DIEUDONNÉ RINGS ASSOCIATED WITH $K(n)_* \underline{k(n)}_*$

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ABSTRACT. We use Dieudonné theory for periodically graded Hopf rings to determine the Dieudonné ring structure of the $\mathbb{Z}/2(p^n - 1)$ -graded Morava K -theory $\overline{K(n)}_*(-)$, with p an odd prime, when applied to the Ω -spectrum $\underline{k(n)}_*$ (and to $\underline{K(n)}_*$). We also expand these results in order to accommodate the case of the full Morava K -theory $K(n)_*(-)$.

1. INTRODUCTION

By Dieudonné theory we mean the definition of functors between certain categories of Hopf algebras and the corresponding categories of modules (over specific rings), called their Dieudonné modules. Such functors have been defined for some objects in the categories of graded Hopf algebras ([16] and [23]), ungraded Hopf algebras ([4]) and periodically graded Hopf algebras ([19]). Additionally, functors from the categories of Hopf *rings* have also been devised: for the graded case ([9]) and the ungraded and periodically graded cases ([20], [21] and [5]).

Morava K -theories provide examples of Hopf rings that are periodically graded, and thus can be identified through the right functor with their *Dieudonné rings* (modules over some rings, with additional product structure). In this paper, we present such Dieudonné rings for some Hopf rings that arise from the Morava K -theories applied to their Ω -spectra.

An important setback in the determination of those Dieudonné rings is that, as defined in section 3, for periodically graded Hopf rings we have a Dieudonné theory only if the ring is concentrated in even degrees (this happens for practical computational reasons that are explained below). This is not the case of $\overline{K(n)}_* \underline{k(n)}_*$, for example, where we do find elements with odd degree. Nevertheless, it is possible in these special cases to discover what a Dieudonné ring for such Hopf rings should be, and those are presented and explained. The case of Hopf rings arising from $K(n)_*(-)$ offers additional difficulties.

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2. HOPF RINGS ASSOCIATED WITH MORAVA K -THEORIES

Morava K -theories are represented by specific ring spectra (see [12]). Fixing an odd prime p , we denote by $K(n)$ the spectrum of the periodic n -th Morava K -theory and by $k(n)$ the spectrum of the connective n -th periodic Morava K -theory, the period in each case being $2(p^n - 1)$. There exists, for each n , an element v_n with degree equal to the period, and we have as coefficient rings $K(n)_* \simeq \mathbb{F}_p[v_n, v_n^{-1}]$ and $k(n)_* \simeq \mathbb{F}_p[v_n]$.

Let $\underline{K(n)}_*$ and $\underline{k(n)}_*$ be the Ω -spectra for the spectra $K(n)$ and $k(n)$. Since $K(n)_* \underline{(-)}_*$ has Künneth isomorphisms, for each r both $K(n)_* \underline{K(n)}_r$ and $K(n)_* \underline{k(n)}_r$ have coproducts and are thus coalgebras. In addition, they are Hopf algebras (group objects in the category of coalgebras) and both $K(n)_* \underline{K(n)}_*$ and $K(n)_* \underline{k(n)}_*$ are Hopf rings (graded ring objects in the category of coalgebras).

The structure of these two Hopf rings, which are $2(p^n - 1)$ -graded, is described in [24], [14] and [2]. $K(n)_* \underline{K(n)}_*$ is generated by elements $e_1 \in K(n)_1 \underline{K(n)}_1$, $a_{(i)} \in K(n)_{2p^i} \underline{K(n)}_1$ (for $i < n$), $b_{(i)} \in K(n)_{2p^i} \underline{K(n)}_2$, $[v_n] \in K(n)_0 \underline{K(n)}_{-2(p^n-1)}$, and $[v_n^{-1}] \in K(n)_0 \underline{K(n)}_{2(p^n-1)}$. We have that $K(n)_* \underline{k(n)}_*$ is a sub-Hopf ring of $K(n)_* \underline{K(n)}_*$. (The element $[v_n^{-1}]$ is not defined in $K(n)_* \underline{k(n)}_*$.) The b_i come from the complex orientation of the Morava K -theories. The a_i are standard homology elements.

Proposition 2.1 ([24], [1]). *Let p be an odd prime and $n \geq 1$.*

Then there exist elements $e_1 \in K(n)_1 \underline{k(n)}_1$, $a_i \in K(n)_{2i} \underline{k(n)}_1$ (for $i < p^n$), $b_i \in K(n)_{2i} \underline{k(n)}_2$.

