A SEPARABLE DEFORMATION OF THE QUATERNION GROUP ALGEBRA

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ABSTRACT. The Donald-Flanigan conjecture asserts that for any finite group G and any field k , the group algebra kG can be deformed to a separable algebra. The minimal unsolved instance, namely the quaternion group Q_8 over a field k of characteristic 2 was considered as a counterexample. We present here a separable deformation of kQ_8 . In a sense, the conjecture for any finite group is open again.

1. INTRODUCTION

In their paper [1], J. D. Donald and F. J. Flanigan conjectured that any group algebra kG of a finite group G over a field k can be deformed to a semisimple algebra even in the modular case, namely where the order of G is not invertible in k . A more customary formulation of the Donald-Flanigan (DF) conjecture is by demanding that the deformed algebra $[kG]_t$ should be separable; i.e., it remains semisimple when tensored with the algebraic closure of its base field. If, additionally, the dimensions of the simple components of $[kG]_t$ are in one-to-one correspondence with those of the complex group algebra $\mathbb{C}G$, then $[kG]_t$ is called a *strong* solution to the problem.

The DF conjecture was solved for different families of groups, using different methods:

- when the group G has a cyclic p -Sylow subgroup and k is an algebraically closed field of characteristic p [12];
- when G has a normal abelian p -Sylow subgroup in characteristic p [6];
- dihedral groups [3] and semi-dihedral groups [2] in characteristic 2;
- reflection groups in any characteristic (with six exceptions) [7, 8, 11].

In [5], it is claimed that the group algebra kQ_8 , where

$$Q_8 = \langle \sigma, \tau \mid \sigma^4 = 1, \tau\sigma = \sigma^3\tau, \sigma^2 = \tau^2 \rangle$$

is the quaternion group of order 8 and k is a field of characteristic 2, does not admit a separable deformation. This result allegedly gives a counterexample to the DF conjecture. However, as observed by M. Schaps, the proof apparently contains an error (see §7).

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The aim of this paper is to present a separable deformation of kQ_8 , where k is any field of characteristic 2, reopening the DF conjecture. The same approach works for the generalized quaternion group algebras and will be introduced separately.

2. PRELIMINARIES

Let $k[[t]]$ be the ring of formal power series over k , and let $k((t))$ be its field of fractions. Recall that the deformed algebra $[kG]_t$ has the same underlying $k((t))$ -vector space as $k((t)) \otimes_k kG$, with multiplication defined on basis elements

$$(2.1) \quad g_1 * g_2 := g_1 g_2 + \sum_{i \geq 1} \Psi_i(g_1, g_2) t^i, \quad g_1, g_2 \in G$$

and extended $k((t))$ -linearly (such that t is central). Here $g_1 g_2$ is the group multiplication. The functions $\Psi_i : G \times G \rightarrow kG$ satisfy certain cohomological conditions induced by the associativity of $[kG]_t$ [4, §§1, 2].

Note that the set of equations (2.1) determines a multiplication on the free $k[[t]]$ -module Λ_t spanned by the elements $\{g\}_{g \in G}$ such that $kG \simeq \Lambda_t / \langle t\Lambda_t \rangle$ and $[kG]_t \simeq \Lambda_t \otimes_{k[[t]]} k((t))$. In a more general context, namely over a domain R which is not necessarily local, the R -module Λ_t which determines the deformation, is required only to be *flat* rather than free [3, §1].

In what follows, we shall define the deformed algebra $[kG]_t$ by using generators and relations. These will implicitly determine the set of equations (2.1).

