SPECTRAL DOMINANCE AND COMMUTING CHAINS

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ABSTRACT. A positive semidefinite (PSD) operator A “spectrally dominates” a PSD operator B if $A^t - B^t$ is PSD for all $t > 0$. We (i) give a new characterization of spectral dominance in finite dimensions in terms of a monotonic chain of intermediate, pairwise commuting operators and (ii) determine for which pairs A, B spectral dominance persists under the taking of arbitrary compressions. Earlier results about spectral dominance are proven (in finite dimensions) in new ways, and several corollary observations are made.

1. INTRODUCTION

Let $\mathfrak{H}$ be a Hilbert space, and let A, B be selfadjoint bounded operators acting on $\mathfrak{H}$. We say $A \succcurlyeq B$ if $A - B$ is positive semidefinite (PSD for short), the standard Löwner partial order on the real space of selfadjoint operators. In case $\dim \mathfrak{H} = n < \infty$ we will without saying identify $\mathfrak{H}$ with $\mathbb{C}^n$, and represent A, B as n -by- n matrices: $A, B \in \mathbb{C}^{n \times n}$.

Recall that if f is a monotonic function on $\mathbb{R}$ and $A \succcurlyeq B$, then

$$(1) \quad f(A) \succcurlyeq f(B)$$

may not hold. The functions f defined on $[0, \infty)$ and such that (1) holds for any PSD operators (respectively, n -by- n matrices) A, B satisfying $A \succcurlyeq B$ are called *monotone operator functions* (monotone matrix functions of order n); see [3], [10, Chapter 6], or [1, Chapter 5]. More recent results and references on the subject can be found in [12].

It is known that $f(x) = x^t$ with $0 \leq t \leq 1$ is a monotone operator function (giving the so-called Löwner-Heinz inequality from (1)), while for $t > 1$ it is not a monotone matrix function of any order $n \geq 2$.

A dual approach is to consider conditions to be imposed on the PSD pair A, B so that (1) holds for $f(x) = x^t$ for any $t > 0$, or more generally, for any f non-decreasing on $[0, \infty)$. As it happens, these requirements are equivalent. More specifically, the following result can be extracted from [11]; see also [4]. Notation P_x in its statement means the value of the spectral function on $[x, \infty)$.

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Theorem 1. *The following statements about PSD operators A, B are equivalent:*

- (i) $P_x(A) \succcurlyeq P_x(B)$ for all $x \in \mathbb{R}$,
- (ii) $A^{t_k} \succcurlyeq B^{t_k}$ for some unbounded sequence $t_k (> 0)$,
- (iii) $f(A) \succcurlyeq f(B)$ for any function f non-decreasing on $\mathbb{R}_+$.

If condition (i) of Theorem 1 holds for a pair of selfadjoint operators A, B , we will write $A \succcurlyeq^s B$ and say that A *spectral dominates* B . Due to Theorem 1, in case of PSD A, B we will alternatively use the term *power dominance*. It is clear that $\succcurlyeq^s$ is a partial order, stronger than $\succcurlyeq$.

We are interested in two new aspects of spectral dominance. In Section 2, we give an alternative description of the property $A \succcurlyeq^s B$ applicable in the finite-dimensional setting. Section 4 pursues the conditions under which spectral dominance is inherited under taking compressions. The intermediate Section 3 contains some additional observations and comments.

2. SPECTRAL ORDER AND COMMUTING CHAINS

Throughout this section, we are dealing with the finite-dimensional setting only. Observe that condition (i) of Theorem 1 then simply means that an eigenvector x of A corresponding to its eigenvalue λ is orthogonal to an eigenvector y of B corresponding to its eigenvalue μ whenever $\lambda < \mu$. For convenience of reference we will say that the ordered pair of n -by- n PSD matrices A, B satisfying this condition is *upward orthogonal*.

In what follows, we write $X \smile Y$ if the matrices X and Y commute, and we say that $X_0, \dots, X_m$ is a commuting chain (of matrices) if $X_0 \smile X_1, X_1 \smile X_2, \dots, X_{m-1} \smile X_m$.

