the $M^{q,q}$ -boundedness ($1 < q < 2$) by interpolating it with the boundedness of the class $S_{1,\delta}^0$ on $M^{2,2} = L^2$. This contradicts Theorem 1.1 with $1 < p = q < 2$; hence Theorem 1.1 is still true for $p = q = 1$. This remark is due to Professor Hans G. Feichtinger.

2. MAIN RESULT

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(\xi) d\xi.$$

Let $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$. The symbol class $S_{\rho,\delta}^m$ consists of all $\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$|\partial_\xi^\alpha \partial_x^\beta \sigma(x, \xi)| \leq C_{\alpha,\beta} (1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|}$$

for all $\alpha, \beta \in \mathbb{Z}_+^n = \{0, 1, 2, \dots\}^n$. For $\sigma \in S_{\rho,\delta}^m$, we define the pseudo-differential operator $\sigma(X, D)$ by

$$\sigma(X, D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(x, \xi) \widehat{f}(\xi) d\xi$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. Given a symbol $\sigma \in S_{\rho,\delta}^m$ with $\delta < 1$, the symbol σ^* defined by

$$\begin{aligned} (2.1) \quad \sigma^*(x, \xi) &= \text{Os-} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-iy \cdot \zeta} \overline{\sigma(x+y, \xi+\zeta)} dy d\zeta \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-iy \cdot \zeta} \chi(\epsilon y, \epsilon \zeta) \overline{\sigma(x+y, \xi+\zeta)} dy d\zeta \end{aligned}$$

satisfies $\sigma^* \in S_{\rho,\delta}^m$ and

$$(2.2) \quad (\sigma(X, D)f, g) = (f, \sigma^*(X, D)g) \quad \text{for all } f, g \in \mathcal{S}(\mathbb{R}^n),$$

where $\chi \in \mathcal{S}(\mathbb{R}^{2n})$ satisfies $\chi(0, 0) = 1$ and $(\cdot, \cdot)$ denotes the inner product on $L^2(\mathbb{R}^n)$ ([8, Chapter 2, Theorem 2.6]). Note that oscillatory integrals are independent of the choice of $\chi \in \mathcal{S}(\mathbb{R}^{2n})$ satisfying $\chi(0, 0) = 1$ ([8, Chapter 1, Theorem 6.4]).

We introduce the modulation spaces based on Gröchenig [5]. Fix a function $\gamma \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ (called a window function). Then the short-time Fourier transform $V_\gamma f$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ with respect to γ is defined by

$$V_\gamma f(x, \xi) = (f, M_\xi T_x \gamma) \quad \text{for } x, \xi \in \mathbb{R}^n,$$

where $M_\xi \gamma(t) = e^{i\xi \cdot t} \gamma(t)$ and $T_x \gamma(t) = \gamma(t - x)$. We can express this in a form of the integral

$$V_\gamma f(x, \xi) = \int_{\mathbb{R}^n} f(t) \overline{\gamma(t - x)} e^{-i\xi \cdot t} dt,$$

which actually converges for appropriate functions f on $\mathbb{R}^n$, say $f \in L^p(\mathbb{R}^n)$.

We note that, for $f \in \mathcal{S}'(\mathbb{R}^n)$, $V_\gamma f$ is continuous on $\mathbb{R}^{2n}$ and $|V_\gamma f(x, \xi)| \leq C(1 + |x| + |\xi|)^N$ for some constants $C, N \geq 0$ ([5, Theorem 11.2.3]). Let $1 \leq p, q \leq \infty$. Then the modulation space $M^{p,q}(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{M^{p,q}} = \|V_\gamma f\|_{L^{p,q}} = \left\{ \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |V_\gamma f(x, \xi)|^p dx \right)^{q/p} d\xi \right\}^{1/q} < \infty.$$

We note that $M^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ ([5, Proposition 11.3.1]) and $M^{p,q}(\mathbb{R}^n)$ is a Banach space ([5, Proposition 11.3.5]). The definition of $M^{p,q}(\mathbb{R}^n)$ is independent of the choice of the window function $\gamma \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$; that is, different window functions yield equivalent norms ([5, Proposition 11.3.2]).

