A DUALITY THEOREM FOR GENERALIZED LOCAL COHOMOLOGY

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(Communicated by Bernd Ulrich)

ABSTRACT. We prove a duality theorem for graded algebras over a field that implies several known duality results: graded local duality, versions of Serre duality for local cohomology and of Suzuki duality for generalized local cohomology, and Herzog-Rahimi bigraded duality.

1. INTRODUCTION

Several duality results for local cohomology (notably Grothendieck and Serre dualities) and for generalized local cohomology (Suzuki duality) are classical results commonly used in commutative algebra. A more recent duality theorem was established by Herzog and Rahimi in the context of bigraded algebras. We here present a duality result that encapsulates all these duality statements. Before stating this theorem, we need to set some notation.

Let Γ be an abelian group and R be a Γ -graded polynomial ring over a commutative Noetherian ring R_0 .

For a Γ -graded R -ideal I , we recall from [CCHS] that the grading of R is I -sharp if $H_I^i(R)_\gamma$ is a finitely generated R_0 -module for any $\gamma \in \Gamma$ and any i . If this is the case, then $H_I^i(M)_\gamma$ is also a finitely generated R_0 -module for any finitely generated graded R -module M , any $\gamma \in \Gamma$ and any i .

In the case where $S = S_0[x_1, \dots, x_m, y_1, \dots, y_n]$ is a graded quotient of $R = R_0[X_1, \dots, X_m, Y_1, \dots, Y_n]$, $\mathfrak{p} := (x_1, \dots, x_m)$ and $\mathfrak{q} := (y_1, \dots, y_n)$, we define the grading of S to be $\mathfrak{p}$ -sharp (equivalently $\mathfrak{q}$ -sharp) if the following condition is satisfied: for all $\gamma \in \Gamma$,

$$|\{(\alpha, \beta) \in \mathbf{Z}_{\geq 0}^m \times \mathbf{Z}_{\geq 0}^n : \sum_i \alpha_i \deg(x_i) = \gamma + \sum_j \beta_j \deg(y_j)\}| < \infty.$$

This condition on the degrees of the variables is equivalent to the fact that the grading of R is $(X_1, \dots, X_m)$ -sharp (equivalently $(Y_1, \dots, Y_n)$ -sharp) by [CCHS, 1.2].

We set ${}^*\mathrm{Hom}_R(M, R_0)$ for the graded R_0 -dual of a graded R -module M . This module is graded and ${}^*\mathrm{Hom}_R(M, R_0)_\gamma = \mathrm{Hom}_{R_0}(M_{-\gamma}, R_0)$.

The notion of generalized local cohomology was given by Herzog in his Habilitationsschrift [H]. For an ideal I in a commutative Noetherian ring R , and M

Received by the editors May 21, 2007 and, in revised form, September 13, 2007.

2000 *Mathematics Subject Classification*. Primary 13D45, 14B15, 13D07, 13C14.

Key words and phrases. Generalized local cohomology, duality.

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and N two R -modules, the i -th generalized local cohomology module of M and N with respect to I is denoted by $H_I^i(M, N)$. One has a natural isomorphism $H_I^i(R, N) \simeq H_I^i(N)$ and a spectral sequence $H_I^i(\text{Ext}_R^j(M, N)) \Rightarrow H_I^{i+j}(M, N)$.

Our main result is the following duality result.

Theorem 1.1. *Let $S = k[x_1, \dots, x_m, y_1, \dots, y_n]$ be a Cohen-Macaulay Γ -graded algebra over a field k . Let $\mathfrak{p} := (x_1, \dots, x_m)$ and $\mathfrak{q} := (y_1, \dots, y_n)$. Assume that the grading of S is $\mathfrak{p}$ -sharp (equivalently $\mathfrak{q}$ -sharp), and M and N are finitely generated graded S -modules such that either M has finite projective dimension or N has finite projective dimension and $\text{Tor}_i^S(M, \omega_S) = 0$ for $i > 0$; then*

$$H_{\mathfrak{p}}^i(M, N) \simeq {}^* \text{Hom}_S(H_{\mathfrak{q}}^{\dim S - i}(N, M \otimes_S \omega_S), k).$$

Several previously known duality results are particular cases of this theorem. Namely,

- (i) graded local duality in the case that the base ring is a field corresponds to the case where $\mathfrak{q} = 0$ and $M = S$,
- (ii) Serre duality follows from the case where $\mathfrak{p} = 0$ and $M = S$, via the spectral sequence $H_{\mathfrak{q}}^i(\text{Ext}_S^j(N, \omega_S)) \Rightarrow H_{\mathfrak{q}}^{i+j}(N, \omega_S)$,
- (iii) Suzuki duality in the context of graded algebras over a field (see [CD, 3.1]) corresponds to the case where $\mathfrak{q} = 0$,
- (iv) the Herzog-Rahimi spectral sequence corresponds to the case where $M = S$ (see [HR] for the standard bigraded case, and [CCHS] for the general case) using the spectral sequence $H_{\mathfrak{q}}^i(\text{Ext}_S^j(N, \omega_S)) \Rightarrow H_{\mathfrak{q}}^{i+j}(N, \omega_S)$.

