MODULAR FORMS OF HALF-INTEGRAL WEIGHT WITH FEW NON-VANISHING COEFFICIENTS MODULO ℓ

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ABSTRACT. Bruinier and Ono classified cusp forms of half-integral weight

$$F(z) := \sum_{n=0}^{\infty} a(n)q^n \in S_{\lambda+\frac{1}{2}}(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]]$$

whose Fourier coefficients are not well distributed for modulo odd primes ℓ . Ahlgren and Boylan established bounds for the weight of such a cusp form and used these bounds to prove Newman's conjecture for the partition function for prime-power moduli. In this note, we give a simple proof of Ahlgren and Boylan's result on bounds of cusp forms of half-integral weight.

1. INTRODUCTION AND RESULTS

Much is known about the divisibility of the Fourier coefficients of modular forms of integral weight. When the weight is half-integral, the situation is not as clear. The distribution of the coefficients of modular forms of half-integral weight in congruence classes has been studied by, among others, Balog, Darmon, and Ono [3], Ono and Skinner [8], [9], Bruinier [4], Bruinier and Ono [5], and Ahlgren and Boylan [1], [2].

Let $S_{\frac{k}{2}}(\Gamma_0(N), \chi)$ be the space of cusp forms of weight $\frac{k}{2}$ on the congruence subgroup $\Gamma_0(N)$ with Dirichlet character χ (see Chapter 3 in [7] for the basic definitions of a cusp form). Recently Bruinier and Ono proved in [5] that $F(z) := \sum_{n=0}^{\infty} a(n)q^n \in S_{\lambda+\frac{1}{2}}(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]]$ has a special form (see (1.1)) modulo an odd prime ℓ if $F(z)$ does not satisfy the following property: for every integer r

$$\#\{1 \leq n \leq X \mid a(n) \equiv r \pmod{\ell}\} \gg_{r,M} \begin{cases} \frac{\sqrt{X}}{\log X} & \text{if } r \not\equiv 0 \pmod{\ell}, \\ X & \text{if } r \equiv 0 \pmod{\ell}. \end{cases}$$

Ahlgren and Boylan obtained the following bounds for the weights of cusp forms not satisfying the above property.

Theorem 1.1 ([2]). *Suppose that we have the following hypotheses:*

- (i) $\lambda \geq 2$ is an integer, N is a positive integer with $4 \mid N$, and χ is a real Dirichlet character modulo N .
- (ii) $\ell \geq 5$ is a prime such that $\ell \nmid N$.

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(iii) $F(z)$ is a cusp form of half-integral weight such that

$$F(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{\lambda+\frac{1}{2}}(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]].$$

(iv) $F(z) \not\equiv 0 \pmod{\ell}$, and there are finitely many square-free integers $n_1, n_2, \dots, n_t$ such that

$$(1.1) \quad F(z) \equiv \sum_{i=1}^t \sum_{m=1}^{\infty} a(n_i m^2) q^{n_i m^2} \pmod{\ell}.$$

Write $\lambda = \bar{\lambda} + i_{\lambda}(\ell - 1)$ with $0 \leq \bar{\lambda} \leq \ell - 2$. Then the following are true:

(1) If $\ell \nmid n_i$ for some i , then

$$\bar{\lambda} \leq 2i_{\lambda} + 1.$$

(2) If $\ell \mid n_i$ for all i and $\bar{\lambda} \leq \frac{\ell-3}{2}$, then

$$\bar{\lambda} \leq 2i_{\lambda} - \frac{\ell-1}{2}.$$

(3) If $\ell \mid n_i$ for all i and $\bar{\lambda} \geq \frac{\ell-1}{2}$, then

$$\bar{\lambda} \leq 2i_{\lambda} + \frac{\ell+3}{2}.$$

In this note, we give an elementary proof of Theorem 1.1 (1). They used these bounds to prove Newman's conjecture [2] for the partition function for prime-power moduli. These bounds were proved in [2] by using Galois representations and Shimura-lifting. Here we obtain a simple proof of these bounds by arithmetic of filtrations of modular forms when $\ell \nmid n_i$ for some i .