Theorem 2. *Let A and B be n -by- n PSD matrices. Denote by $\lambda_1 \geq \dots \geq \lambda_n$ all the eigenvalues of A , and by $x_1, \dots, x_n$ an orthonormal set of corresponding eigenvectors. Similarly, let $\mu_1 \geq \dots \geq \mu_n$ denote the eigenvalues of B , with $y_1, \dots, y_n$ being an orthonormal set of corresponding eigenvectors. Then, the following are equivalent:*

- (i) $A^{t_k} \succcurlyeq B^{t_k}$ for some unbounded sequence $t_k (> 0)$;
- (ii) $x_j^* A^t x_j \geq x_j^* B^t x_j$ for all $t \geq 0$ and $j = 1, \dots, n$;
- (iii) $y_j^* A^t y_j \geq y_j^* B^t y_j$ for all $t \geq 0$ and $j = 1, \dots, n$;
- (iv) the pair A, B is upward orthogonal;
- (v) there exist PSD matrices $C_1, \dots, C_k$ such that $A \succcurlyeq C_1 \succcurlyeq C_2 \succcurlyeq \dots \succcurlyeq C_k \succcurlyeq B$ is a commuting chain; and
- (vi) for every non-decreasing function f defined on $\mathbb{R}_+$, we have $f(A) \succcurlyeq f(B)$.

Of course, the equivalence of statements (i), (iv) and (vi) in Theorem 2 follows directly from Theorem 1. We nevertheless give here an independent proof of all the equivalences in Theorem 2, because it is simpler and allows us to incorporate the additional statements (ii), (iii), (v) smoothly in the flow of the proof.

The subtle part of the proof is the implication (iv) $\longrightarrow$ (v). It uses the following auxiliary result.

Lemma 3. *Let the pair A, B be upward orthogonal. Then:*

- (i) $\lambda_i \geq \mu_i$ for all $i = 1, \dots, n$ and

- (ii) *there exists a matrix $C \succcurlyeq B$ and commuting with B such that the pair A, C is upward orthogonal and unitarily reducible.*

Proof. (i) Suppose that $\lambda_i < \mu_i$ for some i . Then $\lambda_j < \mu_k$ for all $j = i, \dots, n$ and $k = 1, \dots, i$, so that the i -dimensional span of $y_1, \dots, y_i$ is orthogonal to the $(n - i + 1)$ -dimensional span of $x_i, \dots, x_n$. This is an impossibility.

(ii) Let k be the smallest index such that $\lambda_n \geq \mu_k$ (such a k exists due to the already-proved part of the lemma). Let C be the Hermitian matrix with the same eigenvector basis as B and the eigenvalues

$$\nu_j = \begin{cases} \mu_j & \text{if } j < k, \\ \lambda_n & \text{otherwise.} \end{cases}$$

It is clear that C commutes with B and $C \succcurlyeq B$. Since $\nu_j > \lambda_k$ if and only if $\mu_j > \lambda_k$, the pair A, C is upward orthogonal as is the pair A, B . On the other hand, the “upward orthogonality” of the pair A, B and our choice of k imply that the eigenvector x_n of A is orthogonal to the span of $y_1, \dots, y_{k-1}$ and therefore lies in the span of $y_k, \dots, y_n$. Since the latter is the eigenspace of C (corresponding to the eigenvalue λ_n), the matrices A and C have common eigenvectors and therefore joint non-trivial invariant subspaces. $\square$

Proof of Theorem 2. Implications (i) $\longrightarrow$ (ii), (i) $\longrightarrow$ (iii) and (vi) $\longrightarrow$ (i) are obvious. If (v) holds and f is as in (vi), then $f(A) \succcurlyeq f(C_1) \succcurlyeq \dots \succcurlyeq f(C_k) \succcurlyeq f(B)$, so that (vi) holds. Thus, (v) $\longrightarrow$ (vi).

Expanding x_j across the orthonormal basis $\{y_k\}_{k=1}^n$, we may rewrite (ii) as

$$\lambda_j^t \geq \sum_{k=1}^n |x_j^* y_k|^2 \mu_k^t, \quad t \geq 0.$$

The latter inequality cannot hold for t sufficiently large, unless $x_j^* y_k = 0$ whenever $\mu_k > \lambda_j$. Thus, (ii) $\longrightarrow$ (iv). The implication (iii) $\longrightarrow$ (iv) can be proved in a similar fashion. It remains to show that (iv) $\longrightarrow$ (v).