2. THE DUALITY THEOREM

If $F_{\bullet}$ and $G_{\bullet}$ are two complexes of R -modules, $\text{Homgr}_R(F_{\bullet}, G_{\bullet})$ is the cohomological complex with modules $C^i = \prod_{p+q=i} \text{Hom}_R(F_p, G_q)$. If either $F_{\bullet}$ or $G_{\bullet}$ is finite, then $\text{Homgr}_R(F_{\bullet}, G_{\bullet})$ is the totalisation of the double complex with $C^{p,-q} = \text{Hom}_R(F_p, G_q)$.

We first consider the case where $R := k[X_1, \dots, X_m, Y_1, \dots, Y_n]$ is a polynomial ring over a field k , $\mathfrak{p} := (X_1, \dots, X_m)$ and $\mathfrak{q} := (Y_1, \dots, Y_n)$.

From this point on, we will assume that R is Γ -graded, for an abelian group Γ , and that the grading of R is $\mathfrak{p}$ -sharp (equivalently $\mathfrak{q}$ -sharp). One has $\omega_R = R[-\sigma]$, where $\sigma := \sum_i \deg(X_i) + \sum_j \deg(Y_j)$.

Let $\mathcal{C}_{\mathfrak{p}}^{\bullet}(M)$ be the Čech complex $0 \rightarrow M \rightarrow \bigoplus_i M_{X_i} \rightarrow \dots \rightarrow M_{X_1 \dots X_m} \rightarrow 0$ and $\mathcal{C}_{\mathfrak{q}}^{\bullet}(M)$ be the Čech complex $0 \rightarrow M \rightarrow \bigoplus_i M_{Y_i} \rightarrow \dots \rightarrow M_{Y_1 \dots Y_n} \rightarrow 0$. Denote by $C^{\bullet}\{i\}$ the complex $C^{\bullet}$ shifted in cohomological degree by i and set $-^* := {}^* \text{Hom}_R(-, k)$. Recall that $-^*$ is exact on the category of graded R -modules. By [CCHS, 1.4 (b)] one has

Lemma 2.1. *If $F_{\bullet}$ is a graded complex of finite free R -modules, then*

- (i) $H_{\mathfrak{p}}^i(F_{\bullet}) = 0$ for $i \neq m$, $H_{\mathfrak{q}}^j(F_{\bullet}) = 0$ for $j \neq n$,
- (ii) *there is a natural graded map of complexes $d : \mathcal{C}_{\mathfrak{p}}^{\bullet}(R)\{m\} \rightarrow \mathcal{C}_{\mathfrak{q}}^{\bullet}(\omega_R)\{n\}^*$ that induces a map of double complexes $\delta : \mathcal{C}_{\mathfrak{p}}^{\bullet}(F_{\bullet})\{m\} \rightarrow \mathcal{C}_{\mathfrak{q}}^{\bullet}(\text{Hom}_R(F_{\bullet}, \omega_R))\{n\}^*$ which gives rise to a functorial isomorphism*

$$H_{\mathfrak{p}}^m(F_{\bullet}) \simeq H_{\mathfrak{q}}^n(\text{Hom}_R(F_{\bullet}, \omega_R))^*.$$

Proof. We remark that the map defined in [CCHS, 1.4 (b)] is induced by the map from $\mathcal{C}_{\mathfrak{p}}^m(R)$ to $\mathcal{C}_{\mathfrak{q}}^n(\omega_R)^*$ defined by $d(X^{-s-1}Y^p)(X^tY^{-q-1}) = 1$ if $s = t$, and $p = q$

and 0 otherwise (here p, q, r and s are tuples of integers, with $p \geq 0$ and $t \geq 0$). Notice that $\mathcal{C}_p^\bullet(R)\{m\}$ is zero in positive cohomological degrees and $\mathcal{C}_q^\bullet(\omega_R)\{n\}^*$ is zero in negative cohomological degrees. Hence this map extends uniquely to a map of complexes. $\square$

