A NOTE ON THE LIMITING WEAK-TYPE BEHAVIOR FOR MAXIMAL OPERATORS

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ABSTRACT. We study the following open question raised by Janakiraman in (2006): for $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\lambda > 0$, what is the limiting behavior of

$$[m(\{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda\})]^{1/p}$$

as $p \rightarrow \infty$? In this note, we give a complete answer to this question.

1. INTRODUCTION

Denote by m the Lebesgue measure. Let M be the Hardy-Littlewood *maximal operator* defined by

$$M(f)(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)| dy \quad (x \in \mathbb{R}^n),$$

where $B(x, r)$ is a (closed or open) ball centered at x with radius r . The *maximal function theorem* [4] reads that, for any $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$,

$$(1.1) \quad m(\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}) \leq \frac{c_1}{\lambda} \int_{\mathbb{R}^n} |f| dm,$$

where $c_1 > 0$ is independent of λ and f . See also [1, Theorem 2.2] for *doubling metric measure* spaces. The reverse to (1.1) also holds: for $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$, there exist $c, \bar{c} > 0$ independent of f and λ such that

$$(1.2) \quad m(\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}) \geq \frac{c}{\lambda} \int_{\{x \in \mathbb{R}^n : |f(x)| > \lambda/\bar{c}\}} |f| dm.$$

The reader may find the sketched proof in [3, p. 43]. The formulae (1.1) and (1.2) give the upper and lower bounds of

$$(1.3) \quad \lambda m(\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}).$$

Recently, Janakiraman [2] has obtained, among other things, the following interesting result about the limiting behavior of (1.3):

$$(1.4) \quad \lim_{\lambda \rightarrow 0} \lambda m(\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}) = \|f\|_1, \quad f \in L^1(\mathbb{R}^n).$$

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Janakiraman further guessed the following ([2, p. 1947]):

$$(A) \quad \lim_{p \rightarrow \infty} [m(\{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda\})]^{1/p} = \begin{cases} \|f\|_\infty, & \text{if } m(\{|f| > 1\}) > 0, \\ 0, & \text{if } |f| < 1 \text{ a.e.}, \\ \chi_{(0,1)}(\lambda), & \text{if } m(\{|f| = 1\}) > 0, \end{cases}$$

where χ_A is the characteristic function for a measurable set A , that is, $\chi_A(x) = 1$ for $x \in A$, and $\chi_A(x) = 0$ otherwise.

In this note, we will prove that the first equality in (A) is true; see Section 2. However, the other two inequalities are not true. Instead, the third equality in (A) is true under an additional assumption; see Theorem 3.1. For the second equality in (A), we give a counterexample; see Example 4.3. For this case, what we have obtained is that: for $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $\|f\|_\infty = 1$,

$$(1.5) \quad \overline{\lim}_{p \rightarrow \infty} [m(\{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda\})]^{1/p} = \overline{\lim}_{p \rightarrow \infty} [m(\{x \in \mathbb{R}^n : |f|^p(x) > \lambda\})]^{1/p},$$

and

$$(1.6) \quad \underline{\lim}_{p \rightarrow \infty} [m(\{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda\})]^{1/p} = \underline{\lim}_{p \rightarrow \infty} [m(\{x \in \mathbb{R}^n : |f|^p(x) > \lambda\})]^{1/p}.$$

Finally, we discuss formula (1.4) for a general measure. We will give an example, showing that if the Lebesgue measure m is replaced by a doubling measure μ , then (1.4) fails; see Section 5.

2. PROOF OF THE FIRST EQUALITY IN (A)

It is known that, for $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, the function $f \in L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and $\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty$.