3. SKETCH OF THE CONSTRUCTION

Consider the extension

$$(3.1) \quad [\beta] : 1 \rightarrow C_4 \rightarrow Q_8 \rightarrow C_2 \rightarrow 1,$$

where $C_2 = \langle \bar{\tau} \rangle$ acts on $C_4 = \langle \sigma \rangle$ by

$$\begin{aligned} \eta : C_2 &\rightarrow \text{Aut}(C_4), \\ \eta(\bar{\tau}) : \sigma &\mapsto \sigma^3 (= \sigma^{-1}), \end{aligned}$$

and the associated 2-cocycle $\beta : C_2 \times C_2 \rightarrow C_4$ is given by

$$\beta(1, 1) = \beta(1, \bar{\tau}) = \beta(\bar{\tau}, 1) = 1, \quad \beta(\bar{\tau}, \bar{\tau}) = \sigma^2.$$

The group algebra kQ_8 (k any field) is isomorphic to the quotient $kC_4[y; \eta] / \langle q(y) \rangle$, where $kC_4[y; \eta]$ is a skew polynomial ring [10, §1.2], whose indeterminate y acts on the ring of coefficients kC_4 via the automorphism $\eta(\bar{\tau})$ (extended linearly) and where

$$(3.2) \quad q(y) := y^2 - \sigma^2 \in kC_4[y; \eta]$$

is central. The above isomorphism is established by identifying τ with the indeterminate y .

Suppose now that $\text{Char}(k) = 2$. The deformed algebra $[kQ_8]_t$ is constructed as follows.

In §4.1 the subgroup algebra kC_4 is deformed to a separable algebra $[kC_4]_t$ which is isomorphic to $K \oplus k((t)) \oplus k((t))$, where K is a separable field extension of $k((t))$ of degree 2.

The next step (§4.2) is to construct an automorphism η_t of $[kC_4]_t$ which agrees with the action of C_2 on kC_4 when specializing $t = 0$. This action fixes all three

primitive idempotents of $[kC_4]_t$. In that way, we obtain the skew polynomial ring $[kC_4]_t[y; \eta_t]$.

In §5 we deform $q(y) = y^2 + \sigma^2$ to $q_t(y)$, a separable polynomial of degree 2 in the center of $[kC_4]_t[y; \eta_t]$.

By factoring out the two-sided ideal generated by $q_t(y)$, we establish the deformation

$$[kQ_8]_t := [kC_4]_t[y; \eta_t] / \langle q_t(y) \rangle.$$

In §6 we show that $[kQ_8]_t$ as above is separable. Moreover, passing to the algebraic closure $\overline{k((t))}$ we have

$$[kQ_8]_t \otimes_{k((t))} \overline{k((t))} \simeq \bigoplus_{i=1}^4 \overline{k((t))} \oplus M_2(\overline{k((t))}).$$

This is a strong solution to the DF conjecture since its decomposition to simple components is the same as

$$\mathbb{C}Q_8 \simeq \bigoplus_{i=1}^4 \mathbb{C} \oplus M_2(\mathbb{C}).$$

4. A DEFORMATION OF $kC_4[y; \eta]$

4.1. We begin by constructing $[kC_4]_t$, $C_4 = \langle \sigma \rangle$. Recall that

$$kC_4 \simeq k[x] / \langle x^4 + 1 \rangle$$

by identifying σ with $x + \langle x^4 + 1 \rangle$. We deform the polynomial $x^4 + 1$ to a separable polynomial $p_t(x)$ as follows.

Let $k[[t]]^*$ be the group of invertible elements of $k[[t]]$ and denote by

$$U := \{1 + zt \mid z \in k[[t]]^*\}$$

its subgroup of 1-units (when $k = \mathbb{F}_2$, U is equal to $k[[t]]^*$).

Let

$$a \in k[[t]] \setminus k[[t]]^*$$

be a non-zero element, and let

$$b, c, d \in U (c \neq d),$$

such that

$$\pi(x) := x^2 + ax + b$$

is an irreducible (separable) polynomial in $k((t))[x]$. Let

$$p_t(x) := \pi(x)(x + c)(x + d) \in k((t))[x].$$

Then the quotient $k((t))[x] / \langle p_t(x) \rangle$ is isomorphic to the direct sum $K \oplus k((t)) \oplus k((t))$, where $K := k((t))[x] / \langle \pi(x) \rangle$. The field extension $K/k((t))$ is separable and of dimension 2.

