A NOTE ON MULTIPLIER ALGEBRAS ON REPRODUCING KERNEL HILBERT SPACES

TAVAN T. TRENT

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ABSTRACT. We construct a simple reproducing kernel space whose multiplier algebra does not satisfy a “corona theorem”.

Let $\mathcal{H}_E$ denote a reproducing kernel Hilbert space on a set E . That is, $\mathcal{H}_E$ is a Hilbert space of functions on E , satisfying:

- (i) for every $x \in E$, there exists a $C_x < \infty$, so that
 $|f(x)| \leq C_x \|f\|_{\mathcal{H}_E}$, for all $f \in \mathcal{H}_E$,
- and (ii) If $f \in \mathcal{H}_E$ and $f(x) = 0$ for all $x \in E$, then
 $\|f\|_{\mathcal{H}_E} = 0$; i.e. $f = 0$ in $\mathcal{H}_E$.

For $g \in \mathcal{H}_E$, define $M_g(f)(x) = g(x)f(x)$ for $x \in E$ and $f \in \mathcal{H}_E$. Let $\mathcal{M}(\mathcal{H}_E) = \{f \in \mathcal{H}_E : M_f \in \mathcal{B}(\mathcal{H}_E)\}$, the multiplier algebra for $\mathcal{H}_E$.

When can we solve linear equations in $\mathcal{M}(\mathcal{H}_E)$? As a very preliminary step, we might try to characterize when $\mathcal{I}(\{f_j\}_{j=1}^n) = \mathcal{M}(\mathcal{H}_E)$, where $\mathcal{I}(\{f_j\}_{j=1}^n)$ is the ideal generated by $\{f_j\}_{j=1}^n \subset \mathcal{M}(\mathcal{H}_E)$. The model characterization is the “corona theorem” of Carleson [1] for $H^\infty(D)$ ($= \mathcal{M}(H^2(D))$). Namely, for $\{f_j\}_{j=1}^n \subset H^\infty(D)$, if $0 < \epsilon^2 \leq \sum_{j=1}^n |f_j(z)|^2$ for all $z \in D$, then

$$\mathcal{I}(\{f_j\}_{j=1}^n) = H^\infty(D).$$

We are interested in whether an abstract “corona theorem” can hold for a general algebra, $\mathcal{M}(\mathcal{H}_E)$. That is, if $\{f_j\}_{j=1}^n \subset \mathcal{M}(\mathcal{H}_E)$ and $0 < \epsilon^2 \leq \sum_{j=1}^n |f_j(x)|^2$ for all $x \in E$, does there exist $\{g_j\}_{j=1}^n \subset \mathcal{M}(\mathcal{H}_E)$ with $\sum_{j=1}^n f_j g_j = 1$?

Of course, by very deep results of Cole (see [2]) and Sibony [3], there are counterexamples to the corona problem of the form $H^\infty(\Omega)$ with $\Omega \subset \mathbb{C}^3$ and $\Omega \subset \mathbb{C}^2$, respectively. Since these algebras can be viewed as multiplier algebras for appropriate Bergman spaces, the answer to the corona question for a general multiplier algebra is “no”.

In this note, we give a simple counterexample to the “corona problem” for a multiplier algebra of functions of one variable. This example is strong enough to give that:

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there is a $g \in \mathcal{M}(\mathcal{H}_E)$, so that $\frac{1}{g} \in \mathcal{H}_E$ and $|g(x)| \geq \epsilon > 0$ for all $x \in E$, but $\frac{1}{g} \notin \mathcal{M}(\mathcal{H}_E)$.

Clearly, such an example cannot occur in $H^\infty(\Omega)$, and, in fact, functions in our space $\mathcal{M}(\mathcal{H}_E)$, will, unfortunately, have no smoothness properties in general.

For $\underline{a} = (a_1, \dots, a_{N^2}) \in l_{N^2}^2$, let $D_{\underline{a}}$ denote the $N^2 \times N^2$ diagonal matrix, whose diagonal entries are $d_{jj} = a_j$ for $j = 1, \dots, N^2$. Then our example is based on the following lemma:

Lemma 1. *For $N = 1, 2, \dots$ there exists an $\underline{a}_N \in l_{N^2}^2$ and an invertible $A_N \in M_{N^2}(\mathbb{C})$ satisfying:*

- (i) $\|A_N D_{\underline{a}_N} A_N^{-1}\|_{B(l_{N^2}^2)} \leq 2$,
- (ii) $\|A_N D_{\underline{a}_N}^{-1} A_N^{-1}\|_{B(l_{N^2}^2)} \geq N$,
- (iii) $\|A_N D_{\underline{a}_N}\|_{B(l_{N^2}^2)} \leq \frac{1}{N^2}$,
- (iv) $\|A_N D_{\underline{a}_N}^{-1}\|_{B(l_{N^2}^2)} \leq \frac{1}{N^2}$, and
- (v) $\|D_{\underline{a}_N}^{-1}\|_{B(l_{N^2}^2)} \leq C_0$, for a fixed $C_0 < \infty$ independent of N .