Let $a_{(i)} = a_{p^i}$ and $b_{(i)} = b_{p^i}$. ψ denotes the coproduct, $$ the algebra (or group) product and $\circ$ the ring product. We have:*

- i) $e_1 \circ (-)$ is the homology suspension map.*
- ii) $\psi(a_i) = \sum_{j=0}^i a_{i-j} \otimes a_j$ and $\psi(b_i) = \sum_{j=0}^i b_{i-j} \otimes b_j$.*
- iii) The standard mod p homology a_i and b_i are all permanent cycles in the Atiyah-Hirzebruch spectral sequence for $K(n)_* \underline{k(n)}_*$ and represent the corresponding a_i and b_i .*
- iv) $e_1 \circ e_1 = -b_{(0)}$.*
- v) $a_{(i)} \circ a_{(j)} = -a_{(j)} \circ a_{(i)}$.*
- vi) $b_{(i)}^{*p} = 0$.*
- vii) $a_{(i)}^{*p} = 0$ for $i < n - 1$.*
- viii) $a_{(n-1)}^{*p} = v_n a_{(0)} - a_{(0)} \circ b_{(0)}^{\circ(p^n-1)} \circ [v_n]$.*
- ix) $v_n e_1 = b_{(0)}^{\circ(p^n-1)} \circ e_1 \circ [v_n]$.*
- x) $b_{(k)}^{\circ p^n} \circ [v_n] = v_n^{p^k} b_{(k)}$ for $k \geq 0$.*

There exists a surjection of Hopf rings $K(n)_* \underline{k(n)}_* \rightarrow K(n)_* K(\mathbb{Z}/(p), *)$ ([2]). This surjection maps the $a_{(i)}$ from $K(n)_* \underline{k(n)}_1$ nontrivially to elements of the same name in $K(n)_* K(\mathbb{Z}/(p), 1)$.

Proposition 2.2 ([18]). *The elements $a_{(i)}$ generate the Hopf ring $K(n)_* K(\mathbb{Z}/(p), *)$.*

The theorems below need some special notation. Write $I = (i_0, \dots, i_{n-1})$, where $i_k \in \{0, 1\}$ and $J = (j_0, j_1, \dots)$, with $0 \leq j_k < p^n$ and $j_k = 0$ for all but finitely many $k \geq 0$.

For any I and J , we define

$$a^I b^J = a_{(0)}^{i_0} \circ \cdots \circ a_{(n-1)}^{i_{n-1}} \circ b_{(0)}^{j_0} \circ b_{(1)}^{j_1} \circ \cdots$$

($a^I = a^I b^0$ and $b^J = a^0 b^J$, with the convention $a^0 b^0 = [1] - [0]$).

Δ_k will be the I or J sequence that has element k equal to 1 and all others zero. For any I , $\rho(I) = \min\{k \mid i_{n-k} \neq 0\}$ ($n \geq k > 0$). (If all i_k are 1, $\rho(I)$ is defined as ∞ .)

For any $J \neq 0$, $m(J) = \min\{k \mid j_k \neq 0\}$.

$E(x)$ denotes the exterior algebra with generator x , $P(x)$ the polynomial algebra generated by x , and $TP_k(x)$ the truncated polynomial algebra $P(x)/(x^{p^k})$ (where $k > 0$) generated by x ($TP_\infty(x) = P(x)$).

Theorem 2.3 ([2]). *Let p be an odd prime. Then*

$$\begin{aligned} K(n)_* \underline{k(n)}_* \simeq & \bigotimes_{\substack{j_0 < p^n - 1 \\ s \geq 0}} E(a^I b^J \circ e_1 \circ [v_n^s]) \\ & \bigotimes_{\substack{j_0 < p^n - 1 \\ s > 0}} E(a^I b^J \circ e_1 \circ b_{(0)}^{\circ(p^n - 1)s}) \\ & \bigotimes_{\substack{(a)\{s \geq 0, \text{ and } i_0 = 0 \text{ or } j_0 < p^n - 1\} \text{ or} \\ (b)\{s = 0, J = (p^n - 1)\Delta_0, i_0 = 1\}}} TP_{\rho(I)}(a^I b^J \circ [v_n^s]) \\ & \bigotimes_{\substack{s > 0, J \neq 0, \text{ and either} \\ (a)\{i_0 = 0 \text{ or } j_0 < p^n - 1\} \text{ or} \\ (b)\{J = (p^n - 1)\Delta_0, i_0 = 1\}}} TP_{\rho(I)}(a^I b^J \circ b_{(m(j))}^{\circ(p^n - 1)s}) \end{aligned}$$

where the I and J are taken as above.

