COMPACT WEIGHTED COMPOSITION OPERATORS ON THE HARDY SPACE

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ABSTRACT. A weighted composition operator $C_{\psi,\varphi}$ takes an analytic map f on the open unit disk of the complex plane to the analytic map $\psi f \circ \varphi$, where φ is an analytic map of the open unit disk into itself and ψ is an analytic map on the open unit disk. This paper studies how the compactness of $C_{\psi,\varphi}$ depends on the interaction between the two maps ψ and φ .

INTRODUCTION

Let φ be an analytic map from the open unit disk D into itself and let ψ be an analytic map on the open unit disk. Then the weighted composition operator $C_{\psi,\varphi}$ on the Hardy space $H^2(D)$ is the operator that takes f to $\psi f \circ \varphi$. That is, $(C_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z))$. If ψ is identically 1, this operator becomes a composition operator; this has been studied for about thirty-five years. However, less is known about more general weighted composition operators. If ψ is a bounded analytic function on the open unit disk, then the multiplication operator M_ψ where $M_\psi(f)(z) = \psi(z)f(z)$ becomes a bounded operator. The composition operator C_φ where $C_\varphi(f)(z) = f(\varphi(z))$ is always bounded, and the weighted composition operator $C_{\psi,\varphi}$ is equal to $M_\psi C_\varphi$. Therefore, when the composition operator C_φ is compact, the operator $C_{\psi,\varphi}$ will be compact. But if ψ is unbounded, then the resulting multiplication operator is not bounded on $H^2(D)$ and hence the weighted composition operator $C_{\psi,\varphi}$ cannot be written as a product of a multiplication operator and a composition operator. Interestingly, it is possible for $C_{\psi,\varphi}$ to be compact with an unbounded ψ . Also when M_ψ is bounded but neither of the operators C_φ and M_ψ are compact, it is still possible for $C_{\psi,\varphi}$ to be compact. For a bounded weight ψ the compactness of the operator is discussed in Theorem 3.1 of [8], and this theorem depends on Alexandrov-Clark measures. Results in this paper depend on more direct methods. Also Lemma 3.3 of [1] discusses the compactness when the weight ψ is smooth and φ is a linear fractional map. This paper attempts to study the compactness of $C_{\psi,\varphi}$ without assuming that the weight function is bounded on the unit disk.

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PRELIMINARIES

The Hardy space $H^2(D)$ is the set of functions analytic on the open unit disk for which

$$(1) \quad \sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

If f is in the space $H^2(D)$, then it can be extended to the unit circle almost everywhere. More precisely, $f^*(e^{i\theta}) = \lim_{r \rightarrow 1} f(re^{i\theta})$ exists for almost all θ , and the mapping $f \rightarrow f^*$ is an isometry of $H^2(D)$ into a closed subspace of $L^2(d\theta/2\pi)$. For a proof and more detailed discussions see [3, page 10] and [4, page 21]. But in Hardy space literature f^* is seldom used. With a slight abuse of notation f is used instead of f^* on the unit circle. Moreover the mapping $f \rightarrow f^*$ behaves nicely with respect to composition. That is, $(f \circ \varphi)^* = f^* \circ \varphi^*$ almost everywhere [3, page 31]. If f is the constant function 1, then $C_{\psi, \varphi}(f) = \psi$, and in order for the operator $C_{\psi, \varphi}$ to be bounded on $H^2(D)$ it is clear that ψ must be in the space $H^2(D)$. Therefore we will always make the assumption that the weight function ψ is in the Hardy space $H^2(D)$.

The following lemma was inspired by a similar result for unweighted composition operators [3, Proposition 3.11].

Lemma 1. *Let $C_{\psi, \varphi}$ be a bounded operator on $H^2(D)$. Then $C_{\psi, \varphi}$ is compact if and only if whenever $\{f_n\}$ is a bounded sequence in $H^2(D)$ and $f_n \rightarrow 0$ uniformly on compact subsets of D then $C_{\psi, \varphi}(f_n) \rightarrow 0$ in $H^2(D)$.*

A proof of this lemma can be found in [6, page 17]. We now use the previous lemma to prove the following theorem.

Theorem 1. *Let $C_{\psi, \varphi}$ be a bounded operator on $H^2(D)$. Suppose φ is continuous up to the boundary of the unit disk. Let $Z = \{\zeta : |\varphi(\zeta)| = 1, \zeta \text{ on the unit circle}\}$. Assume that $\psi(\zeta) = 0$ for each ζ in Z and also that ψ is continuous at ζ . Then the operator $C_{\psi, \varphi}$ is compact.*

Proof. Let $\{f_n\}$ be a bounded sequence in $H^2(D)$ which approaches 0 uniformly on compact subsets of the open unit disk. We will show that $C_{\psi, \varphi}(f_n)$ approaches 0 in $H^2(D)$. This will prove that the operator is compact. The unweighted composition operator C_φ is also bounded; therefore there is a positive M such that $\|f_n\|^2 \|C_\varphi\|^2 < M$.

