SHARP MAXIMAL INEQUALITY FOR STOCHASTIC INTEGRALS

ADAM OSEKOWSKI

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ABSTRACT. Let $X = (X_t)_{t \geq 0}$ be a nonnegative supermartingale and $H = (H_t)_{t \geq 0}$ be a predictable process with values in $[-1, 1]$. Let Y denote the stochastic integral of H with respect to X . The paper contains the proof of the sharp inequality

$$\sup_{t \geq 0} \|Y_t\|_1 \leq \beta_0 \sup_{t \geq 0} X_t \|1\|_1,$$

where $\beta_0 = 2 + (3e)^{-1} = 2,1226\dots$. A discrete-time version of this inequality is also established.

1. INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, which is filtered by a nondecreasing right-continuous family $(\mathcal{F}_t)_{t \geq 0}$ of sub- σ -fields of $\mathcal{F}$. Assume that $\mathcal{F}_0$ contains all the events of probability 0. Suppose $X = (X_t)_{t \geq 0}$ is an adapted real-valued right-continuous semimartingale with left limits. Let Y be the Itô integral of H with respect to X ,

$$Y_t = H_0 X_0 + \int_{(0,t]} H_s dX_s, \quad t \geq 0,$$

where H is a predictable process with values in $[-1, 1]$. Let $\|Y\|_1 = \sup_{t \geq 0} \|Y_t\|_1$ and $X^* = \sup_{t \geq 0} |X_t|$.

The objective of this paper is to compare the first moments of Y and X^* . In [4], Burkholder introduced a method of proving related maximal inequalities for martingales and obtained the following sharp estimate.

Theorem 1.1. *If X is a martingale and Y is as above, then we have*

$$\|Y\|_1 \leq \gamma \|X^*\|_1,$$

where $\gamma = 2,536\dots$ is the unique solution of the equation

$$\gamma - 3 = -\exp\left(\frac{1-\gamma}{2}\right).$$

The constant is the best possible.

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Using Burkholder's techniques, we find the best constant in the inequality (1.1) in the case X where is a nonnegative supermartingale. The main result of the paper is the following.

Theorem 1.2. *Suppose X is a nonnegative supermartingale and Y is as above. Then the inequality*

$$(1.1) \quad \|Y\|_1 \leq \beta_0 \|X^*\|_1$$

holds true with $\beta_0 = 2 + (3e)^{-1} = 2,1226\dots$. The constant is the best possible. It is already the best possible if X is assumed to be a nonnegative martingale.

As usual, the inequality for stochastic integrals is accompanied by its discrete-time version. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, equipped with filtration $(\mathcal{F}_n)_{n \geq 0}$. Let $f = (f_n)_{n \geq 0}$ be an adapted nonnegative supermartingale and $g = (g_n)_{n \geq 0}$ be its transform by a predictable sequence $v = (v_n)_{n \geq 0}$ bounded in absolute value by 1. That is,

$$f_n = \sum_{k=0}^n df_k, \quad g_n = \sum_{k=0}^n v_k df_k, \quad n = 0, 1, 2, \dots$$

By predictability of v we mean that v_0 is $\mathcal{F}_0$ -measurable and for any $k \geq 1$, v_k is measurable with respect to $\mathcal{F}_{k-1}$. Let $f_n^* = \max_{k \leq n} f_k$ and $f^* = \sup_k f_k$.

A discrete-time version of Theorem 1.2 can be stated as follows.

Theorem 1.3. *Let f, g, β_0 be as above. Then we have*

$$(1.2) \quad \|g\|_1 \leq \beta_0 \|f^*\|_1,$$

and the constant β_0 is the best possible. It is already the best possible if f is assumed to be a nonnegative martingale.

The paper is organized as follows. In the next section we describe Burkholder's method. Section 3 is devoted to the proofs of the maximal inequalities. In the last section we complete the proofs of Theorem 1.2 and Theorem 1.3 by showing that the constant β_0 cannot be replaced by a smaller one.