Theorem 2.4 ([24]). *Let p be an odd prime and $s \in \mathbb{Z}$. Then*

$$K(n)_* \underline{K(n)}_* \simeq \bigotimes_{j_0 < p^n - 1} E(a^I b^J \circ e_1 \circ [v_n^s]) \bigotimes_{i_0 = 0 \text{ or } j_0 < p^n - 1} TP_{\rho(I)}(a^I b^J \circ [v_n^s]).$$

In section 5, we will deal with some Dieudonné rings that correspond to the above Hopf rings. We will restrict our attention to the special $\mathbb{Z}/2(p^n - 1)$ graded Morava K -theory $\underline{K(n)}_*(-)$ one gets by setting $v_n = 1$. Section 6 then presents Dieudonné ring objects corresponding to the full $K(n)_*(-)$.

3. DIEUDONNÉ RING THEORY

Fix p and n and let $m = p^n - 1$. Consider bicommutative Hopf algebras that are graded over $\mathbb{Z}/(2m)$ and focus on those Hopf algebras that are concentrated in even degrees (having elements in odd degrees implies, by commutativity, that the Hopf algebra has elements squaring to zero; in fact, the category of $2m$ -graded bicommutative Hopf algebras splits as a direct product of the category of those concentrated in even degrees and a category of primitively generated exterior algebras; see [10]). One such Hopf algebra will be called *connected* if it has an exhaustive coaugmentation filtration, and *geometric-like* if it can be written as $H \cong H_0 \otimes_k H_c$, where H_0 is a group ring over $\mathbb{F}_p$ and H_c is a $2m$ -graded connected Hopf algebra over $\mathbb{F}_p$ (concentrated in even degrees).

If a Hopf algebra H is in the category of ungraded bicommutative connected or geometric-like Hopf algebras, we can define its *Dieudonné Module* by

$$DH = \operatorname{colim}\{\operatorname{Hom}(H(n), H)\}$$

where $H(n) = \mathbb{F}_p[x_1, x_2, \dots, x_n]$, for $n \geq 0$ ($H(0)$ is just $\mathbb{F}_p$), is the Hopf algebra that has the Witt polynomials as primitives (see [20]). Note: The homomorphisms above are supposed to lie in the appropriate category.

In the case of the category whose objects are $2m$ -graded connected bicommutative Hopf algebras over $\mathbb{F}_p$ concentrated in even degrees, a new definition is needed to take care of the grading, as the maps $H(n) \rightarrow H$ in the colimit above are not necessarily in this new category.

We start by giving each indeterminate x_i of degree $2p^i t \pmod{2m}$ for a fixed t satisfying $1 \leq t \leq p^n - 1$ and define $H(s, t) = \mathbb{F}_p[x_0, x_1, \dots, x_s]$ for each s , giving it the unique Hopf algebra structure graded over $2m$ that makes the Witt polynomials primitive. The Verschiebung on these $H(s, t)$ is given by $V(x_i) = x_{i-1}$ for $i > 0$.

The Hopf algebra maps $\bar{V} : H(s+1, t) \rightarrow H(s, t)$ corresponding to the ones used in the ungraded case to define the colimit are not necessarily in the category with $2m$ -graded Hopf algebras, but we can choose some iterate of $\bar{V}$ that does preserve degree: in fact, since the Verschiebung divides degree by p and our grading imposes $p^n = 1$, we have that $\bar{V}^n$ is a map in the right category and so we can define Dieudonné modules for such Hopf algebras H by

$$DH = \operatorname{colim}\{\operatorname{Hom}(H(s, t), H)\}.$$

Notice that each $\bar{V}^n$ is a map from an $H(s, t)$ with $s > n$ to $H(s-n, t)$. This means that, by the definition of colimit, an element $\alpha \in \operatorname{Hom}(H(s, t), H)$ whose image under the colimit map is in DH is completely determined by the restriction of that map to $H(s', t)$, where $0 \leq s' < n$ and $s' = s \pmod{n}$. If we now vary t , we get a collection $\{D_t H\}_{1 \leq t \leq p^n - 1}$. (One should notice that, since $p^n = 1 \pmod{n}$ in our grading, the modules $D_t H$ for $t > p^n - 1$ would start repeating the ones already determined, and so $DH = \{D_t H\}_{1 \leq t \leq p^n - 1}$ is an m -graded module.)

Definition 3.1. The *Dieudonné module* for a $2m$ -graded connected bicommutative Hopf algebra over $\mathbb{F}_p$ concentrated in even degrees is the m -graded module

$$DH = \{D_t H = \operatorname{colim}\{\operatorname{Hom}(H(s, t), H)\}\}$$

together with homomorphisms $F : D_t H \rightarrow D_{pt} H$ and $V : D_t H \rightarrow D_{p/t} H$.

The homomorphisms above come from the Verschiebung and Frobenius on the Hopf algebra.

We call $\mathcal{DM}_m$ the category of m -graded modules M together with maps V and F satisfying $FV = VF = p$ and such that for each $x \in M$ there exists an $r \geq 1$ with $V^r(x) = 0$.

