SOLUTION OF THE KARLIN PROBLEM FOR ZERO-DIMINISHING SEQUENCES SATISFYING A CARLEMAN CONDITION

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ABSTRACT. We describe all zero-diminishing sequences (over the real-valued polynomials on $\mathbb{R}$) which additionally satisfy a Carleman condition and show that they are of the same kind as those in E. Laguerre's theorem from 1884.

1. INTRODUCTION AND MAIN RESULTS

Denote by $\mathcal{P}$ the set of all real algebraic polynomials. A sequence of real numbers $\gamma := \{\gamma_n\}_{n \geq 0}$ is called a zero-diminishing sequence if for any polynomial $p(x) = p_0 + p_1x + \dots + p_mx^m$ with real coefficients (i.e. $p \in \mathcal{P}$) the number of real zeros of the transformed polynomial $T_\gamma p$ with $T_\gamma p(x) := \gamma_0 p_0 + \gamma_1 p_1 x + \dots + \gamma_m p_m x^m$ is not greater than the number of real roots of p . In other words,

$$(1.1) \quad \mathbb{Z}_{\mathbb{R}}(T_\gamma p) \leq \mathbb{Z}_{\mathbb{R}}(p),$$

where $\mathbb{Z}_A(p)$ denotes the number of zeros of the non-constant polynomial p in the set $A \subset \mathbb{C}$, counting multiplicities, while $\mathbb{Z}_A(a) = 0$ for every real number a . Let $\mathcal{ZD}$ denote the set of all zero-diminishing sequences in the sense of (1.1) and which are not of the form $\{\gamma_0, 0, \dots, 0, \dots\}$.

Let $\mathcal{LP}_2^+$ denote the set of Laguerre-Pólya functions, i.e. the entire functions with a representation

$$(1.2) \quad e^{-ax^2 + bx} \prod_{k=1}^m \left(1 + \frac{x}{\alpha_k}\right) e^{-\frac{x}{\alpha_k}},$$

where $m \in \mathbb{N} \cup \{\infty\}$ and

$$a, b, \alpha_k \in \mathbb{R}, \text{ with } a \geq 0, \alpha_k > 0 \text{ for all } k = 1, \dots, m, \text{ and } \sum_{k=1}^m \frac{1}{\alpha_k^2} < \infty;$$

compare [18, p. 8] or [17, Ch. III, §3]. In 1884, E. Laguerre [21, p. 116] proved that for every entire function $\Phi \in \mathcal{LP}_2^+$ we have $\{1/\Phi(n)\}_{n \geq 0} \in \mathcal{ZD}$. We say

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that a sequence $\gamma \in \mathcal{ZD}$ is of Laguerre type if $\gamma_n = \gamma_0/\Phi(n)$, $n \geq 0$, for some $\Phi \in \mathcal{LP}_2^+$, $\gamma_0 > 0$.

This theorem of Laguerre led S. Karlin [20, p. 382] to the following question (which has ever since been referred to as the Karlin problem): are there zero-diminishing sequences which are (essentially) of non-Laguerre type? Here the word “essential” hints at the fact that $\gamma := \{\gamma_n\}_{n \geq 0} \in \mathcal{ZD}$ implies

$$c\gamma := \{c\gamma_n\}_{n \geq 0} \in \mathcal{ZD} \text{ for } c \neq 0 \text{ and } \gamma^* := \{(-1)^n \gamma_n\}_{n \geq 0} \in \mathcal{ZD}.$$

Karlin’s problem seems to be still open.¹

In 1996, A. Bakan, T. Craven, G. Csordas and A. Golub [3] took up this problem and proved in [3, Lem. 3] that if one (and therefore all) of the real sequences $\gamma, -\gamma, \gamma^*, -\gamma^*$ belongs to $\mathcal{ZD}$, then (exactly) one of them has only positive entries (like the ones of Laguerre type). Therefore it will be sufficient to study positive zero-diminishing sequences. The set of those sequences will be denoted by $\mathcal{ZD}^+$.

With every $\gamma \in \mathcal{ZD}^+$ we associate the formal power series

$$\Gamma_\gamma(z) := \sum_{n=0}^{\infty} \gamma_n z^n.$$

It was proved in [5, Lem. 1] that for every $\gamma \in \mathcal{ZD}^+$ there exists a non-negative Borel measure μ_γ on $\mathbb{R}^+ := [0, +\infty)$ with all moments finite such that

$$(1.3) \quad \gamma_n = \int_0^\infty t^n d\mu_\gamma(t), \quad n \geq 0.$$

Representation (1.3) and the Cauchy-Schwarz inequality allow us to estimate the second-order Hankel determinants (see [28, Th. 1.2]) of the shifted zero-diminishing sequence $\{\gamma_{n+m}\}_{n \geq 0}$ with $m \geq 0$ to get the inverse Turan inequalities for $\gamma \in \mathcal{ZD}^+$ (see [13]):

$$(1.4) \quad \begin{vmatrix} \gamma_n & \gamma_{n+1} \\ \gamma_{n+1} & \gamma_{n+2} \end{vmatrix} = \gamma_n \gamma_{n+2} - \gamma_{n+1}^2 \geq 0, \quad n \geq 0.$$

This implies, in particular, that

$$(1.5) \quad 0 \leq \lim_{n \rightarrow \infty} \frac{\gamma_n}{\gamma_{n+1}} = \lim_{n \rightarrow \infty} \gamma_n^{-1/n} < \infty;$$

see [26, 3.37]. Theorem A below solves completely the cases in (1.5) with $Q(\gamma) := \lim_{n \rightarrow \infty} \gamma_n^{-1/n} > 0$, i.e. if $\Gamma_\gamma(z)$ has a positive radius of convergence.

