ON MULTIPLIERS FOR HARDY-SOBOLEV SPACES

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ABSTRACT. It is shown that membership of holomorphic functions in Hardy-Sobolev spaces in the unit ball cannot be characterized by finiteness of any integral norm. In addition, sufficient conditions are given for a holomorphic function to be a pointwise multiplier of a Hardy-Sobolev space.

1. INTRODUCTION

A natural question to ask about a space of holomorphic functions is whether membership of a given holomorphic function in the space can be characterized by finiteness of some integral norm. For example, the Bergman-Sobolev space $A^2_{1/2}$ consisting of all holomorphic functions on the unit ball B in $\mathbb{C}^n$ with square integrable derivatives of order $1/2$ can be characterized as the space of holomorphic functions on B having boundary values that are square integrable on the sphere (the classical Hardy space). On the other hand, membership in the Bergman-Sobolev space A^2_1 , consisting of holomorphic functions with square integrable first derivatives, cannot be characterized by finiteness of any integral.

In this paper, we show that membership in any Hardy-Sobolev space H^p_s cannot be characterized by finiteness of any integral norm (see Theorem 4.3 for the precise statement). This extends a result of Arveson [1] for a particular Hilbert space which may be realized as the Hardy-Sobolev space $H^2_{(n-1)/2}$. As in Arveson's argument, the key technical step is the existence of functions in the ball algebra that are not in Hardy-Sobolev spaces.

A closely related issue is identification of the pointwise multipliers on the spaces under consideration. It was recently shown by Ryan and Stoll [6] that the Hardy-Sobolev space H^p_s forms a Banach algebra when $sp > n$, and therefore is its own multiplier algebra. Characterization of the multipliers on other Hardy-Sobolev spaces remains open, but we give sufficient conditions for a function to be a pointwise multiplier in Section 5. We do not know if our sufficient conditions are also necessary.

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2. PRELIMINARIES

For $0 < q < \infty$ and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, let

$$k_q(\lambda) = (1 - \lambda)^{-q} = \sum \frac{(q)_j}{j!} \lambda^j$$

and for z and ζ in the unit ball B of $\mathbb{C}^n$, let $K_q(z, \zeta) = k_q(\langle z, \zeta \rangle)$. In other words,

$$(2.1) \quad K_q(z, \zeta) = \frac{1}{(1 - \langle z, \zeta \rangle)^q} = \sum \frac{(q)_{|\alpha|}}{\alpha!} z^\alpha \bar{\zeta}^\alpha,$$

where

$$(q)_k = q(q+1) \cdots (q+k-1) = \frac{\Gamma(q+k)}{\Gamma(q)}.$$

In the sum on the right of (2.1), α runs over all multi-indices in $\mathbb{Z}_+^n$, and $|\alpha| = \sum \alpha_j$. When q is n or $n+1$, K_q is the classical Szegő or Bergman kernel, respectively. In general K_q is a positive definite sesquiholomorphic kernel on B , which is the reproducing kernel of a uniquely determined functional Hilbert space of holomorphic functions on B , which we denote by $\mathcal{H}_q$. To be explicit, $\mathcal{H}_q$ consists of all holomorphic functions

$$f(z) = \sum a_\alpha z^\alpha$$

on B such that

$$\|f\|_q^2 = \sum \frac{\alpha!}{(q)_{|\alpha|}} |a_\alpha|^2 < \infty.$$

The associated inner product

$$\left\langle \sum a_\alpha z^\alpha, \sum b_\alpha z^\alpha \right\rangle_q = \sum \frac{\alpha!}{(q)_{|\alpha|}} a_\alpha \bar{b}_\alpha$$

satisfies

$$f(z) = \langle f, K(\cdot, z) \rangle_q$$

for every $f \in \mathcal{H}_q$ and every $z \in B$. The monomials z^α form a complete orthogonal system for $\mathcal{H}_q$, and

$$(2.2) \quad \|z^\alpha\|_q^2 = \frac{\alpha!}{(q)_{|\alpha|}}.$$

It will also be convenient to define

$$K_0(z, \zeta) = -\frac{1}{\langle z, \zeta \rangle} \log(1 - \langle z, \zeta \rangle) = \sum \frac{1}{j+1} \langle z, \zeta \rangle^j$$

where $\log$ denotes the principal branch of the logarithm. As before, K_0 is positive definite on $B \times B$ and is the reproducing kernel of a functional Hilbert space $\mathcal{H}_0$ consisting of all holomorphic functions $f(z) = \sum a_\alpha z^\alpha$ on B satisfying

$$\|f\|_0^2 = \sum \frac{\alpha!(|\alpha|+1)}{|\alpha|!} |a_\alpha|^2 < \infty.$$