Proposition 2.1. *For $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\lambda > 0$,*

$$(2.1) \quad \overline{\lim}_{p \rightarrow \infty} [m(\{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda\})]^{1/p} \leq \|f\|_\infty.$$

Proof. Note that $|f|^p \in L^1(\mathbb{R}^n)$, and by (1.1),

$$(2.2) \quad m(\{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda\}) \leq \frac{c_1}{\lambda} \int_{\mathbb{R}^n} |f|^p dm.$$

Therefore,

$$(2.3) \quad \overline{\lim}_{p \rightarrow \infty} [m(\{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda\})]^{1/p} \leq \overline{\lim}_{p \rightarrow \infty} \left[\frac{c_1}{\lambda} \int_{\mathbb{R}^n} |f|^p dm \right]^{1/p} = \|f\|_\infty.$$

This completes the proof. $\square$

In the rest of this paper, we denote by $\{f > a\}$ the set of all points $x \in \mathbb{R}^n$ such that $f(x) > a$.

Proposition 2.2. *Suppose that $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $m(\{|f| > 1\}) > 0$. Then*

$$(2.4) \quad \lim_{p \rightarrow \infty} [m(\{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda\})]^{1/p} \geq \|f\|_\infty.$$

Proof. Since $|f|^p \in L^1(\mathbb{R}^n)$, we apply (1.2) for $|f|^p$ to obtain that

$$(2.5) \quad m(\{M(|f|^p) > \lambda\}) \geq \frac{c}{\lambda} \int_{\{|f|^p > \lambda/\bar{c}\}} |f|^p dm,$$

which implies that

$$(2.6) \quad \begin{aligned} \lim_{p \rightarrow \infty} [m(\{M(|f|^p) > \lambda\})]^{1/p} &\geq \lim_{p \rightarrow \infty} \left(\frac{c}{\lambda} \int_{\{|f|^p > \lambda/\bar{c}\}} |f|^p dm \right)^{1/p} \\ &\geq \lim_{p \rightarrow \infty} \left(\int_{\{|f|^p > \lambda/\bar{c}\}} |f|^p dm \right)^{1/p}. \end{aligned}$$

On the other hand, the inequality $m(\{|f| > 1\}) > 0$ implies that $\|f\|_\infty > 1$. Thus, for any small $0 < \varepsilon < \|f\|_\infty - 1$, we have

$$0 < m(\{|f| > \|f\|_\infty - \varepsilon\}) < \infty.$$

Since $\|f\|_\infty - \varepsilon > 1$, there exists p_ε such that $(\|f\|_\infty - \varepsilon)^{p_\varepsilon} > \lambda/\bar{c}$. This gives that, for $p > p_\varepsilon$,

$$\{|f| > \|f\|_\infty - \varepsilon\} \subseteq \{|f|^p > \lambda/\bar{c}\}.$$

Therefore, it follows from (2.6) that

$$\begin{aligned} \lim_{p \rightarrow \infty} [m(\{M(|f|^p) > \lambda\})]^{1/p} &\geq \lim_{p \rightarrow \infty} \left(\int_{\{|f|^p > \lambda/\bar{c}\}} |f|^p dm \right)^{1/p} \\ &\geq \|f\|_\infty - \varepsilon. \end{aligned}$$

Hence (2.4) follows by letting $\varepsilon \rightarrow 0^+$. □

By Propositions 2.1 and 2.2, it follows that, for $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\lambda > 0$, if $m(\{|f| > 1\}) > 0$, then

$$(2.7) \quad \lim_{p \rightarrow \infty} [m(\{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda\})]^{1/p} = \|f\|_\infty.$$

This proves the first equality in (A).

3. THE THIRD EQUALITY IN (A)

Theorem 3.1. *Let $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\lambda > 0$. If $m(\{|f| = 1\}) > 0$ and $\|f\|_\infty = 1$, then*

$$(3.1) \quad \lim_{p \rightarrow \infty} [m(\{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda\})]^{1/p} = \chi_{(0,1)}(\lambda).$$

Proof. Case 1. Let $\lambda \geq 1$. Since $\|f\|_\infty = 1$, we have $|f| \leq 1$ a.e. Thus

$$M(|f|^p) \leq \| |f|^p \|_\infty \leq 1$$

for any $p > 0$. Hence, the Lebesgue measure of the set $\{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda\}$ is zero, and (3.1) immediately follows for $\lambda \geq 1$.