We also introduce a special symbol which will act as the counterexample for the boundedness stated in the Introduction. Let $\varphi, \eta \in \mathcal{S}(\mathbb{R}^n)$ be real-valued functions satisfying

$$\begin{aligned} \varphi : \quad & \text{supp } \varphi \subset \{\xi : |\xi| \leq 1/8\}, \quad \int_{\mathbb{R}^n} \varphi(\xi) d\xi = 1, \\ \eta : \quad & \text{supp } \eta \subset \{\xi : 2^{-1/2} \leq |\xi| \leq 2^{1/2}\}, \quad \eta = 1 \text{ on } \{\xi : 2^{-1/4} \leq |\xi| \leq 2^{1/4}\}. \end{aligned}$$

Moreover, we assume that φ is radial. Then we define the symbol σ_δ by

$$(2.3) \quad \sigma_\delta(x, \xi) = \sum_{j=j_0}^{\infty} 2^{jm} \left(\sum_{0 < |k| \leq 2^{j\delta/2}} e^{-ik \cdot (2^{j\delta/2} x - k)} \Phi(2^{j\delta/2} x - k) \right) \eta(2^{-j}\xi),$$

where $\Phi = \mathcal{F}^{-1}\varphi$, $0 < \delta < 1$ and $j_0 \in \mathbb{Z}_+$ is chosen to satisfy

$$1 + 2^{j_0(\delta-1)+1} \leq 2^{1/4}, \quad 1 - 2^{j_0(\delta-1)+1} \geq 2^{-1/4}, \quad 2^{-j_0\delta/2} \sqrt{n} \leq 2^{-3}.$$

The symbol σ_δ^* is constructed from σ_δ using the oscillatory integral (2.1).

Now, we state our main result, which is a precise version of Theorem 1.1 in the Introduction.

Theorem 2.1. *Let $1 < p, q < \infty$, $0 < \delta < 1$ and $m > -|1/q - 1/2|\delta n$. Then the symbols σ_δ and σ_δ^* defined by (2.3) belong to the class $S_{1,\delta}^m$. Moreover, if $q \geq 2$ ($q \leq 2$ resp.), then the corresponding operator $\sigma_\delta(X, D)$ ($\sigma_\delta^*(X, D)$ resp.) is not bounded on $M^{p,q}(\mathbb{R}^n)$.*

The proof of Theorem 2.1 will be given in the next section.

3. PROOF

In what follows, we consider the symbol τ_δ instead of σ_δ for the sake of simplicity. In order to avoid confusion, we repeat the notation in this context, and also introduce a family of functions $\{f_{j,\epsilon,\delta}\}_j$. Let $\varphi, \psi, \eta \in \mathcal{S}(\mathbb{R}^n)$ be real-valued functions satisfying

$$\begin{aligned} \varphi : \quad & \text{supp } \varphi \subset \{\xi : |\xi| \leq 1/8\}, \quad \int_{\mathbb{R}^n} \varphi(\xi) d\xi = 1, \\ \psi : \quad & \text{supp } \psi \subset \{\xi : |\xi| \leq 1/2\}, \quad \psi = 1 \text{ on } \{\xi : |\xi| \leq 1/4\}, \\ \eta : \quad & \text{supp } \eta \subset \{\xi : 2^{-1/2} \leq |\xi| \leq 2^{1/2}\}, \quad \eta = 1 \text{ on } \{\xi : 2^{-1/4} \leq |\xi| \leq 2^{1/4}\}. \end{aligned}$$

Moreover, we assume that φ and ψ are radial. This assumption implies that Φ and Ψ are also real-valued functions, where $\Phi = \mathcal{F}^{-1}\varphi$ and $\Psi = \mathcal{F}^{-1}\psi$. For these Φ, Ψ, η , we define the symbol τ_δ and the functions $f_{j,\epsilon,\delta}$ by

$$(3.1) \quad \tau_\delta(x, \xi) = \sum_{j=j_0}^{\infty} 2^{jm} \left(\sum_{0 < |k| \leq 2^{j\delta}} e^{-ik \cdot (2^{j\delta}x - k)} \Phi(2^{j\delta}x - k) \right) \eta(2^{-j}\xi)$$

and

$$(3.2) \quad f_{j,\epsilon,\delta}(x) = \sum_{0 < |k'| \leq 2^{j\delta}} |k'|^{-n/q-\epsilon} e^{ik' \cdot (x-k')} \Psi(x - k'),$$

where $k, k' \in \mathbb{Z}^n$, $\epsilon > 0$, $0 < \delta < 1/2$, and $j_0 \in \mathbb{Z}_+$ is chosen to satisfy

$$1 + 2^{j_0(2\delta-1)+1} \leq 2^{1/4}, \quad 1 - 2^{j_0(2\delta-1)+1} \geq 2^{-1/4}, \quad 2^{-j_0\delta} \sqrt{n} \leq 2^{-3}.$$

Note that $\sigma_{2\delta}(x, \xi) = \tau_\delta(x, \xi)$.