Applying Lemma 3, we reduce the general case to the situation in which the pair A, B is unitarily reducible. Since upward orthogonality is preserved under unitary similarity, without loss of generality we may suppose that A and B are block diagonal:

$$(2) \quad A = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & \\ & B_2 \end{bmatrix},$$

with the square blocks A_1 and B_1 (and therefore A_2 and B_2) of the same size. The pairs A_1, B_1 and A_2, B_2 are upward orthogonal together with the pair (2). A straightforward induction argument (on the size of the matrices under consideration) completes the proof. $\square$

3. FURTHER REMARKS

3.1. Related inequalities. Among other things, Theorem 1 means that if $A \overset{s}{\succcurlyeq} B$, then $A^p \succcurlyeq B^p$ for all $p \geq 0$. Consequently, $B^r A^p B^r \succcurlyeq B^{p+2r}$ for all $p, r \geq 0$ ($p \geq 0$, $r \in \mathbb{R}$, if B is invertible), and

$$(3) \quad (B^r A^p B^r)^{1/q} \succcurlyeq B^{(p+2r)/q}$$

for $p, r \geq 0$, $q \geq 1$ ($p \geq 0$, $r \in \mathbb{R}$, $q \geq 1$ if B is invertible) by the Löwner-Heinz inequality. Similarly,

$$(4) \quad A^{(p+2r)/q} \succcurlyeq (A^r B^p A^r)^{1/q}$$

for $p, r \geq 0$, $q \geq 1$ ($p \geq 0$, $r \in \mathbb{R}$, $q \geq 1$ if A is invertible). It is interesting to compare these statements with Furuta's result [5] according to which (3) and (4) hold if $A \succcurlyeq B$ and the parameters satisfy the more restrictive conditions $p, r \geq 0$, $q \geq 1$, $(1+2r)q \geq p+2r$. It is also shown in [5] that the latter condition is essential, even for 2×2 matrices A, B . Naturally, in Furuta's counterexample A does not spectral dominate B .

Observe that condition $q \geq 1$ cannot be weakened to $q \geq 0$ even when $A \stackrel{s}{\succcurlyeq} B$. In other words, it is not true that $A \stackrel{s}{\succcurlyeq} B$ implies $B^r A^p B^r \stackrel{s}{\succcurlyeq} B^{p+2r}$. The following example illustrates this point for $p = r = 1$.

Example. Let

$$A = \begin{bmatrix} 2+\epsilon & \epsilon \\ \epsilon & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then for any $\epsilon \in (0, 1)$ both eigenvalues of A exceed 2, so that $A \stackrel{s}{\succcurlyeq} B$. On the other hand, for ϵ small enough the eigenvalues of BAB are close to 2 and 12 while B^3 has the eigenvalues 1 and 8. Since BAB and B^3 do not commute when $\epsilon \neq 0$, there is no spectral dominance relation between them.

See [6] and the references therein for additional classes of operator inequalities equivalent to spectral dominance.

3.2. Explicit construction. The proof of Theorem 2, as given, hides somewhat a nice construction for the inserted, sequentially commuting matrices C_i , described in (v). For this reason we give an explicit example of how this may be done.

Example. We begin with the interspersal pattern

$$\lambda_1 > \lambda_2 > \lambda_3 > \mu_1 = \mu_2 > \lambda_4 > \mu_3 > \lambda_5 > \mu_4 > \mu_5$$

for n -by- n PSD matrices A, B with eigenvalues and eigenvectors labeled as before. Theorem 2 requires that for A, B to be upward orthogonal, x_4 must be orthogonal to y_1, y_2 and x_5 to y_1, y_2, y_3 . We construct a commuting chain of three matrices between A and B . Set

$$\nu_j = \begin{cases} \mu_j & \text{if } j < 4, \\ \lambda_5 & \text{otherwise.} \end{cases}$$

Since x_5 lies in the span of y_4, y_5 we may set the eigenvector associated with $\nu_5 = \lambda_5$ to be equal to x_5 and choose some vector z_4 also contained in the span of y_4, y_5 and orthogonal to x_5 to be the eigenvector for $\nu_4 = \lambda_5$. Keeping all other associated eigenvectors and eigenvalues in B , we have determined C_3 to be the matrix with eigenvalues and eigenvectors as indicated:

$$\begin{array}{ccccc} \mu_1 & \mu_2 & \mu_3 & \lambda_5 & \lambda_5 \\ y_1 & y_2 & z_3 & z_4 & x_5 \end{array}$$