Case 2. Let $0 < \lambda < 1$. Set $A = \{x \in \mathbb{R}^n : |f(x)| = 1\}$. By the assumption, we see that $m(A) > 0$, and $m(A) < \infty$ for $f \in L^1$. Clearly $|f|^p \geq \chi_A$, and so $M(|f|^p) \geq M(\chi_A)$. Therefore,

$$\{x \in \mathbb{R}^n : M(\chi_A)(x) > \lambda\} \subseteq \{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda\}.$$

Hence,

$$\begin{aligned} (3.2) \quad & \lim_{p \rightarrow \infty} [m(\{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda\})]^{1/p} \\ & \geq \lim_{p \rightarrow \infty} [m(\{x \in \mathbb{R}^n : M(\chi_A)(x) > \lambda\})]^{1/p}. \end{aligned}$$

Noting that $0 < \lambda < 1$, we conclude that

$$0 < m(\{x \in \mathbb{R}^n : M(\chi_A)(x) > \lambda\}) < \frac{c_1}{\lambda} \int_{\mathbb{R}^n} \chi_A dm = \frac{c_1}{\lambda} m(A) < \infty,$$

which combines with (3.2) to yield that

$$\lim_{p \rightarrow \infty} [m(\{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda\})]^{1/p} \geq 1.$$

Combining this with (2.1), we obtain the desired proof. $\square$

4. DISCUSSIONS FOR THE SECOND EQUALITY IN (A)

If $\delta := \|f\|_\infty < 1$, then $M(|f|^p) \leq \delta^p \rightarrow 0$ as $p \rightarrow \infty$. For any $\lambda > 0$, the set

$$\{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda\}$$

is empty for p sufficiently large. Thus, we have

$$(4.1) \quad \lim_{p \rightarrow \infty} [m(\{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda\})]^{1/p} = 0$$

for any $\lambda > 0$. At this time, the second equality in (A) does hold. However, it is possible that $|f| < 1$ a.e., but $\|f\|_\infty = 1$. This situation is more awkward. As we will see, the limit in (4.1) may not exist.

Example 4.1. Let $f = (1 - x)\chi_{[0,1]}$ and $0 < \lambda < 1$. Clearly $|f| < 1$ a.e., and $\|f\|_\infty = 1$ and $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Note that, by the Lebesgue differentiation theorem,

$$(4.2) \quad \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y)|^p dy \rightarrow |f(x)|^p$$

for almost all $x \in \mathbb{R}^n$ as $r \rightarrow 0$, and so, by definition,

$$M(|f|^p) \geq |f|^p \quad \text{a.e.}$$

Hence

$$m(\{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda\}) \geq m(\{x \in \mathbb{R}^n : |f(x)|^p > \lambda\}) = 1 - \lambda^{1/p}.$$

A simple computation shows that $\lim_{p \rightarrow \infty} (1 - \lambda^{1/p})^{1/p} = 1$, and so

$$\lim_{p \rightarrow \infty} [m(\{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda\})]^{1/p} \geq 1.$$

Actually the reader may verify that, for this function f ,

$$\lim_{p \rightarrow \infty} [m(\{x \in \mathbb{R}^n : M(|f|^p)(x) > \lambda\})]^{1/p} = \chi_{(0,1)}(\lambda).$$

Proposition 4.2. *For any $f \in L^1(\mathbb{R}^n)$ and for any $\lambda > 0, 0 < k < 1$,*

$$(4.3) \quad m(\{M(f) > \lambda\}) \leq \frac{c_1}{(1-k)\lambda} \int_{\{|f| > k\lambda\}} |f(x)| dx,$$

where c_1 is the same as in (1.1) (independent of k).