Lemma 2.2. *Let M, N and N' be finitely generated graded R -modules and $f : N \rightarrow N'$ be a graded homomorphism. For any integer i , there exist functorial isomorphisms ψ_N^i and $\psi_{N'}^i$ that give rise to the following commutative diagram where the vertical maps are the natural ones:*

$$\begin{array}{ccc}
 H_p^i(M, N) & \xrightarrow{\psi_N^i} & H_q^{m+n-i}(N, M \otimes_R \omega_R)^* \\
 \downarrow & & \downarrow \\
 H_p^i(M, N') & \xrightarrow{\psi_{N'}^i} & H_q^{m+n-i}(N', M \otimes_R \omega_R)^*
 \end{array}$$

$H^{i-m}(H_p^m(\text{Homgr}_R(F_\bullet^M, F_\bullet^N))) \xrightarrow{\cong} H^{i-m}(H_q^n(\text{Homgr}_R(F_\bullet^N, F_\bullet^M \otimes_R \omega_R))^*)$
 $\downarrow$
 $H^{i-m}(H_p^m(\text{Homgr}_R(F_\bullet^M, F_\bullet^{N'}))) \xrightarrow{\cong} H^{i-m}(H_q^n(\text{Homgr}_R(F_\bullet^{N'}, F_\bullet^M \otimes_R \omega_R))^*)$

Proof. Let $F_\bullet^M$ be a minimal graded free R -resolution of M and $f_\bullet : F_\bullet^N \rightarrow F_\bullet^{N'}$ be a lifting of f to minimal graded free R -resolutions of N and N' .

Recall that there is a spectral sequence

$$H^i(H_p^j(\text{Homgr}_R(F_\bullet^M, F_\bullet^N))) \Rightarrow H_p^{i+j}(M, N),$$

by [CD, 2.12] and [B, §6, Théorème 1, b]. As $H_p^i(F_\bullet) = 0$ for $i \neq m$ if $F_\bullet$ is a complex of free R -modules, there is a natural commutative diagram

$$\begin{array}{ccc}
 H_p^i(M, N) & \xrightarrow{\cong} & H^{i-m}(H_p^m(\text{Homgr}_R(F_\bullet^M, F_\bullet^N))) \\
 \downarrow \text{can} & & \downarrow \text{can} \\
 H_p^i(M, N') & \xrightarrow{\cong} & H^{i-m}(H_p^m(\text{Homgr}_R(F_\bullet^M, F_\bullet^{N'}))).
 \end{array}$$

As $F_\bullet^M$ is bounded, Lemma 2.1 provides the functorial isomorphisms

$$\begin{aligned}
 H_p^m(\text{Homgr}_R(F_\bullet^M, F_\bullet^N)) &\simeq H_q^n(\text{Hom}_R(\text{Homgr}_R(F_\bullet^M, F_\bullet^N), \omega_R))^* \\
 &\simeq H_q^n(\text{Homgr}_R(F_\bullet^N, F_\bullet^M \otimes_R \omega_R))^*
 \end{aligned}$$

and equivalent ones for N' . This provides a commutative diagram

$$\begin{array}{ccc}
 H_p^i(M, N) & \xrightarrow{\cong} & H^{i-m}(H_q^n(\text{Homgr}_R(F_\bullet^N, F_\bullet^M \otimes_R \omega_R))^*) \\
 \downarrow \text{can} & & \downarrow \text{can} \\
 H_p^i(M, N') & \xrightarrow{\cong} & H^{i-m}(H_q^n(\text{Homgr}_R(F_\bullet^{N'}, F_\bullet^M \otimes_R \omega_R))^*).
 \end{array}$$

As k is a field, we have a natural isomorphism

$$H^{i-m}(H_q^n(\text{Homgr}_R(F_\bullet^N, F_\bullet^M \otimes_R \omega_R))^*) \simeq H^{m-i}(H_q^n(\text{Homgr}_R(F_\bullet^N, F_\bullet^M \otimes_R \omega_R))^*)$$

for N and a similar one for N' . As $F_{\bullet}^M \otimes_R \omega_R$ is acyclic, it provides a free R -resolution of $M \otimes_R \omega_R$, and we get a commutative diagram

$$\begin{array}{ccc} H^{m-i}(H_q^n(\text{Homgr}_R(F_{\bullet}^N, F_{\bullet}^M \otimes_R \omega_R))) & \xrightarrow{\simeq} & H_q^{m+n-i}(N, M \otimes_R \omega_R) \\ \uparrow \text{can} & & \uparrow \text{can} \\ H^{m-i}(H_q^n(\text{Homgr}_R(F_{\bullet}^{N'}, F_{\bullet}^M \otimes_R \omega_R))) & \xrightarrow{\simeq} & H_q^{m+n-i}(N', M \otimes_R \omega_R) \end{array}$$

that follows from the same argument as in the first part of this proof, applied to $\mathfrak{q}, N, M \otimes_R \omega_R$ and $\mathfrak{q}, N', M \otimes_R \omega_R$ in place of $\mathfrak{p}, M, N$ and $\mathfrak{p}, M, N'$. This finally gives the claimed diagram of natural maps. $\square$