Suppose ζ is in Z . Then $\psi(\zeta) = 0$ and ψ is continuous at ζ . Therefore there is $t_\zeta > 0$ such that $|\psi(\zeta e^{i\theta})|^2 < \epsilon/2M$ whenever $-t_\zeta < \theta < t_\zeta$. Let U_ζ denote the set $\{\zeta e^{i\theta} : -t_\zeta < \theta < t_\zeta\}$ and $U = \bigcup_{\zeta \in Z} U_\zeta$. Suppose T denotes the unit circle and V denotes $T - U$:

$$(2) \quad \begin{aligned} \|C_{\psi, \varphi}(f_n)\|^2 &= \int_T |\psi(e^{i\theta}) f_n(\varphi(e^{i\theta}))|^2 \frac{d\theta}{2\pi} \\ &= \int_U |\psi(e^{i\theta}) f_n(\varphi(e^{i\theta}))|^2 \frac{d\theta}{2\pi} + \int_V |\psi(e^{i\theta}) f_n(\varphi(e^{i\theta}))|^2 \frac{d\theta}{2\pi}. \end{aligned}$$

Since $|\psi(e^{i\theta})|^2 < \epsilon/2M$ for $e^{i\theta}$ in U we get

$$\begin{aligned} \int_U |\psi(e^{i\theta})|^2 |f_n(\varphi(e^{i\theta}))|^2 \frac{d\theta}{2\pi} &\leq (\epsilon/2M) \int_U |f_n(\varphi(e^{i\theta}))|^2 \frac{d\theta}{2\pi} \\ &\leq (\epsilon/2M) \|C_\varphi(f_n)\|^2 \\ &\leq (\epsilon/2M) \|f_n\|^2 \|C_\varphi\|^2 \\ (3) \qquad \qquad \qquad &\leq \epsilon/2. \end{aligned}$$

Since V is a compact subset of the unit circle and φ is continuous on the unit circle, $\varphi(V)$ is a compact subset of the open unit disk. Therefore the sequence $\{f_n\}$ converges to zero uniformly on $\varphi(V)$. Thus for all $e^{i\theta}$ in V and all n large enough,

$$|f_n(\varphi(e^{i\theta}))|^2 \leq \frac{\epsilon}{2\|\psi\|^2}.$$

Therefore,

$$\begin{aligned} \int_V |\psi(e^{i\theta})|^2 |f_n(\varphi(e^{i\theta}))|^2 \frac{d\theta}{2\pi} &\leq \frac{\epsilon}{2\|\psi\|^2} \int_V |\psi(e^{i\theta})|^2 \frac{d\theta}{2\pi} \\ &\leq \frac{\epsilon}{2}; \end{aligned}$$

hence by (2),

$$\int_T |\psi(e^{i\theta})|^2 |f_n(\varphi(e^{i\theta}))|^2 \frac{d\theta}{2\pi} < \epsilon$$

for all n large enough. Thus $\lim_{n \rightarrow \infty} C_{\psi,\varphi}(f_n) = 0$ in norm. Now the compactness of $C_{\psi,\varphi}$ follows from Lemma 1. $\square$

Theorem 1 can be used to construct some interesting examples. We first state the following lemma. The proof can be found in [10, page 31].

Lemma 2. *If φ is a univalent map of the unit disk into itself and if $\varphi(D)$ contains a disc in D that is tangent to the unit circle, then the composition operator C_φ cannot be compact on $H^2(D)$.*

Example 1. Let $\varphi(z) = (z+1)/2$. The image of the unit circle under the map φ is tangent to the unit circle at the point $(1,0)$. Therefore by Lemma 2 the unweighted composition operator C_φ is not compact. Now let $\psi(z) = 1-z$. Then the multiplication operator M_ψ is bounded and therefore $C_{\psi,\varphi} = M_\psi C_\varphi$ is bounded. By Theorem 1 the operator $C_{\psi,\varphi}$ is compact.

Lemma 3. *If $\varphi(z) = (z+1)/2$ and $\psi(z) = ((1-z)/(1+z))^{1/4}$, then the operator $C_{\psi,\varphi}$ is bounded.*

Proof. Let f be in H^2 . The image of the unit circle under the map φ is a circle of radius $1/2$ tangent to the unit circle at 1. Also φ is one to one. Therefore it is possible to choose an open arc V of the unit circle containing the point 1 such that $|\varphi(e^{i\theta})| < 3/4$ for every $e^{i\theta}$ outside V . Let U consist of all $e^{i\theta}$ that do not belong to V . Now,

$$\begin{aligned} (4) \quad \|C_{\psi,\varphi}(f)\|^2 &= \int_T |\psi(e^{i\theta})|^2 |f(\varphi(e^{i\theta}))|^2 \frac{d\theta}{2\pi} \\ (5) \quad &= \int_U |\psi(e^{i\theta})|^2 |f(\varphi(e^{i\theta}))|^2 \frac{d\theta}{2\pi} + \int_V |\psi(e^{i\theta})|^2 |f(\varphi(e^{i\theta}))|^2 \frac{d\theta}{2\pi}. \end{aligned}$$