Assume that the lemma holds. We proceed to construct our counterexample. Let $\mathcal{H}_N \stackrel{\text{def}}{=} \{\underline{x} \in \mathbb{C}^{N^2} : \text{with } \|\underline{x}\|_{\mathcal{H}_N} \stackrel{\text{def}}{=} \|A_N(\underline{x})\|_{l_{N^2}^2}^\infty\}$. For each $N = 1, 2, \dots$, let $E_N = \underbrace{\left\{ \frac{1}{N} e^{\frac{ijt}{2\pi N^2}} : j = 0, \dots, N^2 - 1 \right\}}_{z_{N,j}}$ and $E = \bigcup_{N=1}^\infty E_N \subset \mathbb{C}$. Note that any

pairwise disjoint E_N 's with sufficiently many points will work.

For $f : E \rightarrow \mathbb{C}$ and $N = 1, 2, \dots$, let $\underline{f}_N = (f(z_{N,0}), \dots, f(z_{N,N^2-1}))$. Then $\mathcal{H}_E \doteq \{f : E \rightarrow \mathbb{C} : \|f\|_{\mathcal{H}_E}^2 \stackrel{\text{def}}{=} \sum_{N=1}^\infty \|\underline{f}_N\|_{\mathcal{H}_N}^2 < \infty\}$. Since $\mathcal{H}_E$ is isomorphic to $\bigoplus_{N=1}^\infty \mathcal{H}_N$, we see that $\mathcal{H}_E$ is a Hilbert space of functions on E . Also, for $z_{N,j} \in E$,

$$\begin{aligned} |f(z_{N,j})| &\leq \|\underline{f}_N\|_{l_{N^2}^2} \leq \|A_N^{-1}\|_{B(l_{N^2}^2)} \|A_N(\underline{f}_N)\|_{l_{N^2}^2} \\ &= \|A_N^{-1}\|_{B(l_{N^2}^2)} \|\underline{f}_N\|_{\mathcal{H}_N} \\ &\leq \|A_N^{-1}\|_{B(l_{N^2}^2)} \|f\|_{\mathcal{H}_E}. \end{aligned}$$

Clearly, if $f : E \rightarrow \mathbb{C}$ vanishes on E , then $\|f\|_{\mathcal{H}_E} = 0$. Thus $\mathcal{H}_E$ is a reproducing kernel Hilbert space on E .

Suppose that $f \in \mathcal{M}(\mathcal{H}_E)$; then a computation shows that, for $h \in \mathcal{H}_E$,

$$\begin{aligned} \|M_f(h)\|_{\mathcal{H}_E}^2 &= \sum_{N=1}^{\infty} \|A_N((fh)_N)\|_{l_{N^2}^2}^2 \\ &= \sum_{N=1}^{\infty} \|A_N(D_{\underline{f}_N}(\underline{h})_N)\|_{l_{N^2}^2}^2 \\ &= \sum_{N=1}^{\infty} \|(A_N D_{\underline{f}_N} A_N^{-1})(A_N(\underline{h}_N))\|_{l_{N^2}^2}^2. \end{aligned}$$

From this equality it follows that

$$\|M_f\|_{B(\mathcal{H}_E)} = \sup_{N \in \mathbb{N}} \{\|A_N D_{\underline{f}_N} A_N^{-1}\|_{B(l_{N^2}^2)}\}.$$