Theorem A ([3], 1996). *For any sequence γ of positive real numbers with $Q := Q(\gamma) > 0$ the following statements are equivalent:*

- (a) $\gamma \in \mathcal{ZD}^+$.
- (b) *There exists $\Phi \in \mathcal{LP}_1(\log Q)$ such that $\gamma_n = \frac{\gamma_0}{\Phi(n)}$, $n \geq 0$.*
- (c) *For all $A > 0$ and $p \in \mathcal{P}$ we have $\mathbb{Z}_{[0, A \cdot Q]}(T_\gamma p) \leq \mathbb{Z}_{[0, A]}(p)$.*

Here

$$\mathcal{LP}_1(a) := \left\{ e^{ax} \prod_{k \geq 1} \left(1 + \frac{x}{\alpha_k}\right) \mid \alpha_k > 0 \text{ for } k \geq 1, \text{ and } \sum_{k \geq 1} \frac{1}{\alpha_k} < \infty \right\}.$$

In particular Theorem A implies:

¹It was shown in [5] and [15] that the solution proposed in [14] is incorrect.

²It follows from (1.1) applied to $x^m p(x)$, $p \in \mathcal{P}$, that $\{\gamma_{n+m}\}_{n \geq 0} \in \mathcal{ZD}^+$ for all $m \geq 0$.

Corollary A. *Let $\gamma \in \mathcal{ZD}^+$ be such that Γ_γ has a positive radius of convergence. Then γ is of Laguerre type.*

In view of (1.3) Karlin's problem can be reformulated as follows: for which measures on $[0, +\infty)$ do the moments form a zero-diminishing sequence? Theorem A solves the case when the support $\text{supp } \mu_\gamma$ is bounded³ (compare [3, Lem. 3]), namely $\text{supp } \mu_\gamma \subset [0, 1/Q(\gamma)]$. The goal of the present paper is to extend the method used in [3] to include measures μ_γ in (1.3) having unbounded support, namely to certain cases with $Q(\gamma) = 0$ or, equivalently, to cases where Γ_γ has no positive radius of convergence.

We will see that Karlin's problem can partially be reduced to Schoenberg's theorem about Pólya Frequency densities (PF-densities) (see [17, Ch. IV], [6]). Our main result is the following theorem.

Theorem 1. *Let $\gamma \in \mathcal{ZD}^+$ be such that for any finite system of numbers $0 < \beta_1 \leq \beta_2 \leq \dots \leq \beta_m < \infty$, $m \geq 1$, the sequence $\{(\beta_1^n + \beta_2^n + \dots + \beta_m^n) \gamma_n\}_{n \geq 0}$ corresponds to a determinate Stieltjes moment problem. Then there exists $\Phi \in \mathcal{LP}_2^+$ such that $\gamma_n = \frac{\gamma_0}{\Phi(n)}$, $n \geq 0$.*

Theorem 1 requires that the measure $d\mu(x) := d\mu_\gamma(x/\beta_1) + d\mu_\gamma(x/\beta_2) + \dots + d\mu_\gamma(x/\beta_m)$ should be the unique solution of the Stieltjes moment problem:

$$(1.6) \quad (\beta_1^n + \beta_2^n + \dots + \beta_m^n) \gamma_n = \int_0^\infty t^n d\mu(t), \quad n \geq 0.$$

It is worth mentioning that Theorem 1 applies to each $\gamma \in \mathcal{ZD}^+$ satisfying the assumption of Theorem A; i.e. $\text{supp } \mu_\gamma$ is bounded, because the moment problem on any bounded interval is indeed determinate (see, for example, [23]).

Two less general but probably more easily verified sufficient conditions for $\gamma \in \mathcal{ZD}^+$ to fulfill the assumptions of Theorem 1 are in the following corollary.

Corollary 1. *Let $\gamma \in \mathcal{ZD}^+$ satisfy one of the following properties:*

A. *There exists $a > 0$ such that the sequence $\left\{\frac{\gamma_n}{n+a}\right\}_{n \geq 0}$ corresponds to a determinate Stieltjes moment problem.*

B. *The Carleman condition (see [11, 12]) holds: $\sum_{n \geq 1} \frac{1}{\gamma_n^{1/2n}} = \infty$.*

Then there exists $\Phi \in \mathcal{LP}_2^+$ such that $\gamma_n = \frac{\gamma_0}{\Phi(n)}$, $n \geq 0$.