The Hilbert space $\mathcal{H}_n$ is the classical Hardy space $H^2(B)$, and

$$\|f\|_n^2 = \sup_{0 < r < 1} \int_S |f(rz)|^2 d\sigma(z),$$

where $d\sigma$ is the normalized surface measure on the unit sphere S . For $q > n$, $\mathcal{H}_q$ is a weighted Bergman space:

$$\|f\|_q^2 = \frac{\Gamma(q)}{\pi^n \Gamma(q-n)} \int_B |f(z)|^2 (1-|z|^2)^{q-n-1} dv(z),$$

where dv is the usual Euclidean volume element on $\mathbb{C}^n$. For these matters, see [5, Chapters 3 and 7].

For $0 \leq q < n$, $\mathcal{H}_q$ may be viewed as a Hardy-Sobolev space. For $0 < p < \infty$ and $s \in \mathbb{R}$, we define the Hardy-Sobolev space $H_s^p(B)$ to be the space of holomorphic functions on B satisfying

$$\|f\|_{p,s} = \|D^s f\|_{H^p},$$

where

$$D = 1 + \sum z_j \frac{\partial}{\partial z_j}$$

with fractional powers defined in terms of power series:

$$D^s \left(\sum a_\alpha z^\alpha \right) = \sum (1 + |\alpha|)^s a_\alpha z^\alpha.$$

With this notation, $\mathcal{H}_q = H_{(n-q)/2}^2$ with equivalent, but not equal, norms. For a detailed exposition of these matters, see [3].

3. A FUNCTION IN THE BALL ALGEBRA

Since the spaces $\mathcal{H}_q$ may be realized as Hardy or weighted Bergman spaces when $q \geq n$, it is clear that $\mathcal{H}_q$ contains the ball algebra when $q \geq n$. In this section we will use the homogeneous polynomials of Ryll and Wojtaszczyk [7] to construct a function in the ball algebra that is not in any of the Hilbert spaces $\mathcal{H}_q$ with $0 \leq q < n$.

In [7] it is shown that there is a sequence (p_k) of homogeneous polynomials in the unit sphere of $H^\infty(B)$ such that p_k has degree k and

$$\|p_k\|_n = \|p_k\|_{H^2} \geq \frac{\sqrt{\pi}}{2^n}.$$

Since p_k is homogeneous of degree k , it follows from (2.2) and Stirling's Formula that

$$\|p_k\|_q^2 = \frac{(n)_k}{(q)_k} \|p_k\|_n^2 \geq \frac{(n)_k}{(q)_k} \frac{\pi}{4^n} \geq ck^{n-q}$$

for a positive constant c depending only on n and q . Let

$$(3.1) \quad \varphi = \sum_{j=1}^{\infty} \frac{1}{j^2} p_{2^j}.$$

Since $\|p_k\|_\infty = 1$, the series converges uniformly on $\overline{B}$ to a function φ in the ball algebra. Moreover, since the polynomials p_k are homogeneous of different degrees, they are orthogonal in $\mathcal{H}_q$, so

$$\|\varphi\|_q^2 = \sum \frac{\|p_{2^j}\|_q^2}{j^4} \geq c \sum \frac{2^{j(n-q)}}{j^4}.$$

The sum on the right diverges if $q < n$, so $\varphi \notin \mathcal{H}_q$ for $q < n$. We have proved

Theorem 3.1. *The function φ defined by (3.1) is in the ball algebra, but is not in $\mathcal{H}_q$ for $0 < q < n$.*

For clarity, we have restricted our attention to the Hilbert spaces $\mathcal{H}_q$, but essentially the same construction can be used to show that some other unitarily invariant Hilbert spaces of holomorphic functions do not contain the ball algebra. To be precise, for any sequence (c_j) of nonnegative numbers such that $\sum \frac{c_j}{j!} \lambda^j$ converges on the unit disk, define

$$K(z, \zeta) = \sum \frac{c_j}{j!} \langle z, \zeta \rangle^j.$$

Then K is the reproducing kernel for a unitarily invariant functional Hilbert space $\mathcal{H}_K$ of holomorphic functions on the ball. The result is as follows.