We have the following equivalence between Hopf algebras and Dieudonné modules:

Theorem 3.2 ([19]). *The above map D is a functor that has a right adjoint, and the pair (D, U) forms an equivalence of categories.*

Consider the category of Hopf rings in the $2m$ -graded connected case, that is, whose objects are collections $\{H_i\}$ of $2m$ -graded connected Hopf algebras (concentrated in even degrees) together with maps $H_i \otimes H_j \rightarrow H_{i+j}$ of $2m$ -graded coalgebras. This category is equivalent to a category of Dieudonné rings.

Every bilinear pairing $\circ_{ij} : H_i \otimes H_j \rightarrow H_{i+j}$ in this Hopf ring category induces a pairing $\circ'_{ij} : DH_i \otimes DH_j \rightarrow DH_{i+j}$ in $\mathcal{DM}_m$. These induced pairings satisfy:

- (a) $V(x \circ' y) = Vx \circ' Vy$;
- (b) $Fx \circ' y = F(x \circ' Vy)$;
- (c) $x \circ' Fy = F(Vx \circ' y)$ [20].

Now consider the functor D^R that takes each sequence of Hopf algebras $\{H_i\}_{i \in \mathbb{Z}}$ to the sequence of Dieudonné modules $\{D(H_i)\}_{i \in \mathbb{Z}}$, where D is the previous functor to Dieudonné modules, and such that the products $\circ_{ij} : H_i \otimes H_j \rightarrow H_{i+j}$ carry over to the products $\circ'_{ij} : DH_i \otimes DH_j \rightarrow DH_{i+j}$ as given above.

Theorem 3.3 ([20]). *The functor D^R has a right adjoint U^R and the pair (D^R, U^R) forms an equivalence of categories.*

The case of $2m$ -graded geometric-like Hopf algebras and corresponding Hopf rings is dealt with similarly, with minor changes. Their category is equivalent to a category of Dieudonné rings ([20]).

An alternate proof of the result in the previous theorem is also presented in [5], following [9]; here, one proves that the categories of Hopf algebras alluded to are monoidal due to a defined $\boxtimes$ -product, and that the corresponding categories of Dieudonné modules also have such products that are moreover carried functorially from those at the Hopf algebra level.

4. DIEUDONNÉ RING STRUCTURES FOR EXTERIOR AND POLYNOMIAL ALGEBRAS

Suppose a finitely-generated free m -graded module M over $R = \mathbb{Z}_p[F, V]/(FV = VF = p)$ is given, with basis $\{a_1, \dots, a_n\}$. If $VM = M$, we can define an R -module structure on the exterior algebra $\Lambda(M)$ as well.

Write $\Lambda(M) = \Lambda^0(M) \oplus \Lambda^1(M) \oplus \dots \oplus \Lambda^n(M)$, where each $\Lambda^k(M)$ is generated by the k^{th} exterior powers of the a_j (elements of the form $a_{i_1} \wedge a_{i_2} \wedge \dots \wedge a_{i_k}$, with $1 \leq i_1 < i_2 < \dots < i_k \leq n$).

The R -module structure on $\Lambda(M)$ is given inductively by

$$\begin{aligned} V(a_{i_1} \wedge a_{i_2} \wedge \dots \wedge a_{i_k}) &= V(a_{i_1} \wedge a_{i_2} \wedge \dots \wedge a_{i_{k-1}}) \wedge V(a_{i_k}), \\ F(a_{i_1} \wedge a_{i_2} \wedge \dots \wedge a_{i_k}) &= F(a_{i_1} \wedge a_{i_2} \wedge \dots \wedge a_{i_{k-1}}) \wedge b, \text{ whenever } a_{i_k} = Vb. \end{aligned}$$

The condition $VM = M$ guarantees the definition works for all elements.

Proposition 4.1. *The exterior algebra $\Lambda(M)$ with the previous R -module operations has a structure of a Dieudonné ring.*

Proof. Define the ring product $\circ$ as the wedge product $\wedge$. We have: $FV(a_{i_1} \wedge a_{i_2} \wedge \dots \wedge a_{i_k}) = F(V(a_{i_1} \wedge a_{i_2} \wedge \dots \wedge a_{i_{k-1}}) \wedge Va_{i_k}) = (FV(a_{i_1} \wedge a_{i_2} \wedge \dots \wedge a_{i_{k-1}})) \wedge a_{i_k}$ and $VF(a_{i_1} \wedge a_{i_2} \wedge \dots \wedge a_{i_k}) = V(F(a_{i_1} \wedge a_{i_2} \wedge \dots \wedge a_{i_{k-1}}) \wedge b) = (VF(a_{i_1} \wedge a_{i_2} \wedge \dots \wedge a_{i_{k-1}})) \wedge a_{i_k}$ and thus the Dieudonné module structure properties on $\Lambda(M)$ are guaranteed by M being a Dieudonné module itself.