A and C_3 share the eigenvector x_5 and so are jointly reducible. Thus, in constructing C_2 we consider only the $n = 4$ block matrices determined by the first four eigenvalues and eigenvectors of A and C_3 after unitary reduction. Since $\mu_1, \mu_2 > \lambda_3$, we set $\nu_3, \nu_4 = \lambda_4$ and retain all other eigenvalues and eigenvectors. Since the pair

A, C_3 is upward orthogonal, x_4 is orthogonal to y_1, y_2 and to x_5 by assumption. Then x_4 lies in the span of y_3, z_4 , so we assign x_4 as the eigenvector associated with $\nu_4 = \lambda_4$ and choose z_3 orthogonal to x_5 in the same subspace for eigenvalue ν_3 . Having determined C_2 , we construct C_1 in a similar fashion. Noting that A and C_2 share eigenvectors x_4, x_5 we consider $n = 3$. Because $\lambda_3 > \mu_1$, we set $\nu_1, \nu_2, \nu_3 = \lambda_3$ and keep the other eigenvalues and eigenvectors. The eigenvectors x_1, x_2, x_3 lie in the span of y_1, y_2, y_3 , so we may choose these to be the eigenvectors associated with corresponding ν 's. The commuting chain of matrices is summarized below.

$$\begin{array}{rcl}
 B: & \begin{array}{ccccc} \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \end{array} \\
 \\
 C_3: & \begin{array}{ccccc} \mu_1 & \mu_2 & \mu_3 & \lambda_5 & \lambda_5 \\ y_1 & y_2 & z_3 & z_4 & x_5 \end{array} \\
 \\
 C_2: & \begin{array}{ccccc} \mu_1 & \mu_2 & \lambda_4 & \lambda_4 & \lambda_5 \\ y_1 & y_2 & z_3 & x_4 & x_5 \end{array} \\
 \\
 C_1: & \begin{array}{ccccc} \lambda_3 & \lambda_3 & \lambda_3 & \lambda_4 & \lambda_5 \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\
 \\
 A: & \begin{array}{ccccc} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{array}
 \end{array}$$

By construction, $A \succ C_1 \succ C_2 \succ C_3 \succ B$ and $A \sim C_1, C_1 \sim C_2, C_2 \sim C_3, C_3 \sim B$. It follows that $A \stackrel{s}{\succ} B$.

Each application of Lemma 3 decreases the size of the matrices under consideration by (at least) one. Thus, the number k of intermediate matrices C_j in condition (v) of Theorem 2 can be chosen not to exceed $n - 1$. A better upper bound can be obtained if we first unitarily reduce the pair A, B to the smallest blocks possible. Namely, $k \leq m - 1$, where m is the size of the largest block in such a reduction. In particular, for commuting matrices A and B , $m = 1$, so that no intermediate matrices are needed at all ($k = 0$). One intermediate matrix suffices if and only if the pair A, B is unitarily similar to the orthogonal sum of pairs A_k, B_k such that the smallest eigenvalue of A_k beats the largest eigenvalue of B_k . This happens in particular if $\lambda_n \geq \mu_1$, in which case the intermediate matrix C can be chosen as a scalar multiple of the identity. It also happens if $m = 2$, which is guaranteed for example (see [2, 9], or [7]) when A and B belong to a C^* -algebra generated by two orthogonal projections. On the other hand, even for $n = 3$ it is possible that two intermediate matrices are required. This happens when

$$\lambda_1 > \lambda_2 > \mu_1 > \lambda_3 > \mu_2 > \mu_3,$$

x_3 is orthogonal to y_1 (so that $A \stackrel{s}{\succ} B$) but A, B do not have common eigenvectors and therefore form a unitarily irreducible pair.