Theorem 3.2. *Suppose there is a summable sequence (ε_j) of positive numbers and an increasing sequence (k_j) of positive integers such that $\sum \varepsilon_j^2 \frac{(n)_{k_j}}{c_{k_j}}$ diverges. Then the function $\varphi = \sum \varepsilon_j p_{k_j}$ is in the ball algebra, but is not in $\mathcal{H}_K$.*

Proof. As before, the series $\sum \varepsilon_j p_{k_j}$ converges uniformly on $\overline{B}$ and its sum φ is therefore in the ball algebra. Also as before, the monomials p_k are pairwise orthogonal in $\mathcal{H}_K$, and

$$\|p_k\|_K^2 = \frac{(n)_k}{c_k} \|p_k\|_n^2,$$

so

$$\begin{aligned} \|\varphi\|_K^2 &= \sum_j \varepsilon_j^2 \|p_{k_j}\|_K^2 \\ &= \sum_j \varepsilon_j^2 \frac{(n)_{k_j}}{c_{k_j}} \|p_{k_j}\|_n^2 \\ &\geq c \sum_j \varepsilon_j^2 \frac{(n)_{k_j}}{c_{k_j}} \\ &= \infty, \end{aligned}$$

and therefore $\varphi \notin \mathcal{H}_K$.

4. HARDY-SOBOLEV SPACES

In this section, we show that Hardy-Sobolev spaces do not contain certain holomorphic Lipschitz functions. We begin with a refinement of Theorem 3.1.

Theorem 4.1. *There is a function in the ball algebra that is not in H_s^p for any $1 \leq p \leq \infty$ and $s > 0$.*

Proof. Let φ be the function defined by (3.1). Since the spaces H_s^p are monotonically decreasing in p , it suffices to show that $D^s \varphi \notin H^1$ for any $s > 0$. Let

$$f = D^s \varphi = \sum \frac{(1 + 2^j)^s}{j^2} p_{2^j}$$

with uniform convergence on compact subsets of B . Since the polynomials p_k are orthogonal, for any $0 < r < 1$,

$$\int_S f(rz) \overline{p_{2^j}(z)} d\sigma(z) = r^{2^j} \frac{(1 + 2^j)^s}{j^2} \|p_{2^j}\|_{H^2}^2,$$

so

$$\sup_{0 < r < 1} \left| \int_S f(rz) \overline{p_{2^j}(z)} d\sigma(z) \right| = \frac{(1 + 2^j)^s}{j^2} \|p_{2^j}\|_{H^2}^2 \geq \frac{\pi}{4^n} \frac{(1 + 2^j)^s}{j^2},$$

which tends to ∞ as $j \rightarrow \infty$. On the other hand, since $\|p_k\|_\infty = 1$,

$$\sup_{0 < r < 1} \left| \int_S f(rz) \overline{p_{2j}(z)} d\sigma(z) \right| \leq \sup_{0 < r < 1} \int_S |f(rz)| d\sigma(z) = \|f\|_{H^1} < \infty.$$

This contradiction completes the proof.

For $s > 0$, the *holomorphic Lipschitz space* Λ_s is defined to be the space of all holomorphic functions f on B satisfying

$$\|f\|_{\Lambda_s} = \sup |D^k f(z)| (1 - |z|^s)^{s-k} < \infty,$$

where k is the least integer greater than s .

Theorem 4.2. *Let $0 < p < \infty$ and $s > 0$. For any $0 < \varepsilon < s$ there is a function $\varphi \in \Lambda_\varepsilon$ such that $\varphi \notin H_s^p$.*

Proof. If $\Lambda_\varepsilon \subset H_s^p$, then the inclusion must be continuous by the Closed Graph Theorem. Let

$$q_k = (1 + k)^{-\varepsilon} p_k,$$

where the p_k are the Ryll-Wojtaszczyk polynomials. Thus $\|D^\varepsilon q_k\|_\infty = \|p_k\|_\infty = 1$, and it follows that

$$(4.1) \quad \|q_k\|_{\Lambda_\varepsilon} \leq c$$

for some positive constant c depending on ε and the dimension n . On the other hand, for $0 < p \leq 2$,

$$\begin{aligned} \|q_k\|_{p,s}^p &= \int_S |D^{s-\varepsilon} p_k(z)|^p d\sigma(z) \\ &= (1 + k)^{p(s-\varepsilon)} \int_S |p_k(z)|^p d\sigma(z) \\ &\geq (1 + k)^{p(s-\varepsilon)} \int_S |p_k(z)|^2 d\sigma(z) \\ &\geq (1 + k)^{p(s-\varepsilon)} \frac{\pi}{4^n}. \end{aligned}$$

The first inequality follows from the fact that $p \leq 2$ and $|p_k| \leq 1$ pointwise. Since the right side tends to ∞ as $k \rightarrow \infty$, and, by (4.1), the sequence (q_k) is bounded in Λ_ε , there is no continuous inclusion of Λ_ε into H_s^p , which establishes the theorem when $0 < p \leq 2$. For $p > 2$, it's only necessary to observe that $H_s^p \subset H_s^2$, so the result follows from the known case $p = 2$.

