AN ELEMENTARY APPROACH TO SEVERAL RESULTS ON THE HARDY-LITTLEWOOD MAXIMAL OPERATOR

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ABSTRACT. We give new elementary proofs of theorems due to B. Muckenhoupt, B. Jawerth, and S. Buckley. By means of our approach we answer a question raised by J. Orobítg and C. Pérez.

1. INTRODUCTION

Let w be a weight, i.e., $w \geq 0$ and $w \in L^1_{\text{loc}}(\mathbb{R}^n)$. Given a measurable set E , let $w(E) = \int_E w(x)dx$. The Hardy-Littlewood maximal operator with respect to w is defined by

$$(1) \quad M_w f(x) = \sup_{Q \ni x} \frac{1}{w(Q)} \int_Q |f(y)|w(y)dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x . If the supremum is taken only over cubes Q centered at x , denote the corresponding operator by M_w^c . We drop the subscript w if $w \equiv 1$. Given a weight w and $p > 1$, set $\sigma = w^{-1/(p-1)}$. We say that w satisfies the A_p condition if

$$(2) \quad \|w\|_{A_p} \equiv \sup_Q \frac{w(Q)\sigma(Q)^{p-1}}{|Q|^p} < \infty.$$

In [8], B. Muckenhoupt proved the following fundamental result.

Theorem A. *The Hardy-Littlewood maximal operator M is bounded on L^p_w , $1 < p < \infty$, if and only if $w \in A_p$.*

The original proof of this theorem was based on the deep property of A_p weights saying that A_p implies $A_{p-\varepsilon}$ for some $\varepsilon > 0$. Then R. Coifman and C. Fefferman [3] gave a simplified proof but still depended on this property. Later, E. Sawyer [10] solved the two weight problem for M . In [6], R. Hunt, D. Kurtz and C. Neugebauer established that in the case of equal weights Sawyer's condition is equivalent to the A_p condition, providing a new proof of Muckenhoupt's theorem that completely avoids the implication $A_p \Rightarrow A_{p-\varepsilon}$. After that, a very simple proof was given by M. Christ and R. Fefferman [2]. However, the proof in [2] is based essentially on

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the Calderón-Zygmund decomposition which makes it applicable only to classical maximal operators.

In [7], B. Jawerth obtained an even more elementary proof with the advantage that it can be applied to maximal operators with respect to a general basis $\mathcal{B}$. By a basis we mean a collection of open sets in $\mathbb{R}^n$. Assume that the supremum is taken in (1) and (2) over $B \in \mathcal{B}$ instead of cubes. Denote the corresponding objects by $M_{w,\mathcal{B}}f$ and $A_{p,\mathcal{B}}$.

Theorem B. *Let $1 < p < \infty$. Then $M_{\mathcal{B}}$ is bounded on L_w^p and on $L_{\sigma}^{p'}$ if and only if $w \in A_{p,\mathcal{B}}$, $M_{\sigma,\mathcal{B}}$ is bounded on L_{σ}^p , and $M_{w,\mathcal{B}}$ is bounded on $L_w^{p'}$, where $1/p + 1/p' = 1$.*

Observe that $w \in A_{p,\mathcal{B}}$ if and only if $\sigma \in A_{p',\mathcal{B}}$. Next, in the case where $\mathcal{B}$ consists of cubes, the A_p condition implies the doubling condition (there exists $c > 0$ such that $w(2Q) \leq cw(Q)$ for any cube Q), and hence the boundedness of M_{σ} on L_{σ}^p , and M_w on $L_w^{p'}$ follows from A_p in the standard way by the usual covering argument. Thus, Theorem B contains Theorem A as a particular case.

In [1], S. Buckley found the sharp dependence of $\|M\|_{L_w^p}$ on $\|w\|_{A_p}$ in Muckenhoupt's theorem.

Theorem C. *Let $1 < p < \infty$. Then $\|M\|_{L_w^p} \leq c_{p,n} \|w\|_{A_p}^{1/(p-1)}$, and the exponent $1/(p-1)$ is best possible.*

The original proof of this result is based on the property $A_p \Rightarrow A_{p-\varepsilon}$ and on interpolation. It was mentioned in [1] that some proofs of the boundedness of M , for instance, Jawerth's proof [7], do not yield the sharp exponent $1/(p-1)$.

In this short note we give an extremely simple argument leading to the proofs of both Theorems B and C. This yields a new proof of Theorem A as well. Also we consider similar questions for A_p weights with respect to non-doubling measures.