It is easy to see that any $\gamma \in \mathcal{ZD}^+$ with $Q(\gamma) > 0$ satisfies both conditions, A and B, of Corollary 1.

Before we turn to the proofs of our results we should like to make the following general remarks on zero-diminishing sequences:

1) In 1951, Schoenberg [27] proved that for every $\gamma \in \mathcal{ZD}^+$ of Laguerre type, except for $\gamma = \{\gamma_0 \cdot e^{bn}\}_{n \geq 0}$, $b \in \mathbb{R}$, there exists a representing measure μ_γ in (1.3) of the form $d\mu_\gamma(x) = (\gamma_0/x)f(\log 1/x)dx$, where $f(x)$ is a PF-density (see [20, p. 31]).

2) In 1993, the following related problem was solved in [5]: to characterize all real sequences γ for which

$$(1.7) \quad \mathbb{Z}_{[0,A]}(T_\gamma p) \leq \mathbb{Z}_{[0,A]}(p) \text{ for all } A > 0, \quad p \in \mathcal{P}.$$

³ $\text{supp } \mu := \{x \in \mathbb{R} \mid \forall \varepsilon > 0 : \mu((x - \varepsilon, x + \varepsilon)) > 0\}.$

The paper [3] started out from this work.

3) Note that for $\gamma \in \mathcal{ZD}^+$ the reciprocal sequence $\tau := \{\tau_n\}_{n \geq 0}$, with $\tau_n := 1/\gamma_n$, $n \geq 0$, has an inverse property:

$$(1.8) \quad \mathbb{Z}_{\mathbb{R}}(T_{\tau}p) \geq \mathbb{Z}_{\mathbb{R}}(p), \quad p \in \mathcal{P}.$$

4) In 1914, Pólya and Schur [24] solved a problem related to (1.8): they characterized all sequences τ of real numbers so that the transformation T_{τ} maps real polynomials with only real zeros to polynomials of the same type.

5) Other questions related to Karlin's problem have been discussed in A. Iserles, S.P. Norsett and E.B. Saff [19], T. Craven and G. Csordas [16] and Bakan, Craven and Csordas [4].

2. HOMOGENEOUSLY DETERMINATE MEASURES ON THE POSITIVE HALF-LINE

In the sequel, we write $\mathcal{M}^*(\mathbb{R}^+)$ and $\mathcal{M}^*(\mathbb{R})$ for the sets of all non-negative Borel measures on $\mathbb{R}$ with all moments finite and with non-empty $\text{supp } \mu$ contained in $\mathbb{R}^+$ or $\mathbb{R}$, respectively. We say that $\mu \in \mathcal{M}^*(\mathbb{R})$ is determinate in the sense of Hamburger (in short: $\mu \in \det \mathcal{H}$) if μ is the only measure in $\mathcal{M}^*(\mathbb{R})$ with the same moments as μ , and we say that μ is indeterminate in the sense of Hamburger (in short: $\mu \in \text{indet } \mathcal{H}$) if there exist different measures in $\mathcal{M}^*(\mathbb{R})$ with the same moments as μ . Replacing in the above sentence $\mathcal{M}^*(\mathbb{R})$ by $\mathcal{M}^*(\mathbb{R}^+)$ we get a definition of the sets $\det \mathcal{S}$ and $\text{indet } \mathcal{S}$ of measures which are called determinate and indeterminate in the sense of Stieltjes, correspondingly.

The index of determinacy of $\mu \in \det \mathcal{H}$ is defined as

$$\text{ind}(\mu) := \sup \{k \in \mathbb{N}_0 \mid \mu_k \in \det \mathcal{H}\},$$

where $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ and $d\mu_s(x) := (x^2 + 1)^s d\mu(x)$, $s \in \mathbb{R}$. Note that $\text{ind}(\mu)$ coincides with the index $\text{ind}_i \mu$ introduced in [8, p. 2795].

Let

$$\pi_{\infty} := \{(\beta_1, \beta_2, \dots, \beta_n) \mid 0 < \beta_1 \leq \beta_2 \leq \dots \leq \beta_n < \infty, 1 \leq n < \infty\}.$$

For arbitrary $\mu \in \mathcal{M}^*(\mathbb{R}^+)$ and $\beta := (\beta_1, \beta_2, \dots, \beta_n) \in \pi_{\infty}$ let $\Gamma_{\beta}\mu$ be the measure

$$\Gamma_{\beta}\mu(A) := \sum_{k=1}^n \mu(A/\beta_k), \quad A \in \mathcal{B}(\mathbb{R}),$$

where $\mathcal{B}(\mathbb{R})$ are the Borel sets in $\mathbb{R}$. The measures in

$$(2.1) \quad \det_{\text{h}} \mathcal{S} := \{\mu \in \mathcal{M}^*(\mathbb{R}^+) \mid \Gamma_{\beta}\mu \in \det \mathcal{S} \text{ for all } \beta \in \pi_{\infty}\}$$

are related to the assumptions in Theorem 1 and are called *homogeneously determinate* on $\mathbb{R}^+$. Note that

$$(2.2) \quad \mu \in \det_{\text{h}} \mathcal{S}, \quad \beta \in \pi_{\infty} \Rightarrow \Gamma_{\beta}\mu \in \det_{\text{h}} \mathcal{S}.$$