It remains to check the ring product conditions from section 3. The first two are the direct definitions. The third one comes from the anti-commutativity of $\wedge$. $\square$

Consider now a free m -graded module M over R generated by a countable basis $B = \{b_1, b_2, \dots\}$ such that $VM = M$. We define an R -module structure on the free

polynomial algebra $P(B)$ generated by the b_k . Put

$$\begin{aligned} V(pq) &= V(p)V(q), \\ F(pq) &= F(p)r, \text{ whenever } q = Vr. \end{aligned}$$

As for exterior algebras, the following result follows directly from the definitions.

Proposition 4.2. *The polynomial algebra $P(B)$ with the previous R -module operations has a structure of a Dieudonné ring.*

If we have additional relations c_i on the generators, we can extend the previous Dieudonné ring definition to $P(B)/(c_i)$ if F and V are zero on the c_i (putting $F([p]) = [F(p)]$ and $V([p]) = [V(p)]$). This happens for some truncated polynomial algebras; the heights of the generators have to match in order to annulate V and F in the relations.

5. THE DIEUDONNÉ RINGS ASSOCIATED WITH $\overline{K(n)}_* (-)$

As before, let $I = (i_0, \dots, i_{n-1})$, where $i_k \in \{0, 1\}$. Define $s(I)$ as the sequence $(i_1, i_2, \dots, i_{n-1}, 0)$, a left translation, and $s^{-1}(I)$ as the sequence $(0, i_0, i_1, \dots, i_{n-2})$, a right translation.

Theorem 5.1 ([19], [5]). *The Dieudonné module $D(\overline{K(n)}_*(K(\mathbb{Z}/(p^j), q))$ is a free $\mathbb{Z}/(p^j)$ module on generators $a^I = a_{(0)}^{i_0} \cdots a_{(n-1)}^{i_{n-1}}$, where $i_k \in \{0, 1\}$ and $\sum_{k=0}^{n-1} i_k = q$, in degree $\sum_{k=0}^{n-1} i_k p^k$, with*

$$V(a^I) = \begin{cases} a^{s(I)} & \text{if } i_0 = 0, \\ (-1)^{q-1} p a^{(i_1, i_2, \dots, i_{n-1}, 1)} & \text{if } i_0 = 1, \end{cases}$$

and

$$F(a^I) = \begin{cases} p a^{s^{-1}(I)} & \text{if } i_{n-1} = 0, \\ (-1)^{q-1} a^{(1, i_0, i_1, \dots, i_{n-2})} & \text{if } i_{n-1} = 1. \end{cases}$$

Since the Dieudonné module above is a free $\mathbb{Z}/(p^j)$ module, the condition $V^m(x) = 0$ for some m is indeed satisfied for every x , except for the element $[1]$ given by setting all i_k to zero. $\overline{K(n)}_*(K(\mathbb{Z}/(p^j), q)$ is not connected, but is geometric-like as defined in the previous section.

From this description and Proposition 2.2, it is easy to present the Dieudonné ring for the Hopf ring $\overline{K(n)}_*(K(\mathbb{Z}/(p), *)$.

Consider the free $\mathbb{Z}/(p)$ module M generated by $\{a_{(0)}, \dots, a_{(n-1)}\}$. By the previous theorem, the action of V and F on these generators is given by $V(a_{(0)}) = 0$, $V(a_{(i+1)}) = a_{(i)}$ and $F(a_{(n-1)}) = a_{(0)}$, $F(a_{(i)}) = 0$ for $i \neq n-1$.

Write $a_{(i)} = V^{n-1-i}$. These confirm the above definition of V on the $a_{(i)}$, and the application of F on these elements implies $FV = VF = 0$ and $F = V^{n-1}$. (This last condition guarantees there is only a finite number of $a_{(i)}$.)

Thus, $M \simeq \mathbb{Z}/(p)[V, F]/(FV = VF = 0, F = V^{n-1})$.

Since VM is generated by the $a_{(0)}, a_{(0)}, \dots, a_{(n-2)}$, we can as in section 4 define an R -module structure on $\Lambda(M)$ except for terms of the form $a_{(i_1)} \wedge a_{(i_2)} \wedge \cdots \wedge a_{(i_k)}$ with $i_k = n-1$ (since no element in M hits $a_{(n-1)}$ through V). We can, however, extend that definition by putting

$$F(a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_{k-1}} \wedge a_{n-1}) = -F(a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{n-1} \wedge a_{i_{k-1}})$$

(since i_{k-1} is necessarily different from $n-1$).