2. MODULAR FORMS MOD ℓ

Throughout this section, we fix an integer $N \geq 1$ and a prime $\ell \nmid N$. We will recall the definitions and results that will be needed for our proof. The theory of modular forms modulo ℓ was developed by Serre [10] and Swinnerton-Dyer [11] for forms of level 1. We will recall the definitions and results that will be needed for the next section. For proofs when $N \geq 4$ and $\ell \nmid N$, see Gross [6].

Denote by M_k the space of modular forms of weight k on $\Gamma_1(N)$ with integer coefficients. If $f \in M_k$, let $\tilde{f} \in (\mathbb{Z}/\ell\mathbb{Z})[[q]]$ be the (coefficient-wise) reduction of $f \bmod \ell$. We write $\widetilde{M_k} := \{\tilde{f} \mid f \in M_k\}$ for the space of weight k modular forms modulo ℓ with level N .

The filtration $\omega(\tilde{f})$ of a modular form $\tilde{f} \in \widetilde{M_k}$ is defined to be

$$\omega(\tilde{f}) := \inf\{k' \mid \tilde{f} \in \widetilde{M_{k'}}\}.$$

If $f \in M_k$, we sometimes write $\omega(f)$ instead of $\omega(\tilde{f})$ if no confusion can arise. Note that $\omega(\tilde{f}) = -\infty$ if and only if $f \equiv 0 \pmod{\ell}$. If $f \in M_k$, $g \in M_{k'}$, and $f \equiv g \not\equiv 0 \pmod{\ell}$, then we must have $k \equiv k' \pmod{\ell-1}$. From this it follows that $\omega(\tilde{f}) \equiv k \pmod{\ell-1}$.

For an positive integer m , the operator U_m acts on q -expansions by

$$\sum_n a(n)q^n | U_m = \sum_n a(mn)q^n.$$

Note that if T_ℓ denotes the ℓ -th Hecke operator on M_k , then $f|T_\ell \equiv f|U_\ell \pmod{\ell}$, so U_ℓ acts on the spaces $\widetilde{M}_k$. We define the theta operator by its effect on Fourier expansions:

$$\theta \left(\sum_n a(n)q^n \right) = \sum_n na(n)q^n.$$

We record some basic properties of the filtration of a modular form modulo ℓ .

Lemma 2.1 ([6], [8], [11]). *Let $f \in \widetilde{M}_k$. Write $k = \bar{k} + i_k(\ell - 1)$ with $0 \leq \bar{k} \leq \ell - 2$.*

- (1) *If f is not a constant modulo ℓ , then $\omega(f) > 0$.*
- (2) *$\omega(f^\ell) = \ell \cdot \omega(f)$.*
- (3) *$\bar{k} \leq \omega(f) \leq k$.*
- (4) *$\omega(\theta f) \equiv k + \ell + 1 \pmod{\ell - 1}$.*
- (5) *Suppose that f is not a constant modulo ℓ . Then $\omega(\theta f) \leq \omega(f) + \ell + 1$ with equality if and only if $\omega(f) \not\equiv 0 \pmod{\ell}$.*

3. PROOF

Proof of Theorem 1.1(i). From the assumption take $n_v \in \{n_1, \dots, n_t\}$ such that $\ell \nmid n_v$. Following the argument of Lemma 4.1 of [2], we can find primes $p_{v,1}, \dots, p_{v,s_v}$ greater than ℓ , and a modular form $F_v(z) \in S_{\lambda+\frac{1}{2}}(\Gamma_0(Np_{v,1}^2 \cdots p_{v,s_v}^2, \chi)) \cap \mathbb{Z}[[q]]$ with the property that

$$F_v(z) \equiv \sum_{\gcd(m, \prod_j p_{v,j})=1} a(n_v m^2) q^{n_v m^2} \not\equiv 0 \pmod{\ell}.$$

We have

$$F_v(z) \in S_{\lambda+\frac{1}{2}} \left(\Gamma_0 \left(Nn_v p_{v,1}^2 \cdots p_{v,s_v}^2, \left(\frac{4n_v}{\bullet} \right) \chi \right) \right)$$

and

$$F_v(z) \equiv \sum_{\gcd(m, \prod_j p_{v,j})=1} a(n_v m^2) q^{m^2} \pmod{\ell}$$