Lemma 2.1. *If $\mu \in \det_{\text{h}} \mathcal{S}$, then for arbitrary $\beta \in \pi_{\infty}$ and $1 \leq p < \infty$ the set $\mathcal{P}$ is dense in the space $L_2(\mathbb{R}, (1+x^2)^p d\Gamma_{\beta}\mu)$.*

Proof. First recall M. Riesz's theorem [25], which implies that $\mathcal{P}$ is dense in $L_2(\mathbb{R}, (1+x^2)d\mu)$ iff $\mu \in \det \mathcal{H}$. If $\mu \in \text{indet } \mathcal{H}$ and $\mathcal{P}$ is dense in $L_2(\mathbb{R}, d\mu)$, then μ is called a Nevanlinna extremal measure (N -extremal in short). All N -extremal measures are discrete and satisfy: $\sum_{\lambda \in \text{supp } \mu} (1+\lambda^2)^{-1} < \infty$ (cf. [1, p. 100]). By (2.2) it suffices to prove polynomial denseness in $L_2(\mathbb{R}, (1+x^2)^p d\mu(x))$ for arbitrary

$\mu \in \det_{\mathbf{h}} \mathcal{S}$. However, for such μ , [7, Cor. 1.2] implies that $\mathcal{P}$ is dense in $L_2(\mathbb{R}, d\mu)$ and then, by M. Riesz's theorem: $\mu_{-1} \in \det \mathcal{H}$.

Assume that $l := \text{ind}(\mu_{-1}) < \infty$. Then Lemma 3.7 and Theorem 3.9 in [8] ensure the existence of an N -extremal measure σ with $d\sigma(x) = \sum_{\lambda \in \Lambda} \sigma_{\lambda} \delta(x - \lambda)$ (here $\delta(x)$ denotes the Dirac generalized function), and a subset $\Lambda_0 \subset \Lambda$ with $l + 1$ distinct elements such that

$$\frac{d\mu(x)}{1+x^2} = d\sigma(x) - \sum_{\lambda \in \Lambda_0} \sigma_{\lambda} \delta(x - \lambda) = \sum_{\lambda \in \Lambda \setminus \Lambda_0} \sigma_{\lambda} \delta(x - \lambda).$$

Next we choose $\beta_* \in (1, +\infty) \setminus \{\lambda/\lambda_1 \mid \lambda \in \Lambda, \lambda_1 \in \Lambda \setminus (\{0\} \cup \Lambda_0)\}$ and write $\{\beta_* \cdot a_k\}_{k \geq 0}$ for the sequence of distinct points in the infinite set $(\beta_* \cdot (\Lambda \setminus \Lambda_0)) \setminus \Lambda$. Look at the measure

$$(2.3) \quad d\nu(x) := \frac{d\mu(x)}{1+x^2} + \frac{d\mu(x/\beta_*)}{1+(x/\beta_*)^2} = \sum_{\lambda \in \Lambda \setminus \Lambda_0} \sigma_{\lambda} \delta(x - \lambda) + \sum_{\lambda \in \Lambda \setminus \Lambda_0} \sigma_{\lambda} \delta(x - \beta_* \lambda).$$

According to Lemma D in [8] the measure

$$d\omega(x) := \sum_{\lambda \in \Lambda \setminus \Lambda_0} \sigma_{\lambda} \delta(x - \lambda) + \sum_{k=0}^l \sigma_{a_k} \delta(x - a_k \beta_*)$$

is N -extremal and so $\mathcal{P}$ is not dense in $L_2(\mathbb{R}, (1+x^2)d\omega(x))$. Consequently (see (2.3)) $\mathcal{P}$ is not dense in $L_2(\mathbb{R}, d\Gamma_{\beta} \mu(x))$ as well, for $\beta := (1, \beta_*)$. Corollary 1.2 in [7, p. 211] then gives $\Gamma_{\beta} \mu \in \text{indet } \mathcal{S}$, which contradicts (2.2) and completes the proof of Lemma 2.1. $\square$

It should be noted that there exist measures $\mu \in \det \mathcal{S} \setminus \det_{\mathbf{h}} \mathcal{S}$ such that $\text{ind}(\mu) := +\infty$. Indeed, for $d\mu(x) = \sum_{k \geq 1} e^{-1,5k^2} \delta(x - e^k)$ it follows from [7, Th. 3.8], [22, Th. 17] and [2, Cor. 2.1] that $\mu \in \det \mathcal{S}$, $\text{ind}(\mu) := +\infty$ but $\Gamma_{(0.75, 1)} \mu \notin \det \mathcal{S}$. We omit the details.