Note, further, that from the construction, a number of C_i 's equal to $n + 1$ less the greatest index j such that $\lambda_j \geq \mu_1$ is sufficient for $n > 1$. In the simultaneously reducible case, an improvement may be achieved. The quantity may be calculated for each pair of corresponding blocks in the finest reduced form, relative to their size, and the largest such quantity is sufficient. The fact that $n = 1$ is an exception (no matrices need be inserted) explains that no matrices need be inserted in the

commuting case. Do these bounds, taken together, give the fewest possible inserted matrices? We know of no counterexample.

3.3. Diagonal entry dominance. Each of parts (ii) and (iii) in the statement of Theorem 2 may be interpreted in terms of a simultaneous unitary similarity on A and B that diagonalizes one of them. For example, (ii) indicates that the diagonal form of A raised to the power t has diagonal entries that beat those of B^t .

Observe that condition (i) of Theorem 2 in a trivial way implies that

$$(5) \quad z_j^* A^t z_j \geq z_j^* B^t z_j \text{ for all } t \geq 0 \text{ and } j = 1, \dots, n$$

for *any* orthonormal basis z_j of $\mathbb{C}^n$. Implications (i) $\rightarrow$ (ii) and (i) $\rightarrow$ (iii) of the theorem are just particular cases of this simple observation when the basis involved consists of the eigenvectors of A or B . The next example shows that switching to an arbitrary orthonormal basis in condition (ii) or (iii) does not give an equivalent condition even if, in addition, we suppose that $A \succcurlyeq B$.

Example. Consider the matrices

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{-4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{-4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 43 & 3 \\ 3 & 57 \end{bmatrix}.$$

Direct computations yield that the matrices A, B are PSD, $A \succcurlyeq B$, and

$$(6) \quad A^t = \frac{1}{2} \begin{bmatrix} 4^t + 2^t & -4^t + 2^t \\ -4^t + 2^t & 4^t + 2^t \end{bmatrix},$$

$$(7) \quad B^t = \frac{1}{25} \begin{bmatrix} 3^{t+2} + 16 & 3^{t+2} - 4 \cdot 3^{t+1} + 12 \\ 3^{t+2} - 4 \cdot 3^{t+1} + 12 & 16 \cdot 3^t + 9 \end{bmatrix}.$$

Elementary calculus shows that the diagonal entries of $A^t - B^t$,

$$\frac{1}{2}(4^t + 2^t) - \frac{9}{25}3^t - \frac{16}{25} \text{ and } \frac{1}{2}(4^t + 2^t) - \frac{16}{25}3^t - \frac{9}{25},$$

are monotonically increasing and thus positive functions of $t \geq 0$. However, A does not power dominate B . To see this, take for example $t = 10$ and obtain (to three decimal places)

$$(8) \quad A^{10} - B^{10} \approx 10^5 \begin{bmatrix} 5.035 & -4.954 \\ -4.954 & 4.870 \end{bmatrix},$$

which has eigenvalues, to three decimal places, 9.908×10^5 and -226.924 .

4. TOTAL SPECTRAL DOMINANCE

Recall that for any subspace $\mathcal{L}$ of $\mathfrak{H}$, the *compression* $T_{\mathcal{L}}(A)$ of A onto $\mathcal{L}$ is defined as the linear transformation PAP considered on $\mathcal{L}$, in which P is the orthogonal projection of $\mathfrak{H}$ onto $\mathcal{L}$. If $\mathfrak{H} = \mathbb{C}^n$ and $\mathcal{L}$ is the subspace generated by the coordinate vectors e_i with i from $\alpha \subset \{1, \dots, n\}$, then $T_{\mathcal{L}}(A)$ is simply the principal submatrix of A indexed by α , the standard notation for which is $A[\alpha]$.

We say that a (partial) order $\geq$ on the set of selfadjoint operators is *inherited* if $A \geq B$ implies $T_{\mathcal{L}}(A) \geq T_{\mathcal{L}}(B)$ for all subspaces $\mathcal{L}$ of $\mathfrak{H}$. It is well known (and follows directly from the definition) that the order $\succcurlyeq$ is inherited, and it is therefore natural to ask the same question about $\overset{s}{\succcurlyeq}$.

Observe that for one-dimensional $\mathcal{L}$ the compressions $T_{\mathcal{L}}(A)$ and $T_{\mathcal{L}}(B)$ are just numbers. Thus, $T_{\mathcal{L}}(A) \succ T_{\mathcal{L}}(B)$ in this case implies $T_{\mathcal{L}}(A) \succ^s T_{\mathcal{L}}(B)$. Consequently, the order $\succ^s$ is inherited in the case $n = 2$.