Let μ be a non-negative Radon measure in $\mathbb{R}^n$ with the property that $\mu(\partial Q) = 0$ for any cube Q . Replacing in (1) w by μ (and $w(y)dy$ by $d\mu(y)$), we get the maximal operator M_{μ} . As usual, by M_{μ}^c we denote the centered maximal operator. Given a measurable set E and a weight w , let $w_{\mu}(E) = \int_E w(x)d\mu(x)$. We say that w satisfies the $A_p^{(r)}(\mu)$ ($0 < r \leq 1, p > 1$) condition if

$$\|w\|_{A_p^{(r)}(\mu)} \equiv \sup_Q A_p^{(r)}(Q) = \frac{w_{\mu}(rQ)}{\mu(rQ)} \left(\frac{\sigma_{\mu}(Q)}{\mu(Q)} \right)^{p-1} < \infty.$$

If $r = 1$ we simply write $A_p(\mu)$. Denote by $L_w^p(\mu)$ the space of all measurable f such that

$$\|f\|_{L_w^p(\mu)} \equiv \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) d\mu(x) \right)^{1/p} < \infty.$$

In [9], J. Orobítz and C. Pérez proved the following variant of Muckenhoupt's theorem.

Theorem D. *If $w \in A_p(\mu), p > 1$, then M_{μ}^c is bounded on $L_w^p(\mu)$.*

Using the same idea as was used in proving Theorems B and C, we get that M_{μ}^c is also bounded on $L_w^p(\mu)$ for $A_p^{(r)}(\mu)$ weights, $r < 1$.

Theorem D'. *Let $0 < r < 1$. If $w \in A_p^{(r)}(\mu), p > 1$, then M_{μ}^c is bounded on $L_w^p(\mu)$.*

It was asked in [9] whether the $A_p(\mu)$ condition is necessary for the $L_w^p(\mu)$ -boundedness of M_μ^c as in the classical case. We show (see Example 1 in Section 2 below) that $A_p^{(r)}(\mu) \not\subset A_p(\mu)$, $r < 1$. Therefore, Theorem D' provides a negative answer to this question.

In the next section we prove Theorems B, C and D'. To be more precise, we prove only the difficult parts of Theorems B and C, that is, the sufficiency part in Theorem B and the estimate for $\|M\|_{L_w^p}$ in Theorem C.

2. PROOFS

Proof of Theorem B. By the symmetry between w and σ , it suffices to prove that $M_{\mathcal{B}}$ is bounded on L_w^p . Setting $A_p(B) = w(B)\sigma(B)^{p-1}/|B|^p$, we have

$$\begin{aligned} \frac{1}{|B|} \int_B |f| &= A_p(B)^{\frac{1}{p-1}} \left\{ \frac{|B|}{w(B)} \left(\frac{1}{\sigma(B)} \int_B |f| \right)^{p-1} \right\}^{\frac{1}{p-1}} \\ &\leq \|w\|_{A_{p,\mathcal{B}}}^{\frac{1}{p-1}} \left\{ \frac{1}{w(B)} \int_B M_{\sigma,\mathcal{B}}(f\sigma^{-1})^{p-1} dx \right\}^{\frac{1}{p-1}}, \end{aligned}$$

and hence,

$$M_{\mathcal{B}}f(x) \leq \|w\|_{A_{p,\mathcal{B}}}^{\frac{1}{p-1}} M_{w,\mathcal{B}}(M_{\sigma,\mathcal{B}}(f\sigma^{-1})^{p-1}w^{-1})(x)^{\frac{1}{p-1}}.$$

Therefore,

$$\begin{aligned} \|M_{\mathcal{B}}f\|_{L_w^p} &\leq \|w\|_{A_{p,\mathcal{B}}}^{\frac{1}{p-1}} \|M_{w,\mathcal{B}}(M_{\sigma,\mathcal{B}}(f\sigma^{-1})^{p-1}w^{-1})\|_{L_w^p}^{\frac{1}{p-1}} \\ &\leq \|w\|_{A_{p,\mathcal{B}}}^{\frac{1}{p-1}} \|M_{w,\mathcal{B}}\|_{L_w^{p'}}^{\frac{1}{p-1}} \|M_{\sigma,\mathcal{B}}(f\sigma^{-1})\|_{L_\sigma^p} \\ (3) \quad &\leq \|w\|_{A_{p,\mathcal{B}}}^{\frac{1}{p-1}} \|M_{w,\mathcal{B}}\|_{L_w^{p'}}^{\frac{1}{p-1}} \|M_{\sigma,\mathcal{B}}\|_{L_\sigma^p} \|f\|_{L_w^p}. \end{aligned}$$

This completes the proof. $\square$

Observe that although the original proof of this theorem [7] (the same proof can be found in [4, p. 423]) was also simple, it involved several additional ingredients such as a selection process for sets B_j^k corresponding to the level set $\{2^k < M_{\mathcal{B}}f \leq 2^{k+1}\}$ with the subsequent introduction of an auxiliary discrete measure.

Proof of Theorem C. Suppose that $\mathcal{B}$ is formed by cubes. Note first that in the one-dimensional case $\|M_w\|_{L_w^{p'}}$ and $\|M_\sigma\|_{L_\sigma^p}$ are bounded uniformly in w (see [11]). Hence, in this case (3) immediately yields the desired estimate for $\|M\|_{L_w^p}$.