Let $\mu \diamond \nu$ denote the convolution of $\mu, \nu \in \mathcal{M}^*(\mathbb{R}^+)$ defined as

$$(2.4) \quad (\mu \diamond \nu)(A) = \iint_{\mathbb{R}^2} \chi_A(x \cdot y) d\mu(x) d\nu(y), \quad A \in \mathcal{B}(\mathbb{R}),$$

where χ_A denotes the characteristic function of the set A . Properties of this convolution can be found in [10, Ch. VIII, §1] and [9, p. 244]. In particular, if $s_{\xi}(\mu) := \int_{\mathbb{R}^+} x^{\xi} d\mu(x)$, for some ξ with $\text{Re } \xi \geq 0$, then $s_{\xi}(\mu \diamond \nu) = s_{\xi}(\mu) \cdot s_{\xi}(\nu)$.

Let δ_a denote the Dirac measure at $a \in \mathbb{R}$, i.e. $d\delta_a(x) := \delta(x - a)$. Then for any $A \in \mathcal{B}(\mathbb{R})$ and $a > 0$ we have $(\delta_a \diamond \mu)(A) = \mu(A/a)$. Thus, for $\beta \in \pi_{\infty}$ and $\mu, \nu \in \mathcal{M}^*(\mathbb{R}^+)$, one obtains

$$(2.5) \quad \Gamma_{\beta} \mu = (\delta_{\beta_1} + \delta_{\beta_2} + \dots + \delta_{\beta_n}) \diamond \mu, \quad \Gamma_{\beta}(\mu \diamond \nu) = (\Gamma_{\beta} \mu) \diamond \nu = \mu \diamond (\Gamma_{\beta} \nu).$$

Lemma 2.2. *Let $\mu, \nu \in \mathcal{M}^*(\mathbb{R}^+)$ and assume that for arbitrary $\theta \in (0, 1]$ there exists a finite positive constant C_{θ} such that $\nu(A/\theta) \leq C_{\theta} \cdot \nu(A)$ holds for any $A \in \mathcal{B}(\mathbb{R})$ (in short: $\delta_{\theta} \diamond \nu \leq C_{\theta} \cdot \nu$). Then $\mu \diamond \nu \in \det \mathcal{S}$ implies $\mu \in \det_{\mathbf{h}} \mathcal{S}$.*

Proof. Lemma 2.2 and Remark 2.3 in [9, pp. 246, 247] state that for any $\rho, \omega \in \mathcal{M}^*(\mathbb{R}^+)$, with ω not a positive multiple of δ_0 , one has

$$(2.6) \quad \rho \diamond \omega \in \det \mathcal{S} \quad \Rightarrow \quad \rho \in \det \mathcal{S}.$$

For $a > 0$ use $\rho = \mu \diamond (\delta_a \diamond \nu)$, $\omega = \delta_{1/a}$ in (2.6) and apply the relation $\delta_a \diamond \delta_b = \delta_{a \cdot b}$, $a, b > 0$, to obtain $\rho \diamond \omega = \mu \diamond \nu$ and

$$(2.7) \quad \mu \diamond (\delta_a \diamond \nu) \in \det \mathcal{S}.$$

Moreover, our assumption concerning ν applied to $A = B/b$ and $\theta = a/b \in (0, 1]$ gives

$$(2.8) \quad \delta_a \diamond \nu \leq C_{a/b} \cdot (\delta_b \diamond \nu), \quad 0 < a \leq b < \infty.$$

It now follows from (2.8) and (2.5) that for arbitrary $\beta \in \pi_\infty$ we have

$$\Gamma_\beta \nu \leq C_\beta \cdot (\delta_{\beta_n} \diamond \nu), \quad \text{where } C_\beta := (C_{\beta_1/\beta_n} + \dots + C_{\beta_{n-1}/\beta_n} + C_1),$$

and therefore, by (2.4) and (2.5), $(\Gamma_\beta \mu) \diamond \nu = \mu \diamond (\Gamma_\beta \nu) \leq C_\beta \cdot (\mu \diamond (\delta_{\beta_n} \diamond \nu))$. From (2.7) we deduce $\mu \diamond (\delta_{\beta_n} \diamond \nu) \in \det \mathcal{S}$ so that [7, Th. 3.9] gives $(\Gamma_\beta \mu) \diamond \nu \in \det \mathcal{S}$. Our conditions imposed on ν do not allow it to be a Dirac measure. Thus (2.6) yields $\Gamma_\beta \mu \in \det \mathcal{S}$ for every $\beta \in \pi_\infty$, which is our claim. $\square$

Let $\mathfrak{P}(\mathbb{R}^+)$ be the class of all continuous and non-negative functions g on $(0, +\infty)$ which, at infinity, do not grow faster than some monomial x^q , $q \geq 0$, and have at most finitely many positive zeros. Let

$$(2.9) \quad \mathcal{O} := \left\{ \pm g(x) \cdot \prod_{k=1}^m (x - x_k) \mid 0 < x_1 < x_2 < \dots < x_m < \infty, m \in \mathbb{N}_0, g \in \mathfrak{P}(\mathbb{R}^+) \right\}$$

where $\prod_{k=1}^0 := 1$. The following is an immediate consequence of Lemma 2.1.