2. THE UPPER CLASS OF FUNCTIONS

Throughout this section we deal with the discrete-time setting. We start with some reductions. Standard approximation arguments (see page 350 of [4]) show that it is enough to prove Theorem 1.3 under an additional assumption that the supermartingale f is simple, i.e. for any n the variable f_n takes only a finite number of values and there is N such that $f_N = f_{N+1} = f_{N+2} = \dots$ with probability 1. Then, clearly, every transform g of f is also simple and the pointwise limits f_∞, g_∞ exist. Furthermore, with no loss of generality, we may restrict ourselves to the special transforms g (called ± 1 transforms), namely, those with all v_n being deterministic and taking values in $\{-1, 1\}$; see Lemma A.1 on page 60 in [3] and observe $(F^j)^* = f^*$ on page 61. Finally, note that in order to prove inequality (1.2), it suffices to show that for any f, g as above and any integer n we have

$$\mathbb{E}|g_n| \leq \beta_0 \mathbb{E}f_n^*.$$

To describe Burkholder's method, let us consider the following general problem, first in the martingale setting: let $D = [0, \infty) \times \mathbb{R} \times [0, \infty)$ and $V : D \rightarrow \mathbb{R}$ be any

Borel function satisfying $V(x, y, z) = V(x, y, x \vee z)$. Suppose we want to prove the inequality

$$(2.1) \quad \mathbb{E}V(f_n, g_n, f_n^*) \leq 0$$

for all nonnegative integers n and all pairs (f, g) , where f is a simple nonnegative martingale and g is its ± 1 transform.

The key idea is to study the family $\mathcal{U}$ of all functions $U : D \rightarrow \mathbb{R}$ satisfying the following three properties:

$$(2.2) \quad U(x, y, z) = U(x, y, x \vee z) \quad \text{if } (x, y, z) \in D,$$

$$(2.3) \quad V(x, y, z) \leq U(x, y, z) \quad \text{if } (x, y, z) \in D;$$

and, furthermore, if $(x, y, z) \in D$, $\varepsilon \in \{-1, 1\}$, $\alpha \in (0, 1)$ and $t_1, t_2 \geq -x$ with $\alpha t_1 + (1 - \alpha)t_2 = 0$, then

$$(2.4) \quad \alpha U(x + t_1, y + \varepsilon t_1, z) + (1 - \alpha)U(x + t_2, y + \varepsilon t_2, z) \leq U(x, y, z).$$

The interplay between the class $\mathcal{U}$ and the maximal inequality (2.1) is described in the theorem below. It is a simple modification of Theorems 2.2 and 2.3 in [4] (see also Section 11 in [2] and Theorem 2.1 in [3]) to the case of nonnegative supermartingales. We omit the proof, as it requires only some minor changes.

Theorem 2.1. *The inequality (2.1) holds for all n and all pairs (f, g) as above if and only if the class $\mathcal{U}$ is nonempty. Furthermore, if $\mathcal{U}$ is nonempty, then there exists the least element in $\mathcal{U}$, given by*

$$(2.5) \quad U^0(x, y, z) = \sup\{\mathbb{E}V(f_\infty, g_\infty, f^* \vee z)\}.$$

Here the supremum runs over all the pairs (f, g) , where f is a simple nonnegative martingale, $\mathbb{P}((f_0, g_0) = (x, y)) = 1$ and $dg_k = \pm df_k$ almost surely for all $k \geq 1$.

In the case where f is assumed to be a nonnegative supermartingale, we can proceed in a similar manner. For a given V , consider the inequality (2.1). Suppose we want it to be valid for any n , any nonnegative supermartingale f and any ± 1 transform g . Let $\mathcal{U}'$ be a subclass of $\mathcal{U}$ containing those functions, which satisfy

$$(2.6) \quad U(x, y, z) \geq U(x - \delta, y \pm \delta, z) \quad \text{if } (x, y, z) \in D, \delta \in [0, x].$$

The analogue of Theorem 2.1 is as follows (the straightforward proof is omitted).

Theorem 2.2. *The inequality (2.1) holds for all n and all pairs (f, g) as above if and only if the class $\mathcal{U}'$ is nonempty.*

Now we turn to (1.2) and assume from now on that the function V is given by

$$V(x, y, z) = V(x, y, x \vee z) = y - \beta(x \vee z),$$

where $\beta > 0$ is a fixed number. The inequality (2.1) reads

$$(2.7) \quad \mathbb{E}|g_n| \leq \beta \mathbb{E}f_n^*.$$

Denote by $\mathcal{U}(\beta)$, $\mathcal{U}'(\beta)$ the classes $\mathcal{U}, \mathcal{U}'$ corresponding to this choice of V .