Lemma 3.1. *The symbol τ_δ defined by (3.1) belongs to $S_{1,2\delta}^m$.*

Proof. Since $\text{supp } \eta(2^{-j}\cdot) \subset \{2^{j-1/2} \leq |\xi| \leq 2^{j+1/2}\}$, we see that, for each $\xi \in \mathbb{R}^n$, at most one term in the sum (3.1) is nonzero with respect to j . Note that $2^j \sim |\xi| \sim 1 + |\xi|$ on $\text{supp } \eta(2^{-j}\cdot)$. Let $\alpha, \beta \in \mathbb{Z}_+^n$ and $\xi \in \text{supp } \eta(2^{-j}\cdot)$. Using

$$\begin{aligned} \partial_\xi^\alpha \partial_x^\beta \tau_\delta(x, \xi) &= 2^{jm} \sum_{0 < |k| \leq 2^{j\delta}} \sum_{\beta_1 + \beta_2 = \beta} C_{\beta_1, \beta_2} (2^{j\delta}k)^{\beta_1} e^{-ik \cdot (2^{j\delta}x - k)} \\ &\quad \times 2^{j\delta|\beta_2|} (\partial^{\beta_2} \Phi)(2^{j\delta}x - k) 2^{-j|\alpha|} (\partial^\alpha \eta)(2^{-j}\xi), \end{aligned}$$

we have

$$\begin{aligned} |\partial_\xi^\alpha \partial_x^\beta \tau_\delta(x, \xi)| &\leq C 2^{j(m-|\alpha|)} \|\partial^\alpha \eta\|_{L^\infty} \left(\sum_{0 < |k| \leq 2^{j\delta}} (1 + |2^{j\delta}x - k|)^{-n-1} \right) \\ &\quad \times \left(\sum_{\beta_1 + \beta_2 = \beta} 2^{j\delta(2|\beta_1| + |\beta_2|)} \|(1 + |\cdot|)^{n+1} (\partial^{\beta_2} \Phi)\|_{L^\infty} \right) \\ &\leq C 2^{j(m-|\alpha|+2\delta|\beta|)} \left(\sup_{y \in \mathbb{R}^n} \sum_{k \in \mathbb{Z}^n} (1 + |y - k|)^{-n-1} \right) \\ &\leq C (1 + |\xi|)^{m-|\alpha|+2\delta|\beta|}. \end{aligned}$$

In the case $\xi \notin \bigcup_{j \geq j_0} \text{supp } \eta(2^{-j}\cdot)$, we have nothing to prove. $\square$

Lemma 3.2. *Let $1 < p, q < \infty$, and let $f_{j,\epsilon,\delta}$ be defined by (3.2). Then the following are true:*

(1) *The Fourier transform of $e^{i2^{j(1-\delta)}x_1} f_{j,\epsilon,\delta}(x)$ is*

$$\mathcal{F}[M_{2^{j(1-\delta)}e_1} f_{j,\epsilon,\delta}](\xi) = \sum_{0 < |k'| \leq 2^{j\delta}} |k'|^{-n/q-\epsilon} e^{-ik' \cdot (\xi - 2^{j(1-\delta)}e_1)} \psi(\xi - 2^{j(1-\delta)}e_1 - k'),$$

where $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$.

(2) *There exists a constant $C > 0$ such that*

$$\|f_{j,\epsilon,\delta}(2^{j\delta}\cdot)\|_{M^{p,q}} \leq C 2^{j\delta n(1/q-1)} \quad \text{for all } j \geq j_0.$$

Proof. We consider only (2). Let $\gamma \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$. Since

$$\begin{aligned} & V_\gamma[f_{j,\epsilon,\delta}(2^{j\delta}\cdot)](2^{-j\delta}x, 2^{j\delta}\xi) \\ &= \sum_{0 < |k'| \leq 2^{j\delta}} |k'|^{-n/q-\epsilon} e^{-i|k'|^2} \int_{\mathbb{R}^n} e^{ik'\cdot(2^{j\delta}t)} \Psi(2^{j\delta}t - k') \overline{\gamma(t - 2^{-j\delta}x)} e^{-i(2^{j\delta}\xi)\cdot t} dt \\ &= 2^{-j\delta n} \sum_{0 < |k'| \leq 2^{j\delta}} |k'|^{-n/q-\epsilon} e^{-i|k'|^2} \int_{\mathbb{R}^n} e^{-i(\xi-k')\cdot t} \Psi(t - k') \overline{\gamma(2^{-j\delta}(t-x))} dt \\ &= 2^{-j\delta n} \sum_{0 < |k'| \leq 2^{j\delta}} |k'|^{-n/q-\epsilon} e^{-i|k'|^2} \\ &\quad \times \int_{\mathbb{R}^n} \left\{ (1 + |\xi - k'|^2)^{-n} (I - \Delta_t)^n e^{-i(\xi-k')\cdot t} \right\} \Psi(t - k') \overline{\gamma(2^{-j\delta}(t-x))} dt, \end{aligned}$$