Proof. We divide $|f|$ into two parts: the “good” one, $g := |f|\chi_{\{|f| \leq k\lambda\}}$, and the “bad” one, $b := |f|\chi_{\{|f| > k\lambda\}}$. Since $|f| = g + b$, we see that

$$M(f) \leq M(g) + M(b) \leq k\lambda + M(b).$$

Thus $\{M(f) > \lambda\} \subseteq \{M(b) > (1-k)\lambda\}$, and so

$$(4.4) \quad \begin{aligned} m(\{M(f) > \lambda\}) &\leq m(\{M(b) > (1-k)\lambda\}) \\ &\leq \frac{c_1}{(1-k)\lambda} \int_{\{|f| > k\lambda\}} |f(x)| dx \end{aligned}$$

by virtue of (1.1). □

We are now in a position to prove (1.5) and (1.6), by using (4.3).

Proof of (1.5) and (1.6). Let $0 < \lambda < 1$. Since $M(|f|^p) \geq |f|^p$ a.e., it follows that

$$m(\{M(|f|^p) > \lambda\}) \geq m(\{|f|^p > \lambda\}).$$

Therefore,

$$(4.5) \quad \begin{aligned} \overline{\lim}_{p \rightarrow \infty} [m(\{M(|f|^p) > \lambda\})]^{1/p} &\geq \overline{\lim}_{p \rightarrow \infty} [m(\{|f|^p > \lambda\})]^{1/p}, \\ \underline{\lim}_{p \rightarrow \infty} [m(\{M(|f|^p) > \lambda\})]^{1/p} &\geq \underline{\lim}_{p \rightarrow \infty} [m(\{|f|^p > \lambda\})]^{1/p}. \end{aligned}$$

On the other hand, as $|f|^p \in L^1(\mathbb{R}^n)$, we apply (4.3) to $|f|^p$, and obtain that, using the assumption that $|f| \leq 1$ a.e.,

$$(4.6) \quad \begin{aligned} m(\{M(|f|^p) > \lambda\}) &\leq \frac{c_1}{(1-k)\lambda} \int_{\{|f|^p > k\lambda\}} |f(x)|^p dx \\ &\leq \frac{c_1}{(1-k)\lambda} m(\{|f|^p > k\lambda\}). \end{aligned}$$

Since $0 < \lambda, k < 1$, we have

$$\begin{aligned} [m(\{|f|^p > k\lambda\})]^{1/p} &= \left[m(\{|f|^{\frac{p \ln \lambda}{\ln k + \ln \lambda}} > \lambda\}) \right]^{\frac{\ln k + \ln \lambda}{p \ln \lambda} \times \frac{\ln \lambda}{\ln k + \ln \lambda}} \\ &:= [m(\{|f|^q > \lambda\})]^{\frac{1}{q} \times \frac{\ln \lambda}{\ln k + \ln \lambda}}, \end{aligned}$$

where $q := \frac{p \ln \lambda}{\ln k + \ln \lambda}$. This combines with (4.6) to yield that

$$(4.7) \quad [m(\{M(|f|^p) > \lambda\})]^{1/p} \leq \left(\frac{c_1}{(1-k)\lambda} \right)^{1/p} [m(\{|f|^q > \lambda\})]^{\frac{1}{q} \times \frac{\ln \lambda}{\ln k + \ln \lambda}}.$$

Taking the lower and upper limits in (4.7), we obtain

$$\begin{aligned} \overline{\lim}_{p \rightarrow \infty} [m(\{M(|f|^p) > \lambda\})]^{1/p} &\leq \overline{\lim}_{q \rightarrow \infty} [m(\{|f|^q > \lambda\})]^{\frac{1}{q} \times \frac{\ln \lambda}{\ln k + \ln \lambda}}, \\ \underline{\lim}_{p \rightarrow \infty} [m(\{M(|f|^p) > \lambda\})]^{1/p} &\leq \underline{\lim}_{q \rightarrow \infty} [m(\{|f|^q > \lambda\})]^{\frac{1}{q} \times \frac{\ln \lambda}{\ln k + \ln \lambda}}. \end{aligned}$$