In the following results, given a $\mathbb{Z}/(p^j)$ -module M we define $E[M]$ as the exterior algebra on the elements of M , viewed as a $\mathbb{Z}/(p^j)$ -module: $(\alpha m) \wedge (\beta n) = (\alpha\beta) m \wedge n$ (for $\alpha, \beta \in \mathbb{Z}/(p^j)$ and $m, n \in M$).

Corollary 5.2. *The Dieudonné ring $D(\overline{K(n)})_*(K(\mathbb{Z}/(p), *))$ is the exterior algebra $E[\mathbb{Z}/(p)[V, F]/(FV = VF = 0, F = V^{n-1})$.*

Proof. Only the Dieudonné ring operations need to be defined and checked. From Proposition 2.1, the only ring relations between the $a_{(i)}$ that carry over to $\overline{K(n)}_*(K(\mathbb{Z}/(p), *))$ are the ones relating to the anti-commutativity $a_{(i)} \circ a_{(j)} = -a_{(j)} \circ a_{(i)}$; thus defining on the Dieudonné modules the ring product as a wedge product agrees with these relations and carries over the module structure from M as defined in the previous section. $\square$

We can use Theorem 5.1 above also to describe the general case of $j > 1$.

Corollary 5.3. *The Dieudonné ring $D(\overline{K(n)})_*(K(\mathbb{Z}/(p^j), *))$ is the exterior algebra $E[\mathbb{Z}/(p^j)[V, F]/(FV = VF = p, F = V^{n-1})$.*

Proof. As with the case with $j = 1$ from the previous corollary, we start by defining the action of V and F on the $a_{(i)}$ by putting (following Theorem 5.1) $V(a_{(0)}) = pa_{(n-1)}$, $V(a_{(i+1)}) = a_{(i)}$ and $F(a_{(n-1)}) = a_{(0)}$, $F(a_{(i-1)}) = pa_{(i)}$.

With $a_{(i)} = V^{n-1-i}$, we get the action of F and V , which satisfies $FV = VF = p$ and $F = V^{n-1}$. The definition of V and F on the exterior algebra $E(\mathbb{Z}/(p^j)[V, F])$ is then completely determined except for terms of the form $a_{(i_1)} \wedge a_{(i_2)} \wedge \cdots \wedge a_{(i_k)}$ with $i_k = n-1$, for which we define

$$F(a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_{k-1}} \wedge a_{n-1}) = -F(a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{n-1} \wedge a_{i_{k-1}})$$

as before.

The Hopf ring relations for this case are again anti-commutativity, and they become wedge products at the Dieudonné ring level. $\square$

From these results we will deduce the Dieudonné ring for $\overline{K(n)}_*(\underline{k(n)}_*)$. We cannot do this directly, though, as in the theory developed in section 3; namely, the determination of the right functor taking periodically graded Hopf rings to (periodically graded) Dieudonné rings does not apply in this case, because the Hopf ring $\overline{K(n)}_*(\underline{k(n)}_*)$ is not concentrated in even degrees. We start though by determining the Dieudonné ring for the sub-Hopf ring generated by the $a_{(i)}$ and the $b_{(j)}$ from Proposition 2.1 (reduced to the case where $v_n = 1$). Call this sub-Hopf ring $[\overline{K(n)}_*(\underline{k(n)}_*)]'$. This one is concentrated in even degrees, and so the theory from section 3 can be applied here.

The Hopf ring relations in this reduced case show that all the $a_{(i)}$ are independent from the $b_{(i)}$. All these elements carry over to elements (of the same name) at the Dieudonné module level, and there exists thus a sub-Dieudonné ring of $D([\overline{K(n)}_*(\underline{k(n)}_*)]')$ which is generated by the $a_{(i)} = V^{n-1-i}$ and is thus an exterior algebra on $M = \mathbb{Z}/(p)[V, F]/(FV = VF = 0, F = V^{n-1})$.

The $b_{(i)}$ have no anti-commutative ring $\circ$ relations at the Hopf ring level, so the corresponding Dieudonné ring will be polynomial modulo relations coming from the circularity $b_{(i)}^{op^n} = b_{(i)}$.

We have that the Dieudonné ring $D(\overline{K(n)} * (\underline{k(n)})')$ is periodical in both gradings: the “inner one,” forced by the fact that each Dieudonné module is periodically graded, and the one that comes from the ring grading.

Theorem 5.4. *The Dieudonné ring $D(\overline{K(n)} * (\underline{k(n)})')$ is given by*

$$E[\mathbb{Z}/(p)[V, F]/(FV = VF = 0, F = V^{n-1})] \\ \otimes (P[V^{-1}, V^{-2}, \dots]) / (\{(V^{-k})^{p^n} - V^{-k}, k \geq 0\}).$$

Proof. The Hopf algebra relations among the $b_{(k)}$ are the same (apart from now forming an infinite basis) as those that appeared for the $a_{(k)}$, and thus at the Dieudonné module level the actions of V and F (which mirror respectively Verschiebung and Frobenius) must be as for the $a_{(k)}$, namely $F(b_{(k)}) = 0$ for all $k \geq 0$ and $V(b_{(k+1)}) = b_{(k)}$, $V(b_{(0)}) = 0$.