we have

$$\begin{aligned} |V_\gamma[f_{j,\epsilon,\delta}(2^{j\delta}\cdot)](2^{-j\delta}x, 2^{j\delta}\xi)| &\leq C 2^{-j\delta n} \sum_{|\alpha_1+\alpha_2| \leq 2n} \sum_{0 < |k'| \leq 2^{j\delta}} |k'|^{-n/q-\epsilon} \\ &\quad \times (1 + |\xi - k'|)^{-2n} [|\widetilde{\partial^{\alpha_1}\Psi}| * |(\partial^{\alpha_2}\gamma)(2^{-j\delta}\cdot)|](k' - x), \end{aligned}$$

where $\widetilde{\Psi}(t) = \Psi(-t)$. Hence, we get

$$\begin{aligned} & \|V_\gamma[f_{j,\epsilon,\delta}(2^{j\delta}\cdot)]\|_{L^{p,q}} \\ &= 2^{-j\delta n(1/p-1/q)} \left\{ \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |V_\gamma[f_{j,\epsilon,\delta}(2^{j\delta}\cdot)](2^{-j\delta}x, 2^{j\delta}\xi)|^p dx \right)^{q/p} d\xi \right\}^{1/q} \\ &\leq C 2^{-j\delta n(1/p-1/q+1)} \sum_{|\alpha_1+\alpha_2| \leq 2n} \\ &\quad \times \left\{ \int_{\mathbb{R}^n} \left(\sum_{0 < |k'| \leq 2^{j\delta}} |k'|^{-\frac{n}{q}-\epsilon} (1 + |\xi - k'|)^{-2n} \|\widetilde{\partial^{\alpha_1}\Psi}\| * |(\partial^{\alpha_2}\gamma)(2^{-j\delta}\cdot)|\|_{L^p} \right)^q d\xi \right\}^{\frac{1}{q}} \\ &\leq C 2^{-j\delta n(1/p-1/q+1)} \left(\sum_{|\alpha_1+\alpha_2| \leq 2n} \|\widetilde{\partial^{\alpha_1}\Psi}\|_{L^1} \|(\partial^{\alpha_2}\gamma)(2^{-j\delta}\cdot)\|_{L^p} \right) \\ &\quad \times \left\{ \sum_{k \in \mathbb{Z}^n} \int_{k+[-1/2, 1/2]^n} \left(\sum_{k' \neq 0} |k'|^{-n/q-\epsilon} (1 + |\xi - k'|)^{-2n} \right)^q d\xi \right\}^{1/q} \\ &\leq C 2^{j\delta n(1/q-1)} \left\{ \sum_{k \in \mathbb{Z}^n} \left(\sum_{k' \neq 0} |k'|^{-n/q-\epsilon} (1 + |k - k'|)^{-2n} \right)^q \right\}^{1/q} \leq C 2^{j\delta n(1/q-1)}. \end{aligned}$$

The proof is complete. $\square$

Lemma 3.3. Let τ_δ be defined by (3.1), and let

$$(3.3) \quad g_{j,\epsilon,\delta}(x) = \sum_{0 < |k|, |k'| \leq 2^{j\delta}} |k'|^{-n/q-\epsilon} e^{-ik'\cdot(x-k)} e^{ik'\cdot(x-k')} \Phi(x - k) \Psi(x - k').$$

Then

$$\tau_\delta(X, D)[(M_{2^{j(1-\delta)}e_1} f_{j,\epsilon,\delta})(2^{j\delta}\cdot)](x) = 2^{jm} e^{i2^j x_1} g_{j,\epsilon,\delta}(2^{j\delta}x)$$

for all $j \geq j_0$ and $x \in \mathbb{R}^n$, where $f_{j,\epsilon,\delta}$ is defined by (3.2).