Letting $k \rightarrow 1^-$ and changing q by p , we see that

$$\begin{aligned}\overline{\lim}_{p \rightarrow \infty} [m(\{M(|f|^p) > \lambda\})]^{1/p} &\leq \overline{\lim}_{p \rightarrow \infty} [m(\{|f|^p > \lambda\})]^{1/p}, \\ \underline{\lim}_{p \rightarrow \infty} [m(\{M(|f|^p) > \lambda\})]^{1/p} &\leq \underline{\lim}_{p \rightarrow \infty} [m(\{|f|^p > \lambda\})]^{1/p}.\end{aligned}$$

Combining this with (4.5), we obtain (1.5) and (1.6). $\square$

The following example shows that the limit of $[m(\{M(|f|^p) > \lambda\})]^{1/p}$ as $p \rightarrow \infty$ may not exist.

Example 4.3. Consider the function f given by

$$f = \sum_{n=0}^{\infty} \exp(-2^{-n}) \chi_{(\exp(-2^n), \exp(-2^{n-1}))}.$$

Clearly $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and $0 < f < 1$ a.e. and $\|f\|_\infty = 1$. For $0 < \lambda < 1$, let

$$p_k = (2^{k+1} - 1) \ln(1/\lambda) \quad \text{and} \quad q_k = (2^k + 1) \ln(1/\lambda) \quad (k \geq 1).$$

Note that

$$\exp(-2^{-k}) < \lambda^{1/p_k} = \exp\left(-\frac{1}{2^{k+1} - 1}\right) < \exp(-2^{-k-1}),$$

and so,

$$m(\{f^{p_k} > \lambda\}) = \exp(-2^k).$$

Therefore,

$$(4.8) \quad \lim_{p_k \rightarrow \infty} [m(\{f^{p_k} > \lambda\})]^{1/p_k} = \lim_{k \rightarrow \infty} \exp\left(\frac{2^k}{(2^{k+1} - 1) \ln \lambda}\right) = \exp\left(\frac{1}{2 \ln \lambda}\right).$$

A similar computation for $\{q_k\}_{k \geq 1}$ shows that

$$(4.9) \quad \lim_{q_k \rightarrow \infty} [m(\{f^{q_k} > \lambda\})]^{1/q_k} = \lim_{k \rightarrow \infty} \exp\left(\frac{2^k}{(2^k + 1) \ln \lambda}\right) = \exp\left(\frac{1}{\ln \lambda}\right).$$

By (1.5) and (1.6), we see that the limit of $[m(\{M(|f|^p) > \lambda\})]^{1/p}$ as $p \rightarrow \infty$ does not exist.

5. DISCUSSION FOR (1.4)

There is a natural question: if the Lebesgue measure m is replaced by a more general measure μ , say a doubling measure, and $\mathbb{R}^n$ replaced by a metric space X , does (1.4) still hold? The answer is negative. Here we give an example, showing that the limit of

$$\lambda \mu(\{x \in X : M(f)(x) > \lambda\})$$

exists as $\lambda \rightarrow 0$ for a doubling metric measure space (X, μ) , but does not equal $\|f\|_1$.

Example 5.1. Let $X = (0, \infty)$ with the Euclidean metric, and let μ be a measure determined by $d\mu = 1_{\{x > 0\}} x dx$. It is easy to see that

$$(5.1) \quad \mu(B(x_0, r)) = \begin{cases} 2x_0 r, & \text{if } x_0 \geq r > 0, \\ (x_0 + r)^2/2, & \text{if } r > x_0 > 0. \end{cases}$$

Thus, for any $x_0 \in X$ and $r > 0$,

$$(5.2) \quad \mu(B(x_0, 2r)) \leq 4\mu(B(x_0, r)),$$

showing that (X, μ) defined as above is a doubling metric measure space.