Using the lemma to get the appropriate sequence of $\underline{a}_N$'s, we define $g : E \rightarrow \mathbb{C}$, so that $\underline{g}_N = \underline{a}_N$, for $N = 1, 2, \dots$. Then $g \in \mathcal{H}_E$, since by (iii) of the lemma

$$\begin{aligned} \|g\|_{\mathcal{H}_E}^2 &= \sum_{N=1}^{\infty} \|\underline{g}_N\|_{\mathcal{H}_N}^2 = \sum_{N=1}^{\infty} \|A_N(\underline{a}_N)\|_{l_{N^2}^2}^2 \\ &= \sum_{N=1}^{\infty} \|A_N D_{\underline{a}_N}(\underline{a}_N)\|_{l_{N^2}^2}^2 \leq \sum_{N=1}^{\infty} \frac{N^2}{N^4} < \infty, \end{aligned}$$

where $\underline{a}_N$ denotes the vector in $\mathbb{C}^{N^2}$ consisting of all 1's. A similar argument, using (iv), shows that $\frac{1}{g} \in \mathcal{H}_E$.

From (v), for all $N = 1, 2, \dots$ and $j = 0, \dots, N^2 - 1$, we have $|g(z_{N,j})| \geq \frac{1}{C_0}$. So $|g(x)| \geq \frac{1}{C_0}$, for all $x \in E$.

By our construction,

$$\|M_g\|_{B(\mathcal{H}_E)} = \sup_{N \in \mathbb{N}} \{\|A_N D_{\underline{g}_N} A_N^{-1}\|_{B(l_{N^2}^2)}\} \leq 2, \text{ by (i).}$$

By a similar estimate, using (ii) we get that

$$\|M_{\frac{1}{g}}\|_{B(\mathcal{H}_E)} = \sup_{N \in \mathbb{N}} \{\|A_N D_{\underline{g}_N}^{-1} A_N^{-1}\|_{B(l_{N^2}^2)}\} = \infty.$$

This completes the construction.

We finish with a proof of the lemma.

Proof. For $N = 1, 2, \dots$ construct the $N^2 \times N^2$ matrix

$$B_N = \begin{bmatrix} \frac{1}{1+\frac{1}{N}+\frac{1}{N^3}} & -1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1+\frac{1}{N}+\frac{2}{N^3}} & -1 & 0 & \cdots & 0 \\ 0 & 0 & -\frac{1}{1+\frac{1}{N}+\frac{3}{N^3}} & 1 & -1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{1+\frac{1}{N}+\frac{N^2}{N^3}} \end{bmatrix}.$$

Clearly, $\sigma(B_N) = \{\frac{1}{1+\frac{1}{N}+\frac{k}{N^3}} : 1 \leq k \leq N^2\} \subset D$, the open unit disk.

Since the spectral radius of B_N is strictly less than 1, there exists a $T_N \in M_{N^2}(\mathbb{C})$ so that

$$\|T_N^{-1}B_NT_N\|_{B(l_{N^2}^2)} \leq 1.$$

Now $\|B_N^{-1}\|_{B(l_{N^2}^2)} \geq \prod_{k=1}^{N^2} (1 + \frac{1}{N} + \frac{k}{N^3}) \geq N$ and $\|B_N\|_{B(l_{N^2}^2)} \leq 2$. Set $C_N = T_N^{-1}B_NT_N$. Then

$$\|C_N\|_{B(l_{N^2}^2)} \leq 1,$$

$$\|T_NC_NT_N^{-1}\|_{B(l_{N^2}^2)} \leq 2,$$

$$\text{and } \|T_NC_N^{-1}T_N^{-1}\|_{B(l_{N^2}^2)} \geq N.$$

Since $\sigma(C_N) = \sigma(B_N)$, $\sigma(C_N)$ consists of the N^2 distinct eigenvalues $\{\frac{1}{1+\frac{1}{N}+\frac{k}{N^3}} : 1 \leq k \leq N^2\}$. If we denote by D_N the $N^2 \times N^2$ matrix with the same diagonal as B_N , then C_N is similar to D_N , say $C_N = P_N D_N P_N^{-1}$. Set $A_N = \frac{T_N P_N}{N^2 \|T_N P_N\|_{B(l_{N^2}^2)} (\|D_N\|_{B(l_{N^2}^2)} + \|D_N^{-1}\|_{B(l_{N^2}^2)})}$. Then the lemma follows with $\underline{a}_N = (1 + \frac{1}{N} + \frac{1}{N^3}, 1 + \frac{1}{N} + \frac{2}{N^3}, \dots, 1 + \frac{1}{N} + \frac{N^2}{N^3}) \in \mathbb{C}^{N^2}$. $\square$

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF ALABAMA, BOX 870350, TUSCALOOSA, ALABAMA 35487-0350

E-mail address: ttrent@as.ua.edu