Proof. By Lemma 3.2 (1), we have

$$\begin{aligned} \mathcal{F}[(M_{2^{j(1-\delta)}e_1} f_{j,\epsilon,\delta})(2^{j\delta}\cdot)](\xi) &= 2^{-j\delta n} \mathcal{F}[M_{2^{j(1-\delta)}e_1} f_{j,\epsilon,\delta}](2^{-j\delta}\xi) \\ &= \sum_{0 < |k'| \leq 2^{j\delta}} |k'|^{-n/q-\epsilon} e^{-ik' \cdot (2^{-j\delta}\xi - 2^{j(1-\delta)}e_1)} 2^{-j\delta n} \psi(2^{-j\delta}\xi - (2^{j(1-\delta)}e_1 + k')). \end{aligned}$$

Since $2\delta < 1$, $j \geq j_0$, $1 + 2^{j_0(2\delta-1)+1} \leq 2^{1/4}$ and $1 - 2^{j_0(2\delta-1)+1} \geq 2^{-1/4}$, we see that

$$\begin{aligned} \text{supp } \psi(2^{-j\delta}\cdot - (2^{j(1-\delta)}e_1 + k')) &\subset \{\xi : |\xi - (2^j e_1 + 2^{j\delta} k')| \leq 2^{j\delta-1}\} \\ &\subset \{\xi : 2^j - 2^{2j\delta+1} \leq |\xi| \leq 2^j + 2^{2j\delta+1}\} \\ &\subset \{\xi : 2^{j-1/4} \leq |\xi| \leq 2^{j+1/4}\} \end{aligned}$$

for all $|k'| \leq 2^{j\delta}$. This implies

$$\text{supp } \mathcal{F}[(M_{2^{j(1-\delta)}e_1} f_{j,\epsilon,\delta})(2^{j\delta}\cdot)] \subset \{\xi : 2^{j-1/4} \leq |\xi| \leq 2^{j+1/4}\}$$

for all $j \geq j_0$. Hence, noting that $\text{supp } \eta(2^{-j'}\cdot) \subset \{2^{j'-1/2} \leq |\xi| \leq 2^{j'+1/2}\}$ and $\eta(2^{-j'}\cdot) = 1$ on $\{2^{j'-1/4} \leq |\xi| \leq 2^{j'+1/4}\}$, we obtain

$$\begin{aligned} &\tau_\delta(X, D)[(M_{2^{j(1-\delta)}e_1} f_{j,\epsilon,\delta})(2^{j\delta}\cdot)](x) \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \left\{ 2^{jm} \left(\sum_{0 < |k| \leq 2^{j\delta}} e^{-ik \cdot (2^{j\delta}x - k)} \Phi(2^{j\delta}x - k) \right) \eta(2^{-j}\xi) \right\} \\ &\quad \times \mathcal{F}[(M_{2^{j(1-\delta)}e_1} f_{j,\epsilon,\delta})(2^{j\delta}\cdot)](\xi) d\xi \\ &= 2^{jm} \left(\sum_{0 < |k| \leq 2^{j\delta}} e^{-ik \cdot (2^{j\delta}x - k)} \Phi(2^{j\delta}x - k) \right) \\ &\quad \times \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \mathcal{F}[(M_{2^{j(1-\delta)}e_1} f_{j,\epsilon,\delta})(2^{j\delta}\cdot)](\xi) d\xi \\ &= 2^{jm} \left(\sum_{0 < |k| \leq 2^{j\delta}} e^{-ik \cdot (2^{j\delta}x - k)} \Phi(2^{j\delta}x - k) \right) (M_{2^{j(1-\delta)}e_1} f_{j,\epsilon,\delta})(2^{j\delta}x) \\ &= 2^{jm} e^{i2^j x_1} g_{j,\epsilon,\delta}(2^{j\delta}x) \end{aligned}$$

for all $j \geq j_0$ and $x \in \mathbb{R}^n$. The proof is complete. $\square$

Lemma 3.4 ([5, Corollary 11.2.7]). *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\gamma \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$. Then*

$$(f, g) = \frac{1}{\|\gamma\|_{L^2}^2} \int_{\mathbb{R}^{2n}} V_\gamma f(x, \xi) \overline{V_\gamma g(x, \xi)} dx d\xi \quad \text{for all } g \in \mathcal{S}(\mathbb{R}^n).$$

For $1 \leq p \leq \infty$, p' is the conjugate exponent of p (that is, $1/p + 1/p' = 1$).

Lemma 3.5 ([5, Proposition 11.3.4, Theorem 11.3.6]). *If $1 \leq p, q < \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is dense in $M^{p,q}(\mathbb{R}^n)$ and $M^{p,q}(\mathbb{R}^n)^* = M^{p',q'}(\mathbb{R}^n)$ under the duality*

$$\langle f, g \rangle = \frac{1}{\|\gamma\|_{L^2}^2} \int_{\mathbb{R}^{2n}} V_\gamma f(x, \xi) \overline{V_\gamma g(x, \xi)} dx d\xi$$

for $f \in M^{p,q}(\mathbb{R}^n)$ and $g \in M^{p',q'}(\mathbb{R}^n)$, where $\gamma \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$.