We claim that for $f \in L^1(X, \mu)$,

$$(5.3) \quad \lim_{\lambda \rightarrow 0} \lambda \mu(\{x \in X : M(f)(x) > \lambda\}) = \frac{1}{4} \|f\|_1.$$

Proof of the claim. For any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that

$$\int_0^{r_\varepsilon} |f| d\mu > (1 - \varepsilon) \|f\|_1$$

for $f \in L^1(X, \mu)$. Set

$$R_\varepsilon := \left(1 + \frac{1}{\varepsilon}\right) r_\varepsilon.$$

It follows from (5.1) that, for $x_0 > R_\varepsilon$,

$$(5.4) \quad \begin{aligned} (1 - \varepsilon) \mu(B(x_0, x_0 + r_\varepsilon)) &= \frac{1 - \varepsilon}{2} (2x_0 + r_\varepsilon)^2 < 2x_0^2 = \mu(B(x_0, x_0)) \\ &< (1 + \varepsilon) 2x_0(x_0 - r_\varepsilon) = (1 + \varepsilon) \mu(B(x_0, x_0 - r_\varepsilon)). \end{aligned}$$

For $\lambda > 0$, we divide the set $E_\lambda := \{x \in X : M(f)(x) > \lambda\}$ into three disjoint parts:

$$\begin{aligned} E_\lambda^1 &= \{x \in E_\lambda : x > R_\varepsilon, M(f)(x) > \|f\|_1 / \mu(B(x, x - r_\varepsilon))\}, \\ E_\lambda^2 &= \{x \in E_\lambda : x > R_\varepsilon, M(f)(x) \leq \|f\|_1 / \mu(B(x, x - r_\varepsilon))\}, \\ E_\lambda^3 &= \{x \in E_\lambda : x \leq R_\varepsilon\}. \end{aligned}$$

First we consider the set E_λ^1 . For any $x_0 \in E_\lambda^1$, we have

$$(5.5) \quad \sup_{r > x_0 - r_\varepsilon} \frac{1}{\mu(B(x_0, r))} \int_{B(x_0, r)} |f| d\mu \leq \frac{\|f\|_1}{\mu(B(x_0, x_0 - r_\varepsilon))} < M(f)(x_0).$$

Therefore,

$$\begin{aligned} M(f)(x_0) &= \sup_{r > 0} \frac{1}{\mu(B(x_0, r))} \int_{B(x_0, r)} |f| d\mu \\ &= \sup_{0 < r \leq x_0 - r_\varepsilon} \frac{1}{\mu(B(x_0, r))} \int_{B(x_0, r)} |f| d\mu. \end{aligned}$$

Since $M(f)(x_0) > \lambda$, there exists some r with $0 < r \leq x_0 - r_\varepsilon$ such that

$$\frac{1}{\mu(B(x_0, r))} \int_{B(x_0, r)} |f| d\mu > \lambda.$$

From this, we see that

$$2r^2 < 2x_0 r = \mu(B(x_0, r)) < \|f\|_1 / \lambda,$$

showing that r is bounded from above by $\sqrt{\|f\|_1 / 2\lambda}$. Thus the set E_λ^1 can be covered by a family of balls with radii uniformly bounded. By the basic covering theorem [1, Theorem 1.16] for the doubling space, there is a countable disjoint subfamily of balls $\{B(x_i, r_i)\}$ such that $E_\lambda^1 \subseteq \bigcup_i B(x_i, 5r_i)$. Since μ is doubling, there exists some $C > 0$ such that

$$\mu(B(x_i, 5r_i)) \leq C\mu(B(x_i, r_i)).$$

Hence, using the fact that $B(x_i, r_i) \subset [r_\varepsilon, \infty)$ for each i ,

$$\begin{aligned}
 \mu(E_\lambda^1) &\leq \sum_i \mu(B(x_i, 5r_i)) \leq C \sum_i \mu(B(x_i, r_i)) \\
 (5.6) \quad &< C \sum_i \frac{1}{\lambda} \int_{B(x_i, r_i)} |f| d\mu \leq \frac{C}{\lambda} \int_{r_\varepsilon}^\infty |f| d\mu \\
 &= \frac{C}{\lambda} \left(\|f\|_1 - \int_0^{r_\varepsilon} |f| d\mu \right) \leq \frac{C\varepsilon}{\lambda} \|f\|_1.
 \end{aligned}$$