Remark 2.3. Lemma 2.2 holds more generally if we replace k by a Gorenstein ring of dimension 0, and assume that either M or N has finite projective dimension.

Corollary 2.4. *Assume S is a Cohen-Macaulay ring and a finitely generated graded R -module. Let $F_{\bullet}$ be a graded complex of finite free S -modules, and consider a graded free R -resolution $F_{\bullet\bullet}$ of $F_{\bullet}$. Then the map given by Lemma 2.1,*

$$\delta : \mathcal{C}_{\mathfrak{p}}^{\bullet}(F_{\bullet\bullet})\{m\} \rightarrow \mathcal{C}_{\mathfrak{q}}^{\bullet}(\text{Hom}_R(F_{\bullet\bullet}, \omega_R))\{n\}^*,$$

induces a graded functorial map of double complexes

$$\mathcal{C}_{\mathfrak{p}}^{\bullet}(F_{\bullet})\{\dim S\} \rightarrow \mathcal{C}_{\mathfrak{q}}^{\bullet}(\text{Hom}_S(F_{\bullet}, \omega_S))^*$$

which gives rise to isomorphisms

$$H_{\mathfrak{p}}^i(F_{\bullet}) \simeq {}^* \text{Hom}_S(H_q^{\dim S-i}(\text{Hom}_S(F_{\bullet}, \omega_S)), k).$$

Proof. First notice that S is a Cohen-Macaulay R -module and a graded k -algebra. By Lemma 2.2, the map δ induces commutative diagrams

$$\begin{array}{ccc} H_{\mathfrak{p}}^i(R, F_{j-1}) & \xrightarrow{\simeq} & H_q^{m+n-i}(F_{j-1}, \omega_R)^* \\ \downarrow & & \downarrow \\ H_{\mathfrak{p}}^i(R, F_j) & \xrightarrow{\simeq} & H_q^{m+n-i}(F_j, \omega_R)^* \\ \downarrow & & \downarrow \\ H_{\mathfrak{p}}^i(R, F_{j+1}) & \xrightarrow{\simeq} & H_q^{m+n-i}(F_{j+1}, \omega_R)^* \end{array}$$

This gives the claim together with the isomorphisms $H_{\mathfrak{p}}^i(R, F_{\bullet}) \simeq H_{\mathfrak{p}}^i(F_{\bullet})$ and $H_q^{m+n-i}(F_{\bullet}, \omega_R) \simeq H_q^{\dim S-i}(\text{Hom}_S(F_{\bullet}, \omega_S))$, where the second one follows from the Cohen-Macaulayness of S via the collapsing spectral sequence $H_q^i(\text{Ext}_R^j(F_{\bullet}, \omega_R)) \Rightarrow H_q^{i+j}(F_{\bullet}, \omega_R)$. $\square$

Lemma 2.5. *Assume S is a Cohen-Macaulay ring and a finitely generated graded R -module. Let M and N be finitely generated graded S -modules such that N has finite projective dimension and $\text{Tor}_i^S(M, \omega_S) = 0$ for $i > 0$. Then the map $f \mapsto f \otimes 1_{\omega_S}$ induces natural isomorphisms*

$$\text{Ext}_S^i(M, N) \simeq \text{Ext}_S^i(M \otimes_S \omega_S, N \otimes_S \omega_S).$$

It follows that for any S -ideal I , $H_I^i(M, N) \simeq H_I^i(M \otimes_S \omega_S, N \otimes_S \omega_S)$ for all i .