Put $b_{(k)} = V^{-k}$ (with $k \geq 0$). These are consistent with the actions of F and V above if we declare $FV = VF = 0$ and $V(b_{(0)}) = V(1) = 0$. We can extend this action to the polynomial algebra $P[V^{-1}, V^{-2}, \dots]$, which is thus a $\mathbb{Z}/(p)[V, F]/(FV = VF = 0)$ module, following section 4. To accomodate the ring relations between the $b_{(i)}$ at the Hopf ring level, we have to check their consistency with respect to the actions of V and F .

We have: $V(((V^{-k})^{p^n}) - V^{-k}) = (V(V^{-k})^{p^n}) - V^{-k+1} = (V^{-k+1})^{p^n} - V^{-k+1}$, still one of the ring relations. $F(((V^{-k})^{p^n}) - V^{-k}) = 0 - 0 = 0$ for all k , because of the definition of F for polynomial algebras and the fact that F is zero for each V^{-k} . $\square$

For the Hopf ring $\overline{K(n)} * (\underline{k(n)})$ we can find an equivalent ring object, which is not a Dieudonné ring as previously defined. We obtain it from the previous Dieudonné ring by adding an element corresponding to e_1 .

Consider the complex-valued algebra $\mathbb{Z}/(p)[i, V, F]$ as an R -module by putting $F(i) = 0$ and $V(i) = iV$. Since $FV(i) = VF(i) = 0$, we have that $R[i]$ is also an R -module.

We get:

Theorem 5.5. *The Hopf ring $\overline{K(n)} * (\underline{k(n)})$ is equivalent to the R -algebra $E[\mathbb{Z}/(p)[V, F]/(FV = VF = 0, F = V^{n-1})] \otimes (R[i][V^{-1}, V^{-2}, \dots]) / (\{(V^{-k})^{p^n} - V^{-k}, k \geq 0\})$.*

Proof. Put $e_1 = i$ for the polynomial algebra.

The only ring $\circ$ relation between e_1 and the other generators on the Hopf ring is $e_1 \circ e_1 = -b_{(0)}$. At the polynomial algebra level we get $i^2 = -1$, and so this relation is satisfied. $\square$

Since in $\overline{K(n)} * (-)$ we take $v_n = 1$, the Dieudonné ring structure for $\overline{K(n)} * \underline{K(n)}$ is the same as for $\overline{K(n)} * \underline{k(n)}$, and it reads as in the previous theorem.

6. THE DIEUDONNÉ RINGS ASSOCIATED WITH $K(n) * (-)$

For the full Morava K -theory $K(n) * (-)$ we get additional information pertaining to $[v_n]$ and $[v_n^{-1}]$ that must be incorporated in the Dieudonné ring structures previously defined.

We start with the simpler case of the Hopf subring R_1 of $K(n) * \underline{k(n)}_*$ (from Proposition 2.1) generated by $a_{(0)}, a_{(1)}, \dots, a_{(n-2)}$, ignoring for now the last $a_{(n-1)}$. The relations on these $n-1$ terms are just as before, and thus we first get an exterior algebra given by

$$E[V(\mathbb{Z}/(p^j)[V, F]/(FV = VF = p, F = V^{n-1}))].$$

(We have an extra V on the basis whose order for $a_{(n-1)} = 1$ is not to be figured yet.)

This has a natural $K(n) *$ -module structure, and so

$$DR_1 \simeq K(n) * [E[V(\mathbb{Z}/(p)[V, F]/(FV = VF = p, F = V^{n-1}))].$$

Next consider the Hopf subring R_2 of $K(n) * \underline{k(n)}_*$ generated by all $a_{(k)}$ except $a_{(n-1)}$, plus all the $b_{(k)}$ and the element $[v_n]$. The difference between its Dieudonné ring and that of the reduced case reflects the relation $b_{(k)}^{p^n} \circ [v_n] = v_n^{p^k} b_{(k)}$ for $k \geq 0$.