If $e^{i\theta}$ is in U , then $|\varphi(e^{i\theta})| < 3/4$. Hence $|f(\varphi(e^{i\theta}))| \leq \|f\|/\sqrt{1-9/16}$, and if $e^{i\theta}$ is in V , then $|\psi(e^{i\theta})| < K$ for some K . Now by (5),

$$\|C_{\psi,\varphi}(f)\|^2 \leq \frac{16}{7}\|f\|^2 \int_U |\psi(e^{i\theta})|^2 \frac{d\theta}{2\pi} + K^2 \int_V |f(\varphi(e^{i\theta}))|^2 \frac{d\theta}{2\pi}.$$

Since ψ is in H^2 and the integral of $|f \circ \varphi|^2$ over V is smaller than $\|f\|^2 \|C_\varphi\|^2$,

$$\|C_{\psi,\varphi}(f)\|^2 \leq \|f\|^2 \left(\frac{16}{7} \|\psi\|^2 + K^2 \|C_\varphi\|^2 \right).$$

Thus $C_{\psi,\varphi}$ is bounded. $\square$

Example 2. Let $\varphi(z) = (z+1)/2$ and $\psi(z) = ((1-z)/(1+z))^{1/4}$. Then by Lemma 3, $C_{\psi,\varphi}$ is a bounded operator. The only place where $|\varphi(\zeta)| = 1$ is when $\zeta = 1$ and ψ is continuous on the unit circle. Now Theorem 1 tells us that the operator $C_{\psi,\varphi}$ is compact.

It is interesting to note that in the above example the composition operator C_φ is not compact and the multiplication operator M_ψ is not even bounded, but the weighted composition operator $C_{\psi,\varphi}$ is compact.

In Example 1, we began with a non-compact composition operator C_φ and ended up with a compact weighted composition operator $C_{\psi,\varphi}$ by choosing an appropriate weight. A natural follow up question would be, can we do the opposite? That is, given a compact composition operator C_φ , can we choose a weight ψ so that the operator $C_{\psi,\varphi}$ is non-compact (but bounded)? The answer, at least in some cases, is, interestingly, no.

Theorem 2. Suppose $\overline{\varphi(D)}$ is contained in the open unit disk D . Then for any ψ in the Hardy space $H^2(D)$ the weighted composition operator $C_{\psi,\varphi}$ is compact.

Proof. We will show that $\|C_{\psi,\varphi}|_{z^n H^2(D)}\|$ tends to zero as n tends to infinity. This will prove the compactness of the operator. Since $\overline{\varphi(D)}$ is contained in the open unit disk, $|\varphi(\zeta)| \leq r$ for some $r < 1$ for almost all ζ on the unit circle. Suppose f is in the subspace $z^n H^2(D)$; then $f(z) = z^n g(z)$ for some g in $H^2(D)$ with $\|f\| = \|g\|$.

$$\begin{aligned} \|C_{\psi,\varphi}f\|^2 &= \int_T |\psi(e^{i\theta})|^2 |\varphi(e^{i\theta})|^{2n} |g(\varphi(e^{i\theta}))|^2 \frac{d\theta}{2\pi} \\ &\leq r^{2n} \int_T |\psi(e^{i\theta})|^2 |g(\varphi(e^{i\theta}))|^2 \frac{d\theta}{2\pi}. \end{aligned}$$

But

$$|g(\varphi(e^{i\theta}))| \leq \|g\| \frac{1}{\sqrt{1-r^2}}$$

so that

$$\|C_{\psi,\varphi}f\|^2 \leq r^{2n} \frac{\|g\|^2}{1-r^2} \int_T |\psi(e^{i\theta})|^2 \frac{d\theta}{2\pi}.$$

Now, ψ is in the Hardy space and $\|g\| = \|f\|$; thus,

$$\|C_{\psi,\varphi}f\|^2 \leq \frac{r^{2n}}{1-r^2} \|\psi\|^2 \|f\|^2.$$

Now we have

$$\|C_{\psi,\varphi}|_{z^n H^2(D)}\|^2 \leq \frac{r^{2n}}{1-r^2} \|\psi\|^2.$$

Therefore $\|C_{\psi,\varphi}|_{z^n H^2(D)}\|$ tends to zero as n tends to infinity; thus $C_{\psi,\varphi}$ is compact. $\square$

Can compactness of $C_{\psi,\varphi}$ force the compactness of C_φ ? The answer is yes if $|\psi|$ is essentially bounded below on the unit circle, for then the operator M_ψ is bounded below and compactness of $M_\psi C_\varphi$ implies compactness of C_φ .

The following example shows that the continuity hypothesis on the weight ψ in Theorem 1 cannot be completely removed.

Example 3. Let $\psi(z) = e^{(z+1)/(z-1)}$ and let $\varphi(z) = (z+1)/2$. The radial limit $\lim_{r \rightarrow 1^-} \psi(r) = 0$; therefore $\psi(1) = 0$ and the only place $|\varphi(z)| = 1$ is $z = 1$. Since ψ has modulus 1 almost everywhere on the unit circle, M_ψ is an isometry. Thus $M_\psi C_\varphi$ cannot be compact.

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