It is natural to ask which spaces of holomorphic functions can be characterized by an integral norm. It is well known (see [3]) that the Hilbert spaces $\mathcal{H}_q$ are weighted Bergman spaces when $q > n$, and that $\mathcal{H}_n$ is a Hardy space. It was shown by Arveson [1] that $\mathcal{H}_1$ cannot be characterized by an integral norm when $n > 1$. Here we show that no Hardy-Sobolev space H_s^p with $s > 0$ can be characterized by an integral norm. In particular, $\mathcal{H}_q$ cannot be so characterized for $0 \leq q < n$. This is elementary when $n = 1$, but seems to be new in higher dimensions.

To be precise, for any Borel measure μ on $\overline{B}$ and any increasing function λ on $[0, \infty)$ with $\lambda(0) = 0$, let H_μ^λ denote the set of all holomorphic functions f on B satisfying

$$\Lambda(f) = \limsup_{r \rightarrow 1^-} \int_B \lambda(|f(rz)|) d\mu(z) < \infty.$$

Theorem 4.3. *Let $0 < p \leq \infty$ and $s > 0$. Then $H_s^p \neq H_\mu^\lambda$ as sets for every μ, ϕ .*

Proof. Suppose $H_s^p = H_\mu^\lambda$. Since the constant function 1 is in H_s^p , we have $\int \lambda(1) d\mu < \infty$, so μ is finite. But then for any f in the ball algebra and any $0 < r < 1$, we have

$$\int_B \lambda(|f(rz)|) d\mu(z) \leq \sup |f| \mu(\overline{B}),$$

so $f \in H_\mu^\lambda = H_s^p$, contradicting Theorem 4.2.

5. MULTIPLIERS

A *functional (quasi-)Banach space*¹ $\mathcal{B}$ on a set X is a (quasi-)Banach space of complex-valued functions on X such that for every $x \in X$ the functional $f \mapsto f(x)$ is continuous on $\mathcal{B}$. We call a functional quasi-Banach space $\mathcal{B}$ on X *total* if for each $x \in X$ there is an $f \in \mathcal{B}$ such that $f(x) \neq 0$.

A complex-valued function φ on X is a *multiplier* for $\mathcal{B}$ if the multiplication operator $M_\varphi f = \varphi f$ maps $\mathcal{B}$ into itself, in which case it follows from the Closed Graph Theorem that M_φ is continuous on $\mathcal{B}$. The set $\mathcal{M}(\mathcal{B})$ of multipliers on $\mathcal{B}$ is an algebra of functions on X , which, if $\mathcal{B}$ is total, may be viewed as a (quasi-)Banach algebra by defining the norm of a multiplier φ to be the operator norm of the multiplication operator M_φ .

Lemma 5.1. *Let $\mathcal{B}$ be a total functional quasi-Banach space on X . Then every multiplier φ for $\mathcal{B}$ is bounded pointwise by its multiplier norm.*

Proof. Let M_φ^* be the adjoint of the multiplication operator, which is an operator on the dual space $\mathcal{B}^*$. Let $x \in X$ and let ε_x be the evaluation functional at x . For any $f \in \mathcal{B}$,

$$(M_\varphi^* \varepsilon_x)g = \varepsilon_x(M_\varphi f) = \varphi(x)f(x) = \varphi(x)\varepsilon_x f,$$

so

$$M_\varphi^* \varepsilon_x = \varphi(x)\varepsilon_x.$$

Therefore

$$\|M_\varphi^*\| \|\varepsilon_x\| \geq |\varphi(x)| \|\varepsilon_x\|$$

and since $\varepsilon_x \neq 0$ by hypothesis,

$$|\varphi(x)| \leq \|M_\varphi^*\| \leq \|M_\varphi\|.$$

In the case of Hardy-Sobolev and Bergman spaces, the function space contains the constants, and so any multiplier must itself be a member of the space, and must therefore be holomorphic. Therefore we have