(see Chapter 3 in [7]). Here, $\left(\frac{4n_v}{\bullet} \right)$ denotes the Jacobi symbol. Note that

$$\ell \nmid Nn_v p_{v,1}^2 \cdots p_{v,s_v}^2.$$

First, we consider the case that $\theta(F_v(z)|U_{n_v}) \equiv 0 \pmod{\ell}$. Let

$$T(z) := 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \in M_{\frac{1}{2}}(\Gamma_0(4)).$$

In this case, $F_v(z)|U_{n_v}$ has the form

$$F_v(z)|U_{n_v} \equiv \sum_{\gcd(m, \prod_j p_{v,j})=1} a(n_v \ell^2 m^2) q^{\ell^2 m^2} \pmod{\ell}.$$

This implies that $(F_v(z)|U_{n_v} U_{\ell^2})^{\ell^2} \equiv F_v(z)|U_{n_v} \pmod{\ell}$. Thus, we have

$$2i_\lambda + 1 \geq \ell + 2 \geq \bar{\lambda}$$

since from Lemma 2.1 (2) and (3)

$$\lambda + \frac{1 + \ell^2}{2} \geq \omega(F_v(z)|U_{n_v} \cdot T(z)^{\ell^2}) = \omega(F_v(z)|U_{n_v} U_{\ell^2} \cdot T(z))^{\ell^2} \geq \ell^2.$$

From now we assume that $\theta(F_v(z)|U_{n_v}) \not\equiv 0 \pmod{\ell}$. Let

$$G(z) := F_v(z)|U_{n_v} \cdot T(z)^\ell.$$

Since $4|N$,

$$G(z) \in S_{\lambda + \frac{\ell+1}{2}}(\Gamma_1(Nn_v p_{v,1}^2 \cdots p_{v,s_v}^2)).$$

Then we have

$$\begin{aligned} (3.1) \quad \theta^{\frac{\ell+1}{2}} G(z) &\equiv \theta^{\frac{\ell+1}{2}} (F_v(z)|U_{n_v}) T(z)^\ell \\ &\equiv \sum_{m=1}^{\infty} \theta^{\frac{\ell+1}{2}} (a(n_i m^2) q^{m^2}) T(z)^\ell \equiv \sum_{m=1}^{\infty} (a(n_v m^2) m^{2 \cdot \frac{\ell+1}{2}} q^{m^2}) T(z)^\ell \\ &\equiv \theta(F_v(z)|U_{n_v}) T(z)^\ell \equiv \theta G(z) \pmod{\ell}. \end{aligned}$$

Let $\omega(\theta G(z)) := (m\ell + r)(\ell - 1) + \bar{\lambda} + \frac{\ell+1}{2} + \ell + 1 = (m\ell + r + 1 - m)\ell - r + \bar{\lambda} + \frac{\ell+3}{2}$ by Lemma 2.1 (3), (4) and (5), where m and r are integers such that $m\ell + r \leq i_\lambda$ and $0 \leq r \leq \ell - 1$. Let

$$\beta(r) := \begin{cases} -r & \text{if } m \geq 0, \\ (m\ell + r + 1 - m)\ell - r & \text{if } m < 0. \end{cases}$$

Then $\beta(r) \leq 1$. If $m < 0$, then $m = -1$ and $r = \ell - 1$ or $\ell - 2$ since $\omega(\theta G(z)) = \beta(r) + \bar{\lambda} + \frac{\ell+3}{2} > 0$. If $m \geq 0$ and $-r + \bar{\lambda} + \frac{\ell+3}{2} \leq 0$, then

$$2i_\lambda \geq 2r \geq \ell + 3 > \bar{\lambda}.$$

Thus, we may assume that

$$0 < \beta(r) + \bar{\lambda} + \frac{\ell+3}{2} < \ell + \frac{\ell+3}{2}.$$

Note that

$$\omega(\theta G(z)) - \left(\beta(r) + \bar{\lambda} + \frac{\ell+3}{2} \right) \geq 0 \text{ and } \omega(\theta G(z)) \equiv \beta(r) + \bar{\lambda} + \frac{\ell+3}{2} \pmod{\ell}.$$

If $\beta(r) + \bar{\lambda} + \frac{\ell+3}{2} < \frac{\ell+3}{2}$ or $\ell < \beta(r) + \bar{\lambda} + \frac{\ell+3}{2}$, then

$$\omega(\theta^{\frac{\ell+1}{2}} G(z)) = \omega(\theta G(z)) + (\ell + 1) \frac{\ell - 1}{2} > \omega(\theta G(z))$$

by Lemma 2.1 (5). This contradicts (3.1). Thus, we have

$$(3.2) \quad \frac{\ell+3}{2} \leq \beta(r) + \bar{\lambda} + \frac{\ell+3}{2} \leq \ell.$$