The following example shows that, starting with $n = 3$, inheritance need not hold.

Example. Let A and B be the two PSD matrices below:

$$A = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 7 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & \frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{30}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{5}{\sqrt{30}} & \frac{-1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{30}} & \frac{-1}{\sqrt{30}} & \frac{5}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{bmatrix}.$$

Keeping our notation from before, we have the interspersal

$$\lambda_1 > \lambda_2 > \mu_1 > \lambda_3 > \mu_2 > \mu_3$$

with the pair A, B upward orthogonal. Then $A \succ^s B$. Consider the submatrices $A[1, 2]$ and $B[1, 2]$. Direct computations yield

$$A[1, 2] = \begin{bmatrix} \frac{13}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{35}{3} \end{bmatrix},$$

$$B[1, 2] = \begin{bmatrix} \frac{29}{15} & \frac{23}{15} \\ \frac{123}{15} & \frac{127}{30} \end{bmatrix}.$$

Because $A \succ B$, we know that $A[1, 2] \succ B[1, 2]$. The eigenvalues of $A[1, 2]$ are $\hat{\lambda}_1 \approx 6.613$ and $\hat{\lambda}_2 \approx 3.554$ while those of $B[1, 2]$ are $\hat{\mu}_1 = 5$ and $\hat{\mu}_2 = \frac{7}{6}$. This gives the interspersal

$$\hat{\lambda}_1 > \hat{\mu}_1 > \hat{\lambda}_2 > \hat{\mu}_2.$$

As noted previously, in the 2-by-2 case, for spectral dominance to hold with this pattern of interspersal, $A[1, 2]$ must commute with $B[1, 2]$. This is not the case.

It therefore makes sense to consider yet another order on the set of selfadjoint operators. Namely, we say that A *totally spectral dominates* B if for all subspaces $\mathcal{L}$ of $\mathfrak{H}$, $T_{\mathcal{L}}(A) \succ^s T_{\mathcal{L}}(B)$. Let us also agree to call the selfadjoint operators A, B *identity aligned* if A, B, I are linearly dependent over $\mathbb{C}$ (equivalently: over $\mathbb{R}$).

Theorem 4. *A totally spectral dominates B if and only if either (i) the spectrum of A lies to the right of the spectrum of B, or (ii) A $\succ$ B and A, B are identity aligned.*

Of course, condition (ii) implies that the operators A and B commute. In the finite-dimensional setting, Theorem 4 therefore implies the following simpler statement.

Corollary 5. *Let the pair of n-by-n matrices A, B be unitarily irreducible. Then A totally spectral dominates B if and only if $\lambda_n \geq \mu_1$.*

Observe also that for $n = 2$ the matrices A, B are identity aligned whenever they commute. Therefore, in this case, Theorem 4 follows directly from the previous discussion of spectral dominance and the fact that for $n = 2$ it coincides with total spectral dominance.

The rest of this section is devoted to the proof of Theorem 4 for $n \geq 3$.

Proof of sufficiency. Under condition (i), let us invoke the fact that for any self-adjoint operator X the closure of its numerical range $W(X)$ is the convex hull of its spectrum (see [8, 10] for this and other relevant properties of the numerical range). Consequently, $W(A)$ lies to the right of $W(B)$. This property is inherited by the compressions $T_{\mathcal{L}}(A), T_{\mathcal{L}}(B)$. This in turn implies spectral dominance of $T_{\mathcal{L}}(A)$ over $T_{\mathcal{L}}(B)$.

Under condition (ii), one simply needs to observe that the identity aligned property is inherited by compressions. Thus, for any subspace $\mathcal{L}$, the compressions of A and B onto $\mathcal{L}$ commute. So, $A \succcurlyeq B$ in this setting as well implies not only that $T_{\mathcal{L}}(A) \succcurlyeq T_{\mathcal{L}}(B)$, but in fact $T_{\mathcal{L}}(A) \stackrel{s}{\succcurlyeq} T_{\mathcal{L}}(B)$. $\square$

The proof of necessity is more involved, and we divide it into several steps. First, we prove it in what seems a very narrow setting.