We denote by B the tensor product of B-splines of degree 2; that is,

$$B(t) = \prod_{i=1}^n \chi_{[-1/2, 1/2]} * \chi_{[-1/2, 1/2]}(t_i),$$

where $\chi_{[-1/2, 1/2]}$ is the characteristic function of $[-1/2, 1/2]$. Note that $\text{supp } B \subset [-1, 1]^n$ and $\mathcal{F}^{-1}B(t) = (2\pi)^{-n} \prod_{i=1}^n \{(\sin(t_i/2))/(t_i/2)\}^2 \in M^{p,q}(\mathbb{R}^n)$ for all $1 \leq p, q \leq \infty$. By Lemmas 3.4 and 3.5, if $1 < p, q < \infty$ and $f \in \mathcal{S}(\mathbb{R}^n)$, then

$$\begin{aligned} (3.4) \quad \|f\|_{M^{p,q}} &= \sup_{\|g\|_{M^{p',q'}}=1} |\langle f, g \rangle| \geq \left| \left\langle f, \frac{\mathcal{F}^{-1}B}{\|\mathcal{F}^{-1}B\|_{M^{p',q'}}} \right\rangle \right| \\ &= \frac{1}{\|\mathcal{F}^{-1}B\|_{M^{p',q'}}} \left| \int_{\mathbb{R}^{2n}} V_\gamma f(x, \xi) \overline{V_\gamma[\mathcal{F}^{-1}B](x, \xi)} dx d\xi \right| \\ &= \frac{1}{\|\mathcal{F}^{-1}B\|_{M^{p',q'}}} \left| \int_{\mathbb{R}^n} f(t) \mathcal{F}^{-1}B(t) dt \right|, \end{aligned}$$

where $\gamma \in \mathcal{S}(\mathbb{R}^n)$ such that $\|\gamma\|_{L^2} = 1$.

Lemma 3.6. *Let $1 < p, q < \infty$, and let $g_{j,\epsilon,\delta}$ be defined by (3.3). Then there exists a constant $C > 0$ such that*

$$\|g_{j,\epsilon,\delta}(2^{j\delta} \cdot)\|_{M^{p,q}} \geq C 2^{-j\delta(n/q+\epsilon)} \quad \text{for all } j \geq j_0.$$

Proof. Let B be the tensor product of B-splines of degree 2. Note that $g_{j,\epsilon,\delta}(2^{j\delta} \cdot) \in \mathcal{S}(\mathbb{R}^n)$. By (3.4), we have

$$\begin{aligned} &\|g_{j,\epsilon,\delta}(2^{j\delta} \cdot)\|_{M^{p,q}} \\ &\geq \frac{1}{\|\mathcal{F}^{-1}B\|_{M^{p',q'}}} \left| \int_{\mathbb{R}^n} g_{j,\epsilon,\delta}(2^{j\delta}x) \mathcal{F}^{-1}B(x) dx \right| \\ &= \frac{1}{\|\mathcal{F}^{-1}B\|_{M^{p',q'}}} \left| \left\{ \sum_{0 < |k| \leq 2^{j\delta}} |k|^{-n/q-\epsilon} \int_{\mathbb{R}^n} \Phi(2^{j\delta}x - k) \Psi(2^{j\delta}x - k) \mathcal{F}^{-1}B(x) dx \right\} \right. \\ &\quad \left. + \left\{ \sum_{0 < |k| \leq 2^{j\delta}} \sum_{\substack{0 < |k'| \leq 2^{j\delta} \\ k' \neq k}} |k'|^{-n/q-\epsilon} \right. \right. \\ &\quad \left. \left. \times \int_{\mathbb{R}^n} \left(e^{-ik \cdot (2^{j\delta}x - k)} \Phi(2^{j\delta}x - k) \mathcal{F}^{-1}B(x) \right) \left(e^{ik' \cdot (2^{j\delta}x - k')} \Psi(2^{j\delta}x - k') \right) dx \right\} \right| \\ &= \frac{1}{\|\mathcal{F}^{-1}B\|_{M^{p',q'}}} |I + II|. \end{aligned}$$