Lemma 2.3. *Let $\mu \in \det_h \mathcal{S}$, $f(x) = g(x) \cdot \prod_{k=1}^m (x - x_k) \in \mathcal{O}$, where $g \in \mathfrak{P}(\mathbb{R}^+)$ belongs to $\bigcap_{\lambda > 0} L_1(\mathbb{R}, d\mu(x/\lambda))$. Then for any finite system $0 < y_1 < y_2 < \dots < y_n < \infty$, there exists a polynomial sequence $P_j(x) = (1/j + Q_j(x)^2) \cdot \prod_{k=1}^m (x - x_k)$ for $j \geq 1$, with $Q_j \in \mathcal{P}$, which satisfies*

$$(2.10) \quad \lim_{j \rightarrow \infty} \int_0^\infty P_j(ty_k) d\mu(t) = \int_0^\infty f(ty_k) d\mu(t), \quad 1 \leq k \leq n.$$

Proof. If $g \in \mathfrak{P}(\mathbb{R}^+)$, then $(1+x)^r g \in L_1(\mathbb{R}, d\mu(x/\lambda))$ for arbitrary $r, \lambda > 0$, so that, according to Lemma 2.1, there exists a polynomial sequence $\{Q_j\}_{j \geq 1}$ which approximates $\sqrt{g(x)}$ in the space $L_2(\mathbb{R}, (1+x)^m d\nu(x))$, with $\nu := \Gamma_\beta \mu$, $\beta := (y_1, y_2, \dots, y_n)$. Then, with $A := \prod_{k=1}^m (1+x_k)$ we obtain

$$\begin{aligned} & \frac{1}{A} \int_0^\infty \left| g(t) \prod_{k=1}^m (t - x_k) - Q_j(t)^2 \prod_{k=1}^m (t - x_k) \right| d\nu(t) \leq \int_0^\infty |g(t) - Q_j(t)^2| (1+t)^m d\nu(t) \\ & \leq \left(\int_0^\infty |\sqrt{g(t)} - Q_j(t)|^2 (1+t)^m d\nu(t) \cdot \int_0^\infty |\sqrt{g(t)} + Q_j(t)|^2 (1+t)^m d\nu(t) \right)^{1/2} \rightarrow 0, \end{aligned}$$

and

$$(1/j) \cdot \int_0^\infty \prod_{k=1}^m |t - x_k| d\nu(t) \leq (A/j) \cdot \int_0^\infty (1+t)^m d\nu(t) \rightarrow 0, \quad \text{as } j \rightarrow \infty.$$

This implies (2.10). $\square$

3. AUXILIARY LEMMAS

Let $\mathcal{ZD}_\infty^+$ be the subset of $\gamma \in \mathcal{ZD}^+$ with $Q(\gamma) = 0$. We need to prove Theorem 1 only for $\gamma \in \mathcal{ZD}_\infty^+$ with $\mu_\gamma \in \det_h \mathcal{S}$ (see (1.3)). For such γ the transformation T_γ can be written as

$$(3.1) \quad T_\gamma p(x) = \int_0^\infty p(xt) d\mu_\gamma(t), \quad p \in \mathcal{P}.$$

We can extend the domain of definition of T_γ to

$$f \in \mathcal{O}_\gamma := \mathcal{O} \cap \left(\bigcap_{\lambda > 0} L_1(\mathbb{R}, d\mu_\gamma(x/\lambda)) \right)$$

by

$$(3.2) \quad T_\gamma f(x) := \int_0^\infty f(xt) d\mu_\gamma(t), \quad x > 0.$$

For arbitrary $A \subset \mathbb{R}$ and $g : A \mapsto \mathbb{R}$ let $S_A(g)$ denote the number of sign changes of g on A . More precisely (see [17, Ch. IV, §1]),

$$S_A(g) := \sup \{n \geq 1 \mid \exists \{t_i\}_{i=0}^n \subset A, t_0 < t_1 < \dots < t_n : g(t_j)g(t_{j+1}) < 0, 0 \leq j \leq n-1\}$$

with $\sup \emptyset := 0$ and $S_+(g) := S_{(0,+\infty)}(g)$. Observe that the definition of $\mathfrak{P}(\mathbb{R}^+)$ implies that the number of sign changes on $(0, +\infty)$ of a function of the form (2.9) is exactly m . An obvious application of Lemma 2.3 then gives