Lemma 6. *Let A and B be commuting 3-by-3 PSD matrices. If A totally power dominates B and $\lambda_3 < \mu_1$, then A and B are identity aligned.*

Proof. If one of the matrices A, B is a scalar multiple of the identity, the pair A, B obviously is identity aligned. Therefore, we may suppose that A, B are not scalar multiples of I .

Due to the commutativity of A and B , we can choose the orthonormal basis consisting of joint eigenvectors of A and B . In other words, we may suppose that (in our standard notation of Theorem 2) y_1, y_2, y_3 is simply a permutation of x_1, x_2, x_3 . Since $\lambda_3 < \mu_1$, y_1 must be different from x_3 . Relabel the remaining element of the basis $\{x_1, x_2, x_3\}$ by z .

Consider a 2-dimensional subspace $\mathcal{L}$ of $\mathbb{C}^3$, denoting by u a unit vector in its orthogonal complement. For u sufficiently close to z , the spectra of $T_{\mathcal{L}}(A)$ and $T_{\mathcal{L}}(B)$ will contain points close to λ_1 and μ_3 , respectively. Consequently, the largest eigenvalue of $T_{\mathcal{L}}(B)$ is greater than the smallest eigenvalue of $T_{\mathcal{L}}(A)$. From here and the total power dominance of A over B it follows that $T_{\mathcal{L}}(A)$ commutes with $T_{\mathcal{L}}(B)$. Since A and B also commute, we conclude that

$$(9) \quad PAQB P = PBQA P,$$

where P is the orthogonal projection onto $\mathcal{L}$ and $Q = I - P$ is the complementary projection.

Now impose the condition that u is not orthogonal to any of the x_j , and therefore is not an eigenvector of A or B . Then $A\mathcal{L}$ is not a subspace of $\mathcal{L}$, so that the set of $x \in \mathcal{L}$ such that $Ax \in \mathcal{L}$ is at most one-dimensional. The same is true for the set of $x \in \mathcal{L}$ such that $Bx \in \mathcal{L}$. Then by simple dimension considerations, there is a $v \in \mathcal{L}$ for which $Av, Bv \notin \mathcal{L}$, so that $QAv, QBv \neq 0$.

Then

$$QAv = c_1 u, \quad QBv = c_2 u,$$

in which c_1, c_2 are non-zero scalars. From here and (9) it follows that

$$P(c_2 A - c_1 B)u = 0;$$

that is, u is an eigenvector of $c_2A - c_1B$. Due to the choice of u , this is possible only when $c_2A - c_1B$ is a scalar multiple of the identity matrix, that is, when A and B are identity aligned. $\square$

Next, an auxiliary statement.

Lemma 7. *Let A and B be n -by- n matrices such that A totally spectral dominates B while $\lambda_i < \mu_j$ for some $i, j = 1, \dots, n$. Then y_j (an eigenvector of B corresponding to the eigenvalue μ_j) is also an eigenvector of A , and x_i (an eigenvector of A corresponding to the eigenvalue λ_i) is also an eigenvector of B .*

Proof. We will prove the statement regarding y_j ; the proof for x_i is similar.

Consider a two-dimensional subspace $\mathcal{L}$ spanned by y_j and a vector x the choice of which will be discussed later. Then the linear transformation $T_{\mathcal{L}}(B)$ has the diagonal matrix

$$\begin{bmatrix} \mu_j & 0 \\ 0 & \tilde{\mu} \end{bmatrix}$$

in the orthonormal basis with the first vector y_j . Choosing x sufficiently close to x_i we may guarantee that one of the eigenvalues of $T_{\mathcal{L}}(A)$ is so close to λ_i that it is smaller than μ_j . Since $T_{\mathcal{L}}(A) \succ^s T_{\mathcal{L}}(B)$, it follows that $\tilde{\mu} < \mu_j$ and that $T_{\mathcal{L}}(A) \sim T_{\mathcal{L}}(B)$. Consequently, the vector Ay_j is orthogonal with $x - y_j^*xy_j$.

Letting x run through a basis of $\mathbb{C}^n$ (which can be done while keeping it as close to x_i as needed for the reasoning above to be valid), we conclude that Ay_j is collinear to y_j . $\square$

Note that Corollary 5 is proved at this stage.