We first consider I . Note that our assumptions $\int_{\mathbb{R}^n} \varphi(\xi) d\xi = 1$, $\text{supp } \varphi \subset \{|\xi| \leq 1/8\}$ and $\psi = 1$ on $\{|\xi| \leq 1/4\}$ give $\varphi * \psi = 1$ on $\{|\xi| \leq 1/8\}$. Since $\text{supp } B \subset \{|\xi| \leq \sqrt{n}\}$ and $2^{-j\delta} \sqrt{n} \leq 2^{-j_0\delta} \sqrt{n} \leq 1/8$, by Plancherel's theorem, we see that

$$\begin{aligned} & \sum_{0 < |k| \leq 2^{j\delta}} |k|^{-n/q-\epsilon} \int_{\mathbb{R}^n} \Phi(2^{j\delta}x - k) \Psi(2^{j\delta}x - k) \mathcal{F}^{-1}B(x) dx \\ &= \sum_{0 < |k| \leq 2^{j\delta}} |k|^{-n/q-\epsilon} \int_{\mathbb{R}^n} (\Phi \Psi)(2^{j\delta}x - k) \overline{\mathcal{F}^{-1}B(x)} dx \\ &= \sum_{0 < |k| \leq 2^{j\delta}} |k|^{-n/q-\epsilon} \frac{1}{(2\pi)^{2n}} \int_{|\xi| \leq \sqrt{n}} 2^{-j\delta n} e^{-ik \cdot (2^{-j\delta}\xi)} \varphi * \psi(2^{-j\delta}\xi) \overline{B(\xi)} d\xi \\ &= C_n 2^{-j\delta n} \sum_{0 < |k| \leq 2^{j\delta}} |k|^{-n/q-\epsilon} \int_{\mathbb{R}^n} e^{-i(2^{-j\delta}k) \cdot \xi} B(\xi) d\xi \\ &= C_n 2^{-j\delta n} \sum_{0 < |k| \leq 2^{j\delta}} |k|^{-n/q-\epsilon} \prod_{i=1}^n \left(\frac{\sin k_i / 2^{j\delta+1}}{k_i / 2^{j\delta+1}} \right)^2. \end{aligned}$$

We next consider II . Using

$$\mathcal{F}[T_k M_{-k} \Phi] = e^{i|k|^2} T_{-k} M_{-k} \varphi \quad \text{and} \quad \mathcal{F}[T_{k'} M_{-k'} \Psi] = e^{i|k'|^2} T_{-k'} M_{-k'} \psi,$$

we have

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(e^{-ik \cdot (2^{j\delta}x - k)} \Phi(2^{j\delta}x - k) \mathcal{F}^{-1}B(x) \right) \left(e^{ik' \cdot (2^{j\delta}x - k')} \Psi(2^{j\delta}x - k') \right) dx \\ &= \int_{\mathbb{R}^n} (T_k M_{-k} \Phi(x) 2^{-j\delta n} [\mathcal{F}^{-1}B](2^{-j\delta}x)) \overline{T_{k'} M_{-k'} \Psi(x)} dx \\ &= \frac{e^{i(|k|^2 - |k'|^2)}}{(2\pi)^{2n}} \int_{\mathbb{R}^n} [(T_{-k} M_{-k} \varphi) * (B(2^{j\delta} \cdot))](\xi) \overline{[T_{-k'} M_{-k'} \psi](\xi)} d\xi \\ &= \frac{e^{i(|k|^2 - |k'|^2)}}{(2\pi)^{2n}} \int_{\mathbb{R}^n} [(M_{-k} \varphi) * (B(2^{j\delta} \cdot))](\xi + k) \overline{[M_{-k'} \psi](\xi + k')} d\xi. \end{aligned}$$

Since $\text{supp } \varphi \subset \{|\xi| \leq 1/8\}$, $\text{supp } B(2^{j\delta} \cdot) \subset \{|\xi| \leq 2^{-j\delta} \sqrt{n}\} \subset \{|\xi| \leq 1/8\}$, we see that $\text{supp } (M_{-k} \varphi) * (B(2^{j\delta} \cdot)) \subset \{|\xi| \leq 1/4\}$. On the other hand, $\text{supp } M_{-k'} \psi \subset \{|\xi| \leq 1/2\}$. Hence, if $k \neq k'$, then

$$\frac{e^{i(|k|^2 - |k'|^2)}}{(2\pi)^{2n}} \int_{\mathbb{R}^n} [(M_{-k} \varphi) * (B(2^{j\delta} \cdot))](\xi + k) \overline{[M_{-k'} \psi](\xi + k')} d\xi = 0;$$

that is, $II = 0$. Therefore, since $\prod_{i=1}^n (\sin x_i / x_i)^2 \geq C$ on $[-1/2, 1/2]^n$, we get