Lemma 3.1. *Let $\gamma \in \mathcal{ZD}_\infty^+$, $\mu_\gamma \in \det_h \mathcal{S}$ and $f \in \mathcal{O}_\gamma$. Then*

$$(3.3) \quad S_+(T_\gamma f) \leq S_+(f).$$

Proof. Assume there exists $f \in \mathcal{O}_\gamma$ satisfying $r := S_+(T_\gamma f) > S_+(f) =: m \geq 0$. Since $S_A(g) \equiv S_A(-g)$ we can assume that f in (2.9) has a form: $f(x) = g(x) \prod_{k=1}^m (x - x_k)$, where $g \in \mathfrak{P}(\mathbb{R}^+) \cap \bigcap_{\lambda > 0} L_1(\mathbb{R}, d\mu(x/\lambda))$. Choose points $0 < y_0 < y_2 < \dots < y_r < \infty$ such that

$$T_\gamma f(y_i) \cdot T_\gamma f(y_{i+1}) < 0, \quad 0 \leq i \leq r-1.$$

If we approximate f by the polynomial sequence $\{P_j\}_{j \geq 1}$ from Lemma 2.3, then by construction $\mathbb{Z}_\mathbb{R}(P_j) = m$, $j \geq 1$, and for sufficiently large j ,

$$T_\gamma P_j(y_i) \cdot T_\gamma P_j(y_{i+1}) < 0, \quad 0 \leq i \leq r-1,$$

i.e. $\mathbb{Z}_\mathbb{R}(T_\gamma P_j) \geq r > m$, which contradicts $\gamma \in \mathcal{ZD}_\infty^+$. $\square$

Lemma 3.2. *Let $\gamma \in \mathcal{ZD}_\infty^+$ and $\mu_\gamma \in \det_h \mathcal{S}$. Then $\mu_\gamma(\{0\}) = 0$.*

Proof. Assume that $a := \mu_\gamma(\{0\}) > 0$. Then

$$(3.4) \quad d\mu_\gamma(x) = a \cdot \delta(x) + d\nu_\gamma(x); \quad \nu_\gamma \in \mathcal{M}^*(\mathbb{R}^+); \quad \nu_\gamma([0, \varepsilon]) \rightarrow 0, \quad \varepsilon \downarrow 0.$$

Choose $0 < \alpha_1 < \alpha_2 < 1$ and let $p(x) = (x - \alpha_1)(x - \alpha_2)$. Then the polynomial $q(x)$ with $T_\gamma q(x) = p(x)$ satisfies $\lim_{x \rightarrow +\infty} q(x) = +\infty$ and therefore one can choose a finite positive number $m > -\min_{x \in \mathbb{R}^+} q(x)$. For every $\varepsilon > 0$ let

$$f_q^\varepsilon(x) := (m + q(x)) \cdot \chi_{(\varepsilon, +\infty)}(x) + \left(-A + \frac{x}{\varepsilon}(A + q(\varepsilon) + m) \right) \cdot \chi_{[0, \varepsilon]}(x),$$

where $A := m \cdot (\gamma_0 - a)/a - q(0)$. Clearly $f_q^\varepsilon \in C(\mathbb{R}^+)$, positive for $x \geq \varepsilon$ and linear on $[0, \varepsilon]$. It follows that f_q^ε can vanish at most once in the interval $(0, \varepsilon)$. Therefore $f_q^\varepsilon \in \mathcal{O}$ and Lemma 3.1 applies to give

$$(3.5) \quad S_+(T_\gamma f_q^\varepsilon) \leq S_+(f_q^\varepsilon) \leq 1, \quad \varepsilon > 0.$$

Then for any $x > 0$,

$$\begin{aligned} T_\gamma f_q^\varepsilon(x) &= \int_0^\infty f_q^\varepsilon(xt) d\mu_\gamma(t) \\ &= -a \cdot A + \int_0^\infty (m + q(t)) d\nu_\gamma(t/x) + \int_0^\varepsilon [f_q^\varepsilon(t) - q(t) - m] d\nu_\gamma(t/x); \end{aligned}$$

hence,

$$(3.6) \quad T_\gamma f_q^\varepsilon(x) = p(x) - m \cdot \nu_\gamma([0, \varepsilon/x]) + \int_0^\varepsilon [f_q^\varepsilon(t) - q(t)] d\nu_\gamma(t/x), \quad \varepsilon > 0.$$

The function $f_q^\varepsilon(t) - q(t)$ is uniformly bounded on $0 \leq t \leq \varepsilon$, so that (3.4) and (3.6) imply $\lim_{\varepsilon \downarrow 0} T_\gamma f_q^\varepsilon(x) = p(x)$ uniformly over $x \geq \delta$ for every $\delta > 0$. Thus, for sufficiently small $\varepsilon > 0$: $S_+(T_\gamma f_q^\varepsilon) \geq S_+(p) = 2$, a contradiction to (3.5). $\square$

4. PROOF OF THEOREM 1

We first study the shifted sequence $\omega := \{\gamma_1, \gamma_2, \dots, \gamma_n, \dots\}$, which also belongs to $\mathcal{ZD}_\infty^+$ (see remark made before (1.4)) and corresponds to the measure μ_ω given by $d\mu_\omega = x \cdot d\mu_\gamma(x)$. Since the product of a finite number of functions in $\mathcal{O}$ belongs to $\mathcal{O}$ and $\log x - \log a = (x - a) \cdot \frac{\log x - \log a}{x - a} \in \mathcal{O}$ for $x, a > 0$, we have

$$(4.1) \quad \{p(\log x) \mid p \in \mathcal{P}\} \subset \mathcal{O}_\omega = \mathcal{O} \cap \left(\bigcap_{\lambda > 0} L_1(\mathbb{R}, d\mu_\omega(x/\lambda)) \right).$$