The rest of this section is devoted to the last part of Theorem 1.3. Let $\beta(\text{possup})$ (resp. $\beta(\text{posmar})$) be the smallest constant β in the inequality (2.7), when f is assumed to run over the class of all nonnegative supermartingales (resp. nonnegative martingales).

Theorem 2.3. *We have $\beta(\text{posmar}) = \beta(\text{possup})$.*

Proof. We only need the inequality $\beta = \beta(\text{posmar}) \geq \beta(\text{possup})$, as the reverse one is trivial. By Theorem 2.2, it suffices to prove that the class $\mathcal{U}'(\beta)$ is nonempty. Theorem 2.1 guarantees the existence of the minimal element U^0 of the class $\mathcal{U}(\beta)$, given by (2.5). By definition we get the following properties of U^0 :

$$(2.8) \quad U^0(x, y, z) = U^0(x, -y, z),$$

$$(2.9) \quad U^0(1, -1, 1) = U^0(1, 1, 1) \leq 0,$$

$$(2.10) \quad U^0(\alpha x, \alpha y, \alpha z) = \alpha U^0(x, y, z) \text{ for any } \alpha > 0.$$

The equality (2.8) is clear, (2.9) follows from the fact that for any pair (f, g) as in Theorem 2.1, starting from $(1, 1)$ or from $(1, -1)$, we have that g is a ± 1 transform of f and therefore, by (2.7), we have $\mathbb{E}V(f_n, g_n, f_n^*) \leq 0$ for any n . For (2.10), we use the fact that V is homogeneous.

We will prove that the function $U : D \rightarrow \mathbb{R}$ given by

$$(2.11) \quad U(x, y, z) = U^0(x, y, z) - U^0(1, 1, 1)x$$

belongs to $\mathcal{U}'$. The conditions (2.2), (2.3) and (2.4) hold true for U , since they are satisfied for U^0 and, by (2.9), we have $U \geq U^0$. It remains to prove (2.6). Note that U satisfies $U(x, y, z) = U(x, -y, z)$, $U(1, -1, 1) = U(1, 1, 1) = 0$ and is homogeneous. Fix $y \in \mathbb{R}$, $0 \leq x \leq z$, $\varepsilon \in \{-1, 1\}$ and let $\delta \in (0, x]$, $t > z - x$. Use (2.4) with $t_1 = -\delta$, $t_2 = t$ and $\alpha = t/(t + \delta)$ to obtain

$$\frac{t}{t + \delta} U(x - \delta, y - \varepsilon\delta, z) + \frac{\delta}{t + \delta} U(x + t, y + \varepsilon t, z) \leq U(x, y, z).$$

By homogeneity of U , this gives

$$(2.12) \quad \frac{t}{t + \delta} U(x - \delta, y - \varepsilon\delta, z) + \frac{\delta(x + t)}{t + \delta} U\left(1, \frac{y + \varepsilon t}{x + t}, 1\right) \leq U(x, y, z).$$

Now we let $t \rightarrow \infty$; the inequality (2.6) will follow if we show that

$$(2.13) \quad \liminf_{s \rightarrow 1} U(1, s, 1) \geq U(1, 1, 1) = 0.$$

For $s > 1$, use (2.4) with $x = z = 1$, $y = s$, $\varepsilon = -1$, $t_1 = -1$, $t_2 = (s - 1)/2$ and get

$$U(1, s, 1) \geq \frac{s - 1}{s + 1} U(0, s + 1, 1) + \frac{2}{s + 1} U\left(\frac{s + 1}{2}, \frac{s + 1}{2}, \frac{s + 1}{2}\right) \geq \frac{s - 1}{s + 1} (s + 1 - \beta),$$

the latter inequality being a consequence of (2.3) and the homogeneity of U . For $0 < s < 1$, apply (2.12) to $x = z = 1$, $y = s$, $\varepsilon = -1$, $\delta = (1 - s)/2$ and $t = 2s/(1 - s)$ (so that $(y + \varepsilon t)/(x + t) = -s$) to obtain

$$U(1, s, 1) \geq \frac{2}{s + 1} U\left(\frac{1 + s}{2}, \frac{1 + s}{2}, 1\right).$$

Now we use the fact that, by (2.4), the function $s \mapsto U(s, s, 1)$ is concave and therefore continuous. This completes the proof of (2.13) and, in consequence, we have $U \in \mathcal{U}'(\beta)$, so this class is nonempty. All that is left is to use Theorem 2.2. $\square$

Thus, to establish the inequality (1.2), we need to find an element U in $\mathcal{U}(\beta_0)$. This will be done in the next section.