Next we consider the set E_λ^2 . By (5.4), we see that

$$\begin{aligned}
 E_\lambda^2 &\subseteq \{x \in X : x > R_\varepsilon, \lambda \leq \|f\|_1 / \mu(B(x, x - r_\varepsilon))\} \\
 &\subseteq \{x \in X : x > R_\varepsilon, \mu(B(x, x)) \leq (1 + \varepsilon) \|f\|_1 / \lambda\} \\
 &= \left(R_\varepsilon, [(1 + \varepsilon) \|f\|_1 / (2\lambda)]^{1/2} \right) := I_1.
 \end{aligned}$$

It follows that

$$(5.7) \quad \mu(E_\lambda^2) \leq \mu(I_1) = (1 + \varepsilon) \|f\|_1 / (4\lambda) - R_\varepsilon^2 / 2.$$

Combining (5.6) and (5.7), we obtain

$$\begin{aligned}
 \lambda \mu(\{M(f) > \lambda\}) &= \lambda (\mu(E_\lambda^1) + \mu(E_\lambda^2) + \mu(E_\lambda^3)) \\
 &\leq C\varepsilon \|f\|_1 + (1 + \varepsilon) \|f\|_1 / 4 - \lambda R_\varepsilon^2 / 2 + \lambda \mu(B(0, R_\varepsilon)) \\
 (5.8) \quad &= C\varepsilon \|f\|_1 + (1 + \varepsilon) \|f\|_1 / 4.
 \end{aligned}$$

In order to obtain the inverse inequality, we observe that for $x_0 \in E_\lambda^1 \cup E_\lambda^2$,

$$\begin{aligned}
 M(f)(x_0) &\geq \frac{1}{\mu(B(x_0, x_0))} \int_{B(x_0, x_0)} |f| d\mu = \frac{1}{\mu(B(x_0, x_0))} \int_0^{2x_0} |f| d\mu \\
 &\geq \frac{1}{\mu(B(x_0, x_0))} \int_0^{r_\varepsilon} |f| d\mu > \frac{(1 - \varepsilon) \|f\|_1}{\mu(B(x_0, x_0))}
 \end{aligned}$$

by using the fact that $x_0 > R_\varepsilon > r_\varepsilon$. This implies that

$$\begin{aligned}
 E_\lambda^1 \cup E_\lambda^2 &= \{x \in X : x > R_\varepsilon, M(f)(x) > \lambda\} \\
 &\supseteq \{x \in X : x > R_\varepsilon, (1 - \varepsilon) \|f\|_1 / \mu(B(x, x)) > \lambda\} \\
 &= \left(R_\varepsilon, [(1 - \varepsilon) \|f\|_1 / (2\lambda)]^{1/2} \right) := I_2.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \lambda \mu(\{M(f) > \lambda\}) &\geq \lambda \mu(E_\lambda^1 \cup E_\lambda^2) \\
 (5.9) \quad &\geq \lambda \mu(I_2) = (1 - \varepsilon) \|f\|_1 / 4 - \lambda R_\varepsilon^2 / 2.
 \end{aligned}$$

Finally, letting $\lambda \rightarrow 0$ in both (5.8) and (5.9) and noting that ε is arbitrary, we arrive at (5.3). This finishes the proof of our claim. $\square$

One may further consider the measure μ on $\mathbb{R}^n$ where $d\mu = |x|^\alpha dx$ for $\alpha > 0$. It can be proven that for suitable $\alpha > 0$, the limit

$$\lim_{\lambda \rightarrow 0} \lambda \mu(\{x \in X : M(f)(x) > \lambda\})$$

exists but does not equal $\|f\|_1$ for $f \in L^1(\mathbb{R}^n, \mu)$. The reader may find examples for which even the above limit does not exist for doubling metric measure spaces.

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