Lemma 5.2. *The multiplier algebra of a Hardy-Sobolev space is contained in H^∞ , and the H^∞ norm of a multiplier is less than or equal to its operator norm.*

Conversely, it is clear that any H^∞ function is a multiplier for any Hardy or Bergman space and that the multiplier norm is less than or equal to the H^∞ norm. Therefore

Lemma 5.3. *The multiplier algebra for any Hardy or Bergman space is precisely H^∞ , and the multiplier and H^∞ norms are equal.*

¹A *quasi-norm* on a vector space V is a function $v \mapsto \|v\|$ from V to $[0, \infty)$ that vanishes only at the zero vector, is positively homogeneous, and satisfies $\|v + w\| \leq C(\|v\| + \|w\|)$ for some positive constant C . A *quasi-Banach space* is a Cauchy complete quasi-normed vector space.

The above lemma does not extend to any Hardy-Sobolev space H_s^p with $s > 0$. For these spaces, there are bounded functions that are not multipliers. Indeed, by Theorem 4.2, there is a holomorphic Lipschitz function that is not in H_s^p , and therefore not a multiplier for H_s^p . This extends a result Arveson [1], who showed that there is a function in the ball algebra that is not in $H_{(n-1)/2}^2 = \mathcal{H}_1$ and is therefore not a multiplier for that space. Moreover, there are multipliers on H_s^p with multiplier norm strictly greater than the H^∞ norm.

Theorem 5.4. *Let $0 < p < \infty$ and let $s > 0$. For any $C > 0$ there is a multiplier φ with $\|\varphi\|_\infty \leq 1$ and $\|M_\varphi\| \geq C$.*

Proof. Suppose the theorem fails. Then there is a C such that for every multiplier φ we have

$$(5.1) \quad \|M_\varphi\| \leq C\|\varphi\|_\infty.$$

By Theorem 4.2, there is a function φ in the ball algebra that is not a multiplier. Let p_k be a sequence of polynomials that converges uniformly to φ . By (5.1), the sequence M_{p_k} is a Cauchy sequence in the operator norm, which must then be norm convergent to a continuous operator T on H_s^p . It follows that

$$(Tf)(z) = \lim_{k \rightarrow \infty} (M_{p_k}f)(z) = \lim_{k \rightarrow \infty} p_k(z)f(z) = \varphi(z)f(z)$$

for every $f \in H_s^p$ and every $z \in B$, and so $M_\varphi = T$ is continuous on H_s^p , a contradiction.

In order to identify multipliers on Hardy-Sobolev spaces, we will need an elementary variant of Hölder's Inequality.

Lemma 5.5. *Let $0 < p, q, r < \infty$ with $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then for any $f \in H^r$ and $g \in H^q$ the product fg is in H^p and*

$$\|fg\|_p \leq \|f\|_r \|g\|_q.$$

Proof. Apply Hölder's Inequality to the functions $|f|^p$ and $|g|^p$.

We next give a sufficient condition for multipliers on H_s^p when s is small.

Theorem 5.6. *Let $0 < p < \infty$ and let s be an integer with $0 < s < n/p$. Any $\varphi \in H_s^{n/s} \cap H^\infty$ is a multiplier for H_s^p .*

Before proceeding with the proof, we note that the spaces $H_s^{n/s}$ are monotonically decreasing in the parameter s :

$$H_t^{n/t} \subset H_s^{n/s} \quad \text{for } t \geq s.$$

Moreover, each of the spaces $H_s^{n/s}$ is contained in BMOA, and when $s \geq n$, is contained in the ball algebra. In particular, it follows that intersecting with H^∞ is redundant when $s \geq n$. See [3] and [2] for details on these matters.

Proof. Let $f \in H_s^p$ and let γ be a multi-index with $|\gamma| \leq s$. Then

$$\partial^\gamma(\varphi f) = \sum_{\alpha+\beta=\gamma} c_{\alpha,\beta} \partial^\alpha \varphi \partial^\beta f,$$

where $c_{\alpha,\beta}$ are positive constants. It therefore suffices to show that

$$\partial^\alpha \varphi \partial^\beta f \in H^p$$

for all multi-indices α and β with $|\alpha| + |\beta| \leq s$. Let's first dispense with the case $\alpha = 0$. By hypothesis, $\varphi \in H^\infty$, so

$$\|\varphi \partial^\beta f\|_p \leq \|\varphi\|_\infty \|\partial^\beta f\|_p < \infty.$$

For $\alpha \neq 0$, we have

$$\partial^\alpha \varphi \in H_{s-|\alpha|}^{n/s} \subset H^r,$$

where $r = n/|\alpha|$. The last inclusion is a consequence of the Sobolev Embedding Lemma for Hardy-Sobolev spaces (see [3]). Similarly,

$$\partial^\beta f \in H_{s-|\beta|}^p \subset H_{|\alpha|}^p \subset H^q,$$

where

$$\frac{n}{q} = \frac{n}{p} - |\alpha| = \frac{n}{p} - \frac{n}{r}.$$

Thus

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r},$$

so it follows from Lemma 5.5 that $\partial^\alpha \varphi \partial^\beta f \in H^p$, which completes the proof.