In (3.2) we set $x = e^u$, $t = e^{-v}$, and write $F(u) := f(e^u)$, $V_\omega F(u) := T_\omega f(e^u)$ and $dL(v) := -d\mu_\omega(e^{-v})$. Then (3.2) becomes

$$(4.2) \quad V_\omega F(u) = \int_{-\infty}^{+\infty} F(u - v) dL(v),$$

where L can be looked at as the function $L(x) = \mu_\omega([e^{-x}, +\infty))$, $x \in \mathbb{R}$. Now Lemma 3.1 and (4.1) give

$$(4.3) \quad S_{\mathbb{R}}(V_\omega p) \leq S_{\mathbb{R}}(p), \quad p \in \mathcal{P}.$$

Recalling the proof of [17, Th. 4.1] we note that essentially nothing changes if, in that proof, we replace the convolution of $\pi^{-1/2}e^{-t^2}$ with the variation-diminishing frequency function φ by

$$\alpha(x) := \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-(x-t)^2} dL(t).$$

Indeed, the two-sided Laplace transform of $\alpha(x)$,

$$\int_{\mathbb{R}} e^{-sx} \alpha(x) dx = e^{\frac{s^2}{4}} \int_{\mathbb{R}} e^{-sx} dL(x) = e^{\frac{s^2}{4}} \int_0^\infty x^{s+1} d\mu_\gamma(x),$$

is analytic in the domain $\operatorname{Re} s > -1$ and (4.3) gives

$$S_{\mathbb{R}}\left(\int_{\mathbb{R}} \alpha(x - t) p(t) dt\right) \leq S_{\mathbb{R}}(p), \quad p \in \mathcal{P}.$$

This establishes the existence of

$$E(s) = E(0)e^{-as^2+bs} \prod_{k \geq 1}^m \left(1 + \frac{s}{\alpha_k}\right) e^{-\frac{s}{\alpha_k}},$$

where $a \geq 0$, $b \in \mathbb{R}$, $\alpha_k \geq 1$, $k \geq 1$, $m \in \mathbb{N} \cup \{\infty\}$ and $\sum_{k \geq 1} 1/\alpha_k^2 < \infty$, such that:

$$(4.4) \quad \int_{\mathbb{R}} e^{-sx} dL(x) = \int_0^\infty x^{s+1} d\mu_\gamma(x) = \frac{1}{E(s)},$$

for $\operatorname{Re} s > -1$. Lemma 3.2 allows us to pass to the limit $\operatorname{Re} s \rightarrow -1$ in the second equation of (4.4) so that (4.4) holds indeed for all $\operatorname{Re} s \geq -1$ and consequently $\alpha_k > 1$ for all $k \geq 1$. Therefore (4.4) is actually the required representation for $\gamma = \{\gamma_0, \gamma_1, \dots\}$ if we set $\Phi(s) := E(s-1)$.

5. PROOF OF COROLLARY 1

5.1. Proof for condition A. For any sequence γ fulfilling condition A we show that the corresponding measure μ_γ (see (1.3)) belongs to $\det_{\mathbf{h}} \mathcal{S}$; i.e. γ satisfies the assumptions of Theorem 1. For arbitrary $a > 0$ let $\tau_a \in \mathcal{M}^*(\mathbb{R}^+)$ be defined as follows: $d\tau_a(x) := x^{a-1} \cdot \chi_{[0,1]}(x) dx$. Since $s_n(\tau_a) = 1/(n+a)$ we find $s_n(\mu_\gamma \diamond \tau_a) = \gamma_n/(n+a)$, $n \in \mathbb{N}_0$, from which by condition A it follows that

$$(5.1) \quad \mu_\gamma \diamond \tau_a \in \det \mathcal{S}.$$

But for arbitrary $A \in \mathcal{B}(\mathbb{R})$ and $\theta \in (0, 1]$,

$$\tau_a(A/\theta) = \int_{A/\theta} x^{a-1} \cdot \chi_{[0,1]}(x) dx = \frac{1}{\theta^a} \cdot \int_A x^{a-1} \cdot \chi_{[0,\theta]}(x) dx \leq \frac{1}{\theta^a} \cdot \tau_a(A),$$

which shows that the measure τ_a satisfies the condition of Lemma 2.2. Therefore (5.1) and Lemma 2.2 imply $\mu_\gamma \in \det_{\mathbf{h}} \mathcal{S}$.

5.2. Proof for condition B. Since $\lim_{n \rightarrow \infty} \gamma_n = \infty$ for every $\gamma \in \mathcal{ZD}_\infty^+$ the assumption implies $\sum_{n \geq 1} (\gamma_n/(n+a))^{-1/(2n)} = \infty$, $a > 0$, so that Carleman's theorem [28, Th. 1.11] gives $\{\gamma_n/(n+a)\}_{n \geq 0} \in \det \mathcal{S}$, $a > 0$. Condition A in Corollary 1 is therefore fulfilled, and this completes the proof of part B.

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