The polynomial algebra $P([V^{-1}, V^{-2}, \dots]) / (\{(V^{-k})^{p^n} - V^{-k}, k \geq 0\})$, which reflects the relations among the $b_{(k)}$, is given a natural $K(n) *$ -module structure. $R[v_n]$ as a module over $K(n) *$ is viewed as the polynomial algebra $R[v_n]$. For this Hopf ring, we get the Dieudonné ring DR_2 as $K(n) * [E[V(\mathbb{Z}/(p)[V, F]/(FV = VF = p, F = V^{n-1}))] \otimes (R[v_n, v_n^{-1}, V^{-1}, V^{-2}, \dots]) / (\{(V^{-k})^{p^n} - V^{-k}, k \geq 0\}) \otimes R[v_n]$ modulo the relations $1 \otimes ((V^{-k})^{p^n}) \otimes [v_n] - 1 \otimes (v_n^{p^k} V^{-k}) \otimes 1$ for all $k \geq 0$.

Consider now the Hopf subring R_2 of $K(n) * \underline{k(n)}_*$ generated by all $a_{(k)}$ except $a_{(n-1)}$, plus all the $b_{(k)}$ and the element $[v_n]$, adding now e_1 . For this case we must take into account the Hopf ring relation $v_n e_1 = b_{(0)}^{p^n-1} \circ e_1 \circ [v_n]$. As in the previous chapter, we get a complex polynomial algebra for the $b_{(k)}$ and e_1 . A Dieudonné ring object for R_3 is $K(n) * [E[V(\mathbb{Z}/(p)[V, F]/(FV = VF = 0, F = V^{n-1}))] \otimes (R[i][v_n, v_n^{-1}, V^{-1}, V^{-2}, \dots]) / (\{(V^{-k})^{p^n} - V^{-k}, k \geq 0\}) \otimes R[v_n]$ modulo the relations $1 \otimes ((V^{-k})^{p^n}) \otimes [v_n] - 1 \otimes (v_n^{p^k} V^{-k}) \otimes 1$ (for all $k \geq 0$) and $1 \otimes (v_n i) \otimes 1 - 1 \otimes i \otimes v_n$.

We can now present Dieudonné rings for $K(n) * \underline{k(n)}_*$ and $K(n) * \underline{K(n)}_*$.

Theorem 6.1. *The Hopf ring $K(n) * \underline{k(n)}_*$ is equivalent to the R -module $K(n) * [E[\mathbb{Z}/(p)[V, F]/(FV = VF = 0, F = V^{n-1})] \otimes (R[i][v_n, v_n^{-1}, V^{-1}, V^{-2}, \dots]) / (\{(V^{-k})^{p^n} - V^{-k}, k \geq 0\}) \otimes R[v_n]$ modulo $1 \otimes ((V^{-k})^{p^n}) \otimes [v_n] - 1 \otimes (v_n^{p^k} V^{-k}) \otimes 1$ (for all $k \geq 0$), $\otimes (v_n i) \otimes 1 - 1 \otimes i \otimes v_n$ and $F \otimes 1 \otimes 1 - (v_n V^{n-1}) \otimes 1 \otimes 1 - V^{n-1} \otimes 1 \otimes v_n$.*

Proof. The sequence $R_1 \subset R_2 \subset R_3 \subset K(n) * \underline{k(n)}_*$ gives a sequence of Dieudonné ring objects $DR_1 \subset DR_2 \subset DR_3 \subset DK(n) * \underline{k(n)}_*$ whose first three terms were determined above.

For the conclusion, we have to add a condition at the Dieudonné ring level that takes into account the ring relation $a_{(n-1)}^{p^n} = v_n a_{(0)} - a_{(0)} \circ b_{(0)}^{p^n-1} \circ [v_n]$.

Since on Dieudonné modules F emulates the Frobenius, the relation above gives F of the element 1 (that corresponds to $a_{(n-1)}$), and so we get the relation

$$F \otimes 1 \otimes 1 - (v_n V^{n-1}) \otimes 1 \otimes 1 - V^{n-1} \otimes 1 \otimes v_n.$$

□

For $K(n) * \underline{K(n)}_*$ the extra element v_n^{-1} appears.

Theorem 6.2. *The Hopf ring $K(n) * (\underline{K(n)})_*$ is equivalent to the R -module $K(n) * [E[\mathbb{Z}/(p)[V, F]/(FV = VF = 0, F = V^{n-1})] \otimes (R[i][v_n, v_n^{-1}, V^{-1}, V^{-2}, \dots]) / ((V^{-k})^{p^n} - V^{-k}, k \geq 0) \oplus R[v_n, v_n^{-1}]$ modulo $1 \otimes ((V^{-k})^{p^n}) \otimes [v_n] - 1 \otimes (v_n^k V^{-k}) \otimes 1$ (for all $k \geq 0$), $1 \otimes (v_n i) \otimes 1 - 1 \otimes i \otimes v_n$ and $F \otimes 1 \otimes 1 - (v_n V^{n-1}) \otimes 1 \otimes 1 - V^{n-1} \otimes 1 \otimes v_n$.*

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