$$\begin{aligned} \frac{1}{\|\mathcal{F}^{-1}B\|_{M^{p',q'}}} |I + II| &= \frac{C_n 2^{-j\delta n}}{\|\mathcal{F}^{-1}B\|_{M^{p',q'}}} \left| \sum_{0 < |k| \leq 2^{j\delta}} |k|^{-n/q-\epsilon} \prod_{i=1}^n \left(\frac{\sin k_i / 2^{j\delta+1}}{k_i / 2^{j\delta+1}} \right)^2 \right| \\ &= C_n 2^{-j\delta n} \sum_{0 < |k| \leq 2^{j\delta}} |k|^{-n/q-\epsilon} \prod_{i=1}^n \left(\frac{\sin k_i / 2^{j\delta+1}}{k_i / 2^{j\delta+1}} \right)^2 \\ &\geq C_n 2^{-j\delta n} 2^{-j\delta(n/q+\epsilon)} \sum_{0 < |k| \leq 2^{j\delta}} 1 \geq C 2^{-j\delta(n/q+\epsilon)}. \end{aligned}$$

The proof is complete. $\square$

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Assume that $1 < p, q < \infty$, $0 < \delta < 1$ and $m > -|1/q - 1/2|\delta n$. Let σ_δ be defined by (2.3). Then $\sigma_\delta(x, \xi) = \tau_{\delta/2}(x, \xi)$, where $\tau_{\delta/2}$ is defined by (3.1). By Lemma 3.1, we see that $\sigma_\delta \in S_{1,\delta}^m$. This implies $\sigma_\delta^* \in S_{1,\delta}^m$, where σ_δ^* is defined by (2.1).

We first consider the case $q \geq 2$. In this case, $m > (1/q - 1/2)\delta n$. Since $m > (1/q - 1/2)\delta n$, we can take $\epsilon > 0$ such that $m - \epsilon\delta/2 > (1/q - 1/2)\delta n$. We assume that $\sigma_\delta(X, D)$ is bounded on $M^{p,q}(\mathbb{R}^n)$. Then, by Lemmas 3.2, 3.3, 3.6 and the modulation invariance of the norm $\|\cdot\|_{M^{p,q}}$, we see that

$$\begin{aligned} C2^{j(m-\delta(n/q+\epsilon)/2)} &\leq 2^{jm}\|g_{j,\epsilon,\delta/2}(2^{j\delta/2}\cdot)\|_{M^{p,q}} = \|2^{jm}e^{i2^jx_1}g_{j,\epsilon,\delta/2}(2^{j\delta/2}\cdot)\|_{M^{p,q}} \\ &= \|\tau_{\delta/2}(X, D)[(M_{2^{j(1-\delta/2)}e_1}f_{j,\epsilon,\delta/2})(2^{j\delta/2}\cdot)]\|_{M^{p,q}} \\ &= \|\sigma_\delta(X, D)[(M_{2^{j(1-\delta/2)}e_1}f_{j,\epsilon,\delta/2})(2^{j\delta/2}\cdot)]\|_{M^{p,q}} \\ &\leq \|\sigma_\delta(X, D)\|_{\mathcal{L}(M^{p,q})}\|(M_{2^{j(1-\delta/2)}e_1}f_{j,\epsilon,\delta/2})(2^{j\delta/2}\cdot)\|_{M^{p,q}} \\ &= \|\sigma_\delta(X, D)\|_{\mathcal{L}(M^{p,q})}\|f_{j,\epsilon,\delta/2}(2^{j\delta/2}\cdot)\|_{M^{p,q}} \leq C2^{j\delta n(1/q-1)/2} \end{aligned}$$

for all $j \geq j_0$, where $f_{j,\epsilon,\delta/2}$ and $g_{j,\epsilon,\delta/2}$ are defined by (3.2) and (3.3). However, since $m - \epsilon\delta/2 > (1/q - 1/2)\delta n$, this is a contradiction. Hence, σ_δ belongs to $S_{1,\delta}^m$, but $\sigma_\delta(X, D)$ is not bounded on $M^{p,q}(\mathbb{R}^n)$.

We next consider the case $q \leq 2$. In this case, $m > -(1/q - 1/2)\delta n$. Since $q' \geq 2$ and $m > (1/q' - 1/2)\delta n$, by Theorem 2.1 with $q \geq 2$, we see that $\sigma_\delta(X, D)$ is not bounded on $M^{p',q'}(\mathbb{R}^n)$. By duality and (2.2), if $\sigma_\delta^*(X, D)$ is bounded on $M^{p,q}(\mathbb{R}^n)$, then $\sigma_\delta(X, D)$ is bounded on $M^{p',q'}(\mathbb{R}^n)$. Hence, σ_δ^* belongs to $S_{1,\delta}^m$, but $\sigma_\delta^*(X, D)$ is not bounded on $M^{p,q}(\mathbb{R}^n)$. The proof is complete. $\square$

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