This implies that if $m = -1$ and $r = \ell - 2$, then $\bar{\lambda} = \ell - 2$. Suppose that $m = -1$ and $r = \ell - 2$. Then, $\omega(G(z)) \equiv 0 \pmod{\ell}$ by Lemma 2.1 (5) since $\omega(\theta G(z)) < \ell + 1$. So, if $\omega(G(z)) := (i_\lambda - t)(\ell - 1) + \frac{\ell+1}{2} + \ell - 2$ for some nonnegative integer $t \leq i_\lambda + 1$, then

$$\omega(G(z)) \equiv -i_\lambda + t + \frac{\ell+1}{2} - 2 \equiv 0 \pmod{\ell}.$$

We have

$$2i_\lambda \geq 2(i_\lambda - t) \geq \ell - 3 = \bar{\lambda} - 1.$$

Thus, from now on, we assume that $r = \ell - 1$ if $m = -1$.

Now we consider the value $\omega(\theta^{\frac{\ell+1}{2}} G(z))$. Combining Lemma 2.1 (3) and (4), we obtain

$$\begin{aligned} (3.3) \quad &\omega(\theta^{\ell - (\beta(r) + \bar{\lambda} + \frac{\ell+3}{2}) + 2} G(z)) \\ &= \omega(\theta G(z)) + \left(\ell - \left(\beta(r) + \bar{\lambda} + \frac{\ell+3}{2} \right) + 1 \right) (\ell + 1) - \alpha(\ell - 1) \\ &= (m\ell + r - \beta(r) + \frac{\ell+3}{2} - m - \bar{\lambda} - \alpha) \ell + 1 + \alpha - r - \beta(r), \end{aligned}$$

where α is a nonnegative integer. Since $(\ell - (\beta(r) + \bar{\lambda} + \frac{\ell+3}{2}) + 2) + (\beta(r) + \bar{\lambda}) = \frac{\ell+1}{2}$, Lemma 2.1 (5) implies that

$$\omega(\theta^{\ell - (\beta(r) + \bar{\lambda} + \frac{\ell+3}{2}) + 2} G(z)) + (\beta(r) + \bar{\lambda})(\ell + 1) \geq \omega(\theta^{\frac{\ell+1}{2}} G(z)) = \omega(\theta G(z)).$$

Thus, from (3.3) we have

$$\frac{\ell + 1}{2} \geq \alpha.$$

Note that $\omega(\theta^{\ell - (\beta(r) + \bar{\lambda} + \frac{\ell+3}{2}) + 2} G(z)) \equiv 1 + \alpha \pmod{\ell}$. Since by (3.2)

$$0 < 1 + \alpha + \beta(r) + \bar{\lambda} \leq \ell,$$

Lemma 2.1 (5) implies that

$$\omega(\theta G(z)) = \omega(\theta^{\frac{\ell+1}{2}} G(z)) = \omega(\theta^{\ell - (\beta(r) + \bar{\lambda} + \frac{\ell+3}{2}) + 2} G(z)) + (\beta(r) + \bar{\lambda})(\ell + 1).$$

Thus, we have

$$(3.4) \quad \alpha = \frac{\ell + 1}{2}.$$

Note that $\omega(\theta^{\ell - (\beta(r) + \bar{\lambda} + \frac{\ell+3}{2}) + 2} G(z)) > 0$ by Lemma 2.1 (1), and that $m = -1$ and $r = \ell - 1$ if $m < 0$. Thus, by using (3.3) and (3.4), we obtain

$$\begin{aligned} m\ell + 2r - m - \bar{\lambda} + 1 &\geq 0 && \text{if } m \geq 0, \\ \bar{\lambda} &\leq -1 && \text{if } m < 0 \text{ and } r = \ell - 1. \end{aligned}$$

We complete the proof by the following inequalities: for $m \geq 0$,

$$2i_\lambda \geq 2(m\ell + r) \geq m\ell + 2r \geq m + \bar{\lambda} - 1 \geq \bar{\lambda} - 1.$$

□

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