For $s = n/p$ we have

Theorem 5.7. *If $s = n/p$ is an integer, and if $\varphi \in H_t^p$ for some $t > s$, then φ is a multiplier for H_s^p .*

Proof. Since $H_t^p \subset H_{n/p}^p \cap H^\infty$, the proof can proceed as that of Theorem 5.6, except for the estimation of terms of the form $(\partial^\alpha \varphi)f$ with $|\alpha| = n/p$. Letting $\varepsilon = t - s$, we have

$$\partial^\alpha \varphi \in H_\varepsilon^p \subset H^q,$$

where $\frac{n}{q} = \frac{n}{p} - \varepsilon$ and

$$f \in H_{n/p}^p \subset BMOA \subset \bigcap_{0 < r < \infty} H^r.$$

In particular, $f \in H^r$ with $r = n/\varepsilon$. Since $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$, it follows that $(\partial^\alpha \varphi)f \in H^p$.

For $s > n/p$, the multipliers for H_s^p are characterized by a result of Ryan and Stoll [6].

Theorem 5.8 (Ryan and Stoll). *Let s be a positive integer, satisfying $s > n/p$, or $s = n/p$ with $p \leq 1$. Then H_s^p is an algebra, and therefore is its own multiplier algebra.*

The above results on multipliers for Hardy-Sobolev spaces extend readily to Bergman-Sobolev spaces. We define $A_{q,s}^p$ to be the space of all holomorphic functions f on B satisfying

$$\|f\|_{p,q,s} = \left(\int_B |D^s f(z)|^p (1 - |z|^2)^{q-1} dv(z) \right)^{1/p} < \infty.$$

Here $s \geq 0$, $0 < p < \infty$, and $q > 0$. It is known that $A_{q_1,s_1}^p = A_{q_2,s_2}^p$ when $q_1 - ps_1 = q_2 - ps_2$.

Theorem 5.9. (1) *If $0 \leq s < q/p$, then the multiplier algebra for $A_{q,s}^p$ is H^∞ .*

(2) *If $\frac{q}{p} \leq s < \frac{n+q}{p}$, then each function in $A_{q,s}^{(n+q)/s} \cap H^\infty$ is a multiplier.*

- (3) If $s = (n + q)/p$, then the multiplier algebra for $A_{q,s}^p$ contains $A_{q,t}^p$ for every $t > s$.
- (4) If $s > (n + q)/p$ or if $p \leq 1$ and $s = (n + q)/p$, then $A_{q,s}^p$ is an algebra, which is therefore its own multiplier algebra.

Proof. Item 1 follows from the fact that $A_{q,s}^p$ is the same as the weighted Bergman space $A_{q-ps,0}^p$ when $ps < q$. When the parameter s is an integer, the remaining items can be proved by the same arguments used in the Hardy space case. For noninteger values of s , we have $A_{q,s}^p = A_{q_1,s_1}^p$ where s_1 is the least integer greater than or equal to s and $q_1 = q + p(s_1 - s)$, so the result follows from the integer case.

We now return to the Hardy-Sobolev spaces in the case $p = 2$. In this case (see [3]),

$$H_s^2 = A_{q,t}^2$$

whenever $t = s + q/2$. In the case $p = 2$, this observation allows us to remove the restriction that the parameter s is an integer, which was needed previously.

Theorem 5.10. (1) If $0 < s < \frac{n}{2}$, then each function in $H_{n/2}^2 \cap H^\infty$ is a multiplier.

(2) The multiplier algebra for $H_{n/2}^2$ contains H_t^2 for every $t > n/2$.

(3) If $s > n/2$, then H_s^2 is an algebra, which is therefore its own multiplier algebra.

Item 1 can be restated as

Corollary 5.11. For $0 < q < \infty$ the multiplier algebra for $\mathcal{H}_q$ contains $\mathcal{H}_0 \cap H^\infty$.

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