In the multi-dimensional case we have only to slightly modify the above argument. Setting $A_p(Q) = w(Q)\sigma(3Q)^{p-1}/|Q|^p$, we have

$$\begin{aligned} \frac{1}{|Q|} \int_Q |f| &= A_p(Q)^{\frac{1}{p-1}} \left\{ \frac{|Q|}{w(Q)} \left(\frac{1}{\sigma(3Q)} \int_Q |f| \right)^{p-1} \right\}^{\frac{1}{p-1}} \\ &\leq 3^{np} \|w\|_{A_p}^{\frac{1}{p-1}} \left\{ \frac{1}{w(Q)} \int_Q M_\sigma^c(f\sigma^{-1})^{p-1} dx \right\}^{\frac{1}{p-1}}. \end{aligned}$$

From this and from the fact that $Mf(x) \leq 2^n M^c f(x)$ we get

$$Mf(x) \leq 2^n 3^{np} \|w\|_{A_p}^{\frac{1}{p-1}} M_w^c(M_\sigma^c(f\sigma^{-1})^{p-1}w^{-1})(x)^{\frac{1}{p-1}}.$$

Using this inequality and the well-known fact (based on the Besicovitch covering theorem) that $\|M_w^c\|_{L_w^{p'}}$ and $\|M_\sigma^c\|_{L_\sigma^p}$ are bounded uniformly in w , exactly as above, we get Buckley's theorem. $\square$

Proof of Theorem D'. Define the maximal operator $M_\mu^{(r)}$ by

$$M_\mu^{(r)}f(x) = \sup_{rQ \ni x} \frac{1}{\mu(Q)} \int_Q |f(y)| d\mu(y),$$

where the supremum is taken over all cubes Q such that $x \in rQ$. By the Besicovitch covering theorem (see [5, pp. 6-7]) we have that the operator $M_\mu^{(r)}$ is bounded on L_μ^p for all $p > 1$, as the usual centered maximal operator (here we essentially use that $r < 1$). We have

$$\begin{aligned} \frac{1}{\mu(Q)} \int_Q |f| d\mu &= A_p^{(r)}(Q)^{\frac{1}{p-1}} \left\{ \frac{\mu(rQ)}{w_\mu(rQ)} \left(\frac{1}{\sigma_\mu(Q)} \int_Q |f| d\mu \right)^{p-1} \right\}^{\frac{1}{p-1}} \\ &\leq \|w\|_{A_p^{(r)}(\mu)}^{\frac{1}{p-1}} \left\{ \frac{1}{w_\mu(rQ)} \int_{rQ} M_{\sigma(d\mu)}^{(r)} (f\sigma^{-1})^{p-1} d\mu \right\}^{\frac{1}{p-1}}, \end{aligned}$$

and hence,

$$(4) \quad M_\mu^c f(x) \leq \|w\|_{A_p^{(r)}(\mu)}^{\frac{1}{p-1}} M_{w(d\mu)}^c (M_{\sigma(d\mu)}^{(r)} (f\sigma^{-1})^{p-1} w^{-1})(x)^{\frac{1}{p-1}}.$$

From this, using the fact that $M_{w(d\mu)}^c$ is bounded on $L_w^{p'}(\mu)$ and $M_{\sigma(d\mu)}^{(r)}$ is bounded on $L_\sigma^p(\mu)$, we get the $L_w^p(\mu)$ -boundedness of M_μ^c . $\square$

Remark. It is well-known (see [12] and [13]) that for $n \geq 2$ the maximal operator M_μ does not bounded on L_μ^p , in general. Therefore, the above proof does not work when $n \geq 2$ and $r = 1$. Note that the proof of Theorem D [9] was based on the property $A_p(\mu) \Rightarrow A_{p-\varepsilon}(\mu)$. We do not know how to avoid this property in the multi-dimensional case. On the other hand, in the case $n = 1$ and $r = 1$ a full analogue of inequality (4) for non-centered maximal functions holds which shows that the $A_p(\mu)$ condition is sufficient (and trivially necessary) for the boundedness of M_μ on $L_w^p(\mu)$.

In order to show that $A_p^{(r)}(\mu) \not\subset A_p(\mu)$, $r < 1$, we give the following example.

Example 1. Let $n = 1$. Set $d\mu = e^{|x|} dx$ and $w(x) = e^{(p-1)|x|}$. Let us show that $w \in A_p^{(r)}(\mu)$ for any $r < 1$ but $w \notin A_p(\mu)$. Indeed, it suffices to estimate $A_p^{(r)}(I)$ for any $I \subset (0, \infty)$. Let $I = (a, a+h)$. A straightforward calculation shows that

$$A_p^{(r)}(I) = \frac{h^{p-1}}{p} \frac{e^{(p-1)(1-r)h/2} (e^{prh} - 1)}{(e^{rh} - 1)(e^h - 1)^{p-1}}.$$

Hence, $\sup_I A_p(I) = \infty$, while $\sup_I A_p^{(r)}(I) < \infty$ for any $r < 1$.

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