3. THE PROOFS OF THE INEQUALITIES (1.1) AND (1.2)

Here we construct the special function U corresponding to the maximal inequality (1.2). This is the main section of the paper.

Let S denote the strip $[-1, 1] \times \mathbb{R}$. Consider the following subsets of S :

$$\begin{aligned} D_1 &= \{(x, y) : 0 \leq x < \frac{2}{3}, x + y \geq \frac{2}{3}\}, \\ D_2 &= \{(x, y) : \frac{2}{3} \leq x \leq 1, x - y \leq \frac{2}{3}\}, \\ D_3 &= \{(x, y) : 0 \leq x < \frac{2}{3}, y \geq 0, x + y \leq \frac{2}{3}\}, \\ D_4 &= \{(x, y) : \frac{2}{3} < x \leq 1, y \geq 0, x - y > \frac{2}{3}\}. \end{aligned}$$

Let the function u be defined on S by the condition $u(x, y) = u(|x|, |y|)$ and

$$u(x, y) = \begin{cases} y - \beta_0 + x\{\exp[-\frac{3}{2}(y + x - \frac{2}{3})] + 1\}, & (x, y) \in D_1, \\ y - \beta_0 + (\frac{4}{3} - x)\exp[-\frac{3}{2}(y - x + \frac{2}{3})] + x, & (x, y) \in D_2, \\ y - \beta_0 - x\log[\frac{3}{2}(x + y)] + 2x, & (x, y) \in D_3, \\ -\beta_0 - \frac{1}{3}(2 - 2x - y)(3 - 3x + 3y)^{1/2} + \frac{14}{9}, & (x, y) \in D_4. \end{cases}$$

A function defined on the strip S is said to be *diagonally concave* if it is concave on the intersection of S with any line of slope 1 or -1 . The proof of the following statement is just a matter of elementary calculations.

Lemma 3.1. *For any real number y we have*

$$\begin{aligned} (3.1) \quad & u(0, y) = |y| - \beta_0, \quad u(1, y) \geq |y| - \beta_0, \\ (3.2) \quad & u \text{ is diagonally concave,} \\ (3.3) \quad & u(1, \cdot) \text{ is convex,} \\ (3.4) \quad & u(1 - \delta, y \pm \delta) \leq u(1, y) \text{ for any } \delta \in [0, 1], \\ (3.5) \quad & u(1, 1) = 0. \end{aligned}$$

Define $U : D \rightarrow \mathbb{R}$ by

$$U(x, y, z) = (x \vee z)u\left(\frac{x}{x \vee z}, \frac{y}{x \vee z}\right).$$

We have the following statement.

Lemma 3.2. *The function U belongs to $\mathcal{U}(\beta_0)$.*

This fact can be proved exactly in the same manner as Lemma 3.1 in [4]. We omit the details. Now we are ready to prove the maximal inequalities.

Proof of the inequality (1.2). It is an immediate consequence of Theorem 2.1, Theorem 2.3 and Lemma 3.2. $\square$

Proof of inequality (1.1). This follows by approximation argument. See Section 16 of [2], where it is shown how the result of Bichteler [1] can be used to deduce the estimates for stochastic integrals from their discrete-time versions. $\square$

4. SHARPNESS

Clearly, we need only to focus on the sharpness of (1.2), since it immediately implies that β_0 is also the best possible in (1.1).

Let $\beta = \beta(\text{posmar})$. By Theorem 2.3, we need to prove $\beta \geq \beta_0$. This can be done by constructing an appropriate example. However, we take a different approach.