Proof of necessity; general finite-dimensional setting. We need only consider pairs A, B satisfying the conditions of Lemma 7. Fix a pair i, j for which $\lambda_i < \mu_j$, and choose the respective eigenvectors x_i, y_j . The eigenvalue $\tilde{\lambda}$ of A corresponding to the eigenvector y_j (we are using Lemma 7 here) is greater than μ_j due to the property $A \succ B$, and is therefore different from λ_i . Thus, the vectors x_i, y_j are orthogonal, and we can introduce a unitary transformation U that maps e_1 to x_i , e_2 to y_j . The matrices A, B under the similarity by U take the form

$$\begin{bmatrix} \lambda_i & \\ & \tilde{\lambda} \end{bmatrix} \oplus A_0 \text{ and } \begin{bmatrix} \tilde{\mu} & \\ & \mu_j \end{bmatrix} \oplus B_0,$$

respectively.

Now let $\mathcal{L}$ be a 3-dimensional subspace spanned by x_i, y_j and yet another vector x orthogonal to the first two. Then the compressions of A and B onto $\mathcal{L}$ with respect to the basis $\{x_i, y_j, x\}$ are diagonal matrices with the diagonal entries $\lambda_i, \tilde{\lambda}, x^*A_0x$ and $\tilde{\mu}, \mu_j, x^*B_0x$, respectively. Since the total power dominance is inherited by definition, we are now in the setting of Lemma 6. According to this lemma,

$$(10) \quad x^*B_0x = \alpha x^*A_0x + \beta,$$

with

$$\alpha = \frac{\mu_j - \tilde{\mu}}{\tilde{\lambda} - \lambda_i}, \quad \beta = \frac{\tilde{\mu}\tilde{\lambda} - \mu_j\lambda_i}{\tilde{\lambda} - \lambda_i}.$$

Letting x run through the unit sphere of $\mathbb{C}^{n-2}$, we conclude from (10) that $B_0 = \alpha A_0 + \beta I$, and therefore $B = \alpha A + \beta I$. $\square$

The transition to infinite dimensions is easy.

Proof of necessity; infinite-dimensional setting. It suffices to consider two self-adjoint operators A, B such that A totally spectral dominates B while

$$a = \min \sigma(A) < b = \max \sigma(B).$$

Pick a unit vector x in the range of the spectral projection of A corresponding to the interval $[a, a + \epsilon]$ and a unit vector y in the range of the spectral projection of B corresponding to the interval $[b - \epsilon, b]$ for some $\epsilon < (a + b)/2$. These vectors are orthogonal, since $A \stackrel{s}{\succ} B$. On the other hand, $\langle Ax, x \rangle < \langle By, y \rangle$. Choosing any unit vector z orthogonal to x, y and letting $\mathcal{L} = \text{span}\{x, y, z\}$, we conclude from the already-proved finite-dimensional version that

$$(11) \quad T_{\mathcal{L}}(B) = \alpha T_{\mathcal{L}}(A) + \beta I.$$

The values of α and β are determined uniquely by the compressions of A and B onto $\text{span}\{x, y\}$ and thus do not depend on the choice of z .

It follows that the numerical range of the compression of $B - \alpha A - \beta I$ onto $(\text{span}\{x, y\})^{\perp}$ is zero. Consequently, the compression itself is equal to zero. Since, moreover, $(B - \alpha A - \beta I)z$ is orthogonal to x, y for any z as above, in fact the restriction of $B - \alpha A - \beta I$ onto $(\text{span}\{x, y\})^{\perp}$ is zero. Combining this conclusion with (11) completes the proof. $\square$

Note that only compressions onto 2-dimensional subspaces were used in the proofs in the case of unitarily irreducible pairs of matrices, and 2- or 3-dimensional subspaces in the general case. Consequently, the following observation holds.

Corollary 8. *Let A, B be a pair of selfadjoint operators. Then A totally spectral dominates B if and only if $T_{\mathcal{L}}(A) \stackrel{s}{\succ} T_{\mathcal{L}}(B)$ for all subspaces $\mathcal{L}$ of dimension not exceeding 3 (2, if A, B is the pair of unitarily irreducible matrices).*

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