Note that $p_{t=0}(x) = x^4 + 1$ and that only lower order terms of the polynomial were deformed. Hence, the quotient $k[[t]][x] / \langle p_t(x) \rangle$ is $k[[t]]$ -free and $k((t))[x] / \langle p_t(x) \rangle$ indeed defines a deformation $[kC_4]_t$ of $kC_4 \simeq k[x] / \langle x^4 + 1 \rangle$. The new multiplication $\sigma^i * \sigma^j$ of basis elements (2.1) is determined by identifying σ^i with $\bar{x}^i := x^i + \langle p_t(x) \rangle$. We shall continue to use the term $\bar{x}$ in $[kC_4]_t$ rather than σ .

Assume further that there exists $w \in k[[t]]$ such that

$$(4.1) \quad (x + w)(x + c)(x + d) = x\pi(x) + a$$

(see Example 4.3). Then $K \simeq ([kC_4]_t)e_1$, where

$$(4.2) \quad e_1 = \frac{(\bar{x} + w)(\bar{x} + c)(\bar{x} + d)}{a}.$$

The other two primitive idempotents of $[kC_4]_t$ are

$$(4.3) \quad e_2 = \frac{c(\bar{x} + d)\pi(\bar{x})}{a(c + d)}, \quad e_3 = \frac{d(\bar{x} + c)\pi(\bar{x})}{a(c + d)}.$$

4.2. Let

$$\eta_t : k((t))[x] \rightarrow k((t))[x]$$

be an algebra endomorphism determined by its value on the generator x as follows:

$$(4.4) \quad \eta_t(x) := x\pi(x) + x + a.$$

We compute $\eta_t(\pi(x))$, $\eta_t(x + c)$ and $\eta_t(x + d)$:

$$\begin{aligned} \eta_t(\pi(x)) &= \eta_t(x)^2 + a\eta_t(x) + b = x^2\pi(x)^2 + x^2 + a^2 + ax\pi(x) + ax + a^2 + b \\ &= \pi(x)(x^2\pi(x) + ax + 1). \end{aligned}$$

By (4.1),

$$(4.5) \quad \eta_t(\pi(x)) = \pi(x) + x(x + w)p_t(x) \in \langle \pi(x) \rangle.$$

Next,

$$\eta_t(x + c) = x\pi(x) + x + a + c.$$

By (4.1),

$$(4.6) \quad \eta_t(x + c) = (x + c)[(x + w)(x + d) + 1] \in \langle x + c \rangle.$$

Similarly,

$$(4.7) \quad \eta_t(x + d) = (x + d)[(x + w)(x + c) + 1] \in \langle x + d \rangle.$$

By (4.5), (4.6) and (4.7), we obtain that $\eta_t(p_t(x)) \in \langle p_t(x) \rangle$, and hence η_t induces an endomorphism of $k((t))[x]/\langle p_t(x) \rangle$, which we continue to denote by η_t . As can easily be verified, the primitive idempotents given in (4.2) and (4.3) are fixed under η_t :

$$(4.8) \quad \eta_t(e_i) = e_i, \quad i = 1, 2, 3,$$

whereas

$$(4.9) \quad \eta_t(\bar{x}e_1) = \eta_t(\bar{x})e_1 = (\bar{x}\pi(\bar{x}) + \bar{x} + a)e_1 = (\bar{x} + a)e_1.$$

Hence, η_t induces an automorphism of K of order 2 while fixing the two copies of $k((t))$ pointwise. Furthermore, one can easily verify that

$$\eta_{t=0}(\bar{x}) = \bar{x}^3.$$

Consequently, the automorphism η_t of $[kC_4]_t$ agrees with the automorphism $\eta(\bar{\tau})$ of kC_4 when $t = 0$. The skew polynomial ring

$$[kC_4]_t[y; \eta_t] = (k((t))[x]/\langle p_t(x) \rangle)[y; \eta_t]$$

is therefore a deformation of $kC_4[y; \eta]$.

Note that by (4.8), the idempotents $e_i, i = 1, 2, 3$ are central in $[kC_4]_t[y; \eta_t]$ and hence

$$(4.10) \quad [kC_4]_t[y; \eta_t] = \bigoplus_{i=1}^3 [kC_4]_t[y; \eta_t]e_i.$$

4.3. **Example.** The following is an example for the above construction.