Proof. The case where $M = N = S$ is a classical property of ω_S . If $M = S$ the result follows by induction on the projective dimension of N , using the “five lemma” and the exact sequences of Ext modules derived from the exact sequences $E : 0 \rightarrow \text{Syz}_1^S(N) \rightarrow F \rightarrow N \rightarrow 0$ and $E \otimes_S \omega_S$. Notice $E \otimes_S \omega_S$ is exact since $\text{Tor}_i^S(P, \omega_S) = 0$ for $i > 0$ if P has finite projective dimension (see [S, 3.17, 4.5]). Hence:

$$(*) \quad \text{Hom}_S(\omega_S, N \otimes_S \omega_S) \simeq N \quad \text{and} \quad \text{Ext}_S^i(\omega_S, N \otimes_S \omega_S) = 0, \quad \forall i > 0.$$

For the general case, let $F_\bullet^M$ be a free S -resolution of M . By (*), the map $f \mapsto f \otimes 1_{\omega_S}$ induces an isomorphism $\text{Hom}_S(F_\bullet^M \otimes_S \omega_S, N \otimes_S \omega_S) \simeq \text{Hom}_S(F_\bullet^M, N)$. Furthermore, as $\text{Ext}_S^i(F_\bullet^M \otimes_S \omega_S, N \otimes_S \omega_S) = 0, \forall i > 0$ and $F_\bullet^M \otimes_S \omega_S$ is acyclic, it follows that $H^i(\text{Hom}_S(F_\bullet^M \otimes_S \omega_S, N \otimes_S \omega_S)) \simeq \text{Ext}_S^i(M \otimes_S \omega_S, N \otimes_S \omega_S)$. This shows the first isomorphism.

It also follows that the map from

$$\mathcal{C}_I^\bullet(\text{Homgr}_S(F_\bullet^M, N)) \quad \text{to} \quad \mathcal{C}_I^\bullet(\text{Homgr}_S(F_\bullet^M \otimes_S \omega_S, N \otimes_S \omega_S)),$$

induced by tensoring by 1_{ω_S} , gives a morphism of spectral sequences which is an isomorphism between the first terms, which are respectively $\mathcal{C}_I^p(\text{Ext}_S^q(M, N))$ and $\mathcal{C}_I^p(\text{Ext}_S^q(M \otimes_S \omega_S, N \otimes_S \omega_S))$. Hence the total homology of these double complexes is isomorphic. This proves our claim by [CD, 2.12]. $\square$

We can now prove the main result of this article.

Proof of Theorem 1.1. Since the grading of S is $\mathfrak{p}$ -sharp, it follows that $H_{\mathfrak{p}}^i(M, N) \simeq H_{\mathfrak{p}}^i(M, N)^{**}$, for all i . Hence by Lemma 2.5 it suffices to treat the case where M has finite projective dimension. Let $F_\bullet^M$ and $F_\bullet^N$ be a minimal graded free S -resolutions of M and N , respectively. Set $F_\bullet := \text{Homgr}_S(F_\bullet^M, F_\bullet^N)$.

By Corollary 2.4, if $F_{\bullet\bullet}$ is a graded free R -resolution of $F_\bullet$, the map

$$\delta : \mathcal{C}_{\mathfrak{p}}^\bullet(F_{\bullet\bullet})\{m\} \rightarrow \mathcal{C}_{\mathfrak{q}}^\bullet(\text{Hom}_R(F_{\bullet\bullet}, \omega_R))\{n\}^*$$

given by Lemma 2.1 induces a graded morphism of double complexes

$$\mathcal{C}_{\mathfrak{p}}^\bullet(\text{Homgr}_S(F_\bullet^M, F_\bullet^N))\{\dim S\} \rightarrow \mathcal{C}_{\mathfrak{q}}^\bullet(\text{Homgr}_S(F_\bullet^N, F_\bullet^M \otimes_S \omega_S))^*$$

that is an isomorphism on the E_1 terms:

$$H_{\mathfrak{p}}^i(\text{Homgr}_S(F_\bullet^M, F_\bullet^N)) \simeq H_{\mathfrak{q}}^{\dim S - i}(\text{Homgr}_S(F_\bullet^N, F_\bullet^M \otimes_S \omega_S))^*.$$

It follows that these two spectral sequences have isomorphic terms at any level, in particular the same abutment. The first one abuts to $H_{\mathfrak{p}}^i(M, N)$ by [CD, 2.12]. The second one abuts to $H_{\mathfrak{q}}^{\dim S - i}(N, M \otimes_S \omega_S)^*$ by [B, §6, Théorème 1, b] and [CD, 2.12], as $\text{Tor}_i^S(M, \omega_S) = 0$ for $i > 0$ (because M has finite projective dimension). $\square$

ACKNOWLEDGMENTS

We thank the referee for indicating a simplification in the proof of Lemma 2.5.

This research took place when the second named author was on sabbatical leave. He would like to express his deep thanks to the Institut de Mathématiques de Jussieu for its kind hospitality.

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