By Theorem 2.1, the class $\mathcal{U}(\beta)$ is nonempty, we can consider its minimal element U^0 and, as we have already proved, the function U given by (2.11) belongs to $\mathcal{U}'(\beta)$. Define $u : S \rightarrow \mathbb{R}$ by

$$(4.1) \quad u(x, y) = U(x, y, 1).$$

The conditions (2.3), (2.4) and (2.6) imply that

$$(4.2) \quad u(x, y) \geq |y| - \beta,$$

$$(4.3) \quad u \text{ is diagonally concave,}$$

$$(4.4) \quad u(x, y) \geq u(x - \delta, y \pm \delta) \quad \text{for } \delta \in [0, x]$$

and, moreover, we have

$$(4.5) \quad u(1, 1) = U(1, 1, 1) = 0.$$

Furthermore, note that for any y , by definition of U^0 ,

$$(4.6) \quad u(0, y) = U^0(0, y, 1) = |y| - \beta,$$

since the only nonnegative martingale starting from 0 is the constant one.

We will show that the existence of u satisfying the properties (4.2) – (4.6) implies $\beta \geq \beta_0$. This will be done in several steps. Set $B(x) = u(1, x + 1/3)$ and $C(x) = u(2/3, x)$.

Step 1. By properties (4.3) and (4.6), we have

$$\begin{aligned} u\left(\frac{2}{3} + \delta, 2k\delta + \delta\right) &\geq (1 - 3\delta)C(2k\delta) + 3\delta B(2k\delta), \\ C((2k + 2)\delta) &\geq \frac{2}{2 + 3\delta}u\left(\frac{2}{3} + \delta, 2k\delta + \delta\right) + \frac{3\delta}{2 + 3\delta}(2k\delta + 2\delta + \frac{2}{3} - \beta), \end{aligned}$$

from which we deduce that

$$(4.7) \quad C((2k + 2)\delta) \geq \frac{2(1 - 3\delta)}{2 + 3\delta}C(2k\delta) + \frac{6\delta}{2 + 3\delta}B(2k\delta) + \frac{3\delta}{2 + 3\delta}(2k\delta + 2\delta + \frac{2}{3} - \beta).$$

Furthermore, (4.3) and (4.4) yield

$$B(2k\delta) \geq u(1 - \delta, 2k\delta + \delta + \frac{1}{3}) \geq (1 - 3\delta)B((2k + 2)\delta) + 3\delta C((2k + 2)\delta).$$

Multiply this inequality throughout by $\alpha > 0$ and add it to (4.7). We obtain

$$\begin{aligned} &C((2k + 2)\delta)(1 - 3\alpha\delta) - \alpha(1 - 3\delta)B((2k + 2)\delta) \\ &\geq \frac{2(1 - 3\delta)}{2 + 3\delta}C(2k\delta) - \left(\alpha - \frac{6\delta}{2 + 3\delta}\right)B(2k\delta) + \frac{3\delta}{2 + 3\delta}((2k + 2)\delta + \frac{2}{3} - \beta), \end{aligned}$$

or, equivalently, after substitution

$$(4.8) \quad \overline{B}(t) = B(t) - t - \frac{2}{3} + \beta, \quad \overline{C}(t) = C(t) - t - \frac{2}{3} + \beta,$$

we get

$$\begin{aligned}
 (4.9) \quad & \overline{C}((2k+2)\delta) - \frac{\alpha(1-3\delta)}{1-3\alpha\delta} \overline{B}((2k+2)\delta) \\
 & \geq \frac{2(1-3\delta)}{(2+3\delta)(1-3\alpha\delta)} \left[\overline{C}(2k\delta) - \frac{2\alpha+3\alpha\delta-6\delta}{2(1-3\delta)} \overline{B}(2k\delta) \right] \\
 & \quad + \frac{2\delta}{1-3\alpha\delta} \left(\alpha - \frac{2}{2+3\delta} \right).
 \end{aligned}$$

Step 2. Now we will use the inequality (4.9) several times. The choice

$$\alpha = \frac{5 \pm \sqrt{9-24\delta}}{2(2+3\delta)}$$

gives

$$\frac{\alpha(1-3\delta)}{1-3\alpha\delta} = \frac{2\alpha+3\alpha\delta-6\delta}{2(1-3\delta)}$$

and using (4.9) for $k-1, k-2, \dots, l$ yields

$$\begin{aligned}
 (4.10) \quad & \overline{C}(2k\delta) - \frac{\alpha(1-3\delta)}{1-3\alpha\delta} \overline{B}(2k\delta) \\
 & \geq \left[\frac{2(1-3\delta)}{(2+3\delta)(1-3\alpha\delta)} \right]^{k-l} \left[\overline{C}(2l\delta) - \frac{\alpha(1-3\delta)}{1-3\alpha\delta} \overline{B}(2l\delta) \right] + \eta,
 \end{aligned}$$

where

$$\begin{aligned}
 \eta &= \frac{2\delta}{1-3\alpha\delta} \left(\alpha - \frac{2}{2+3\delta} \right) \sum_{r=0}^{k-l-1} \left[\frac{2(1-3\delta)}{(2+3\delta)(1-3\alpha\delta)} \right]^r \\
 &= \frac{2(2\alpha+3\alpha\delta-2)}{-9+6\alpha+9\alpha\delta} \left\{ \left[\frac{2(1-3\delta)}{(2+3\delta)(1-3\alpha\delta)} \right]^{k-l} - 1 \right\}.
 \end{aligned}$$

Now fix $K > L \geq 0$ with L/K rational. Then we may find arbitrarily large integers k and l such that $K = 2k\delta$ and $L = 2l\delta$ for some $\delta > 0$. Letting $k, l \rightarrow \infty$, we have $\delta \rightarrow 0$, $\alpha \rightarrow 2^{\pm 1}$, and (4.10) leads to

$$\overline{C}(K) - \alpha \overline{B}(K) + \frac{4(\alpha-1)}{-9+6\alpha} \geq \exp\left(\frac{(K-L)(-9+6\alpha)}{4}\right) \left[\overline{C}(L) - \alpha \overline{B}(L) + \frac{4(\alpha-1)}{-9+6\alpha} \right].$$

Now we come back to the original functions B, C . For $\alpha = 2$, the above inequality takes the form

$$(4.11) \quad C(K) + K + 2 - \beta - 2B(K) \geq \exp\left(\frac{3}{4}(K-L)\right) [C(L) + L + 2 - \beta - 2B(L)],$$

while for $\alpha = 1/2$, we get

$$(4.12) \quad 2C(K) - K + \beta - B(K) \geq \exp\left(-\frac{3}{2}(K-L)\right) [2C(L) - L + \beta - B(L)].$$

Step 3. This is the final part. By (4.2) and (4.4), we have $B(K) \geq K + \frac{1}{3} - \beta$ and $B(K) \geq C(K)$. Plugging these estimates into (4.11) we get that for any L ,

$$(4.13) \quad C(L) + L + 2 - \beta - 2B(L) \leq 0.$$

Furthermore, the conditions (4.3) and (4.6) yield

$$(4.14) \quad C(0) \geq \frac{2}{3}B(0) + \frac{1}{3}u\left(0, -\frac{2}{3}\right) = \frac{2}{3}B(0) + \frac{1}{3}\left(\frac{2}{3} - \beta\right).$$

Combining (4.14) with (4.13) applied to $L = 0$ gives

$$0 \geq C(0) + 2 - \beta - 2B(0) \geq -\frac{4}{3}B(0) - \frac{4}{3}\beta + \frac{20}{9},$$

which implies

$$(4.15) \quad \beta + B(0) \geq \frac{5}{3}.$$

The inequality (4.13), applied to $L = 2/3$, gives

$$(4.16) \quad C\left(\frac{2}{3}\right) \leq \beta - \frac{8}{3},$$

since $B(2/3) = 0$, due to (4.5). Now use (4.12) for $K = 2/3$ and $L = 0$ to obtain

$$2C\left(\frac{2}{3}\right) - \frac{2}{3} + \beta \geq \frac{1}{e}(2C(0) + \beta - B(0)).$$

Combining this estimate with (4.14), (4.15) and (4.16) yields

$$3\beta - 6 \geq \frac{1}{e}\left(\frac{5}{9} + \frac{4}{9}\right) = \frac{1}{e},$$

or $\beta \geq 2 + (3e)^{-1}$. This completes the proof of the sharpness of the inequality (1.2).

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DEPARTMENT OF MATHEMATICS, INFORMATICS AND MECHANICS, UNIVERSITY OF WARSAW, BANACHA 2, 02-097 WARSAW, POLAND

E-mail address: ados@mimuw.edu.pl