Put

$$a := \frac{t+t^2+t^3}{1+t}, \quad b := 1+t^2+t^3, \quad c := \frac{1}{1+t}, \quad d := 1+t+t^2, \quad w := t.$$

These elements satisfy equation (4.1):

$$\begin{aligned} (x+w)(x+c)(x+d) &= (x+t)\left(x+\frac{1}{1+t}\right)(x+1+t+t^2) \\ &= x^3 + \frac{t+t^2+t^3}{1+t}x^2 + (1+t^2+t^3)x + \frac{t+t^2+t^3}{1+t} = x\pi(x) + a. \end{aligned}$$

The polynomial

$$\pi(x) = x^2 + \frac{t+t^2+t^3}{1+t}x + 1+t^2+t^3$$

does not admit roots in $k[[t]]/\langle t^2 \rangle$; thus it is irreducible over $k((t))$.

5. A DEFORMATION OF $q(y)$

The construction of $[kQ_8]_t$ will be completed once the product $\bar{\tau} * \bar{\tau}$ is defined. For this purpose the polynomial $q(y)$ (3.2), which determined the ordinary multiplication τ^2 , will now be developed in powers of t .

For any non-zero element $z \in k[[t]] \setminus k[[t]]^*$, let

$$(5.1) \quad q_t(y) := y^2 + z\bar{x}\pi(\bar{x})y + \bar{x}^2 + a\bar{x} \in [kC_4]_t[y; \eta_t].$$

The decomposition of (5.1) with respect to the idempotents e_1, e_2, e_3 yields

$$(5.2) \quad q_t(y) = (y^2 + b)e_1 + [y^2 + zay + c(c+a)]e_2 + [y^2 + zay + d(d+a)]e_3.$$

We now show that $q_t(y)$ is in the center of $[kC_4]_t[y; \eta_t]$.

First, the leading term y^2 is central since the automorphism η_t is of order 2. Next, by (4.8), the free term $be_1 + c(c+a)e_2 + d(d+a)e_3$ is invariant under the action of η_t and hence central. It is left to check that the term $za(e_2 + e_3)y$ is central. Indeed, since e_2 and e_3 are η_t -invariant, then $za(e_2 + e_3)y$ commutes both with $[kC_4]_t[y; \eta_t]e_2$ and $[kC_4]_t[y; \eta_t]e_3$. Furthermore, by orthogonality,

$$za(e_2 + e_3)y \cdot [kC_4]_t[y; \eta_t]e_1 = [kC_4]_t[y; \eta_t]e_1 \cdot za(e_2 + e_3)y = 0,$$

and hence $za(e_2 + e_3)y$ commutes with $[kC_4]_t[y; \eta_t]$.

Consequently, $\langle q_t(y) \rangle = q_t(y)[kC_4]_t[y; \eta_t]$ is a two-sided ideal.

Now, as can easily be deduced from (5.1),

$$(5.3) \quad q_{t=0}(y) = y^2 + \bar{x}^2 = q(y),$$

where the leading term y^2 remains unchanged. Then

$$[kQ_8]_t := [kC_4]_t[y; \eta_t] / \langle q_t(y) \rangle$$

is a deformation of kQ_8 , identifying $\bar{\tau}$ with $\bar{y} := y + \langle q_t(y) \rangle$.

6. SEPARABILITY OF $[kQ_8]_t$

Finally, we need to prove that the deformed algebra $[kQ_8]_t$ is separable. Moreover, we prove that its decomposition to simple components over the algebraic closure of $k((t))$ resembles that of $\mathbb{C}Q_8$. By (4.10), we obtain

$$(6.1) \quad [kQ_8]_t = \bigoplus_{i=1}^3 [kC_4]_t[y; \eta_t]e_i / \langle q_t(y)e_i \rangle.$$

We handle the three summands in (6.1) separately.

By (5.2),

$$[kC_4]_t[y; \eta_t]e_1 / \langle q_t(y)e_1 \rangle \simeq K[y; \eta_t] / \langle y^2 + b \rangle \simeq K *^f C_2.$$

The rightmost term is the *crossed product* of the group $C_2 := \langle \bar{\tau} \rangle$ acting faithfully on the field $K = [kC_4]_t e_1$ via η_t (4.9), with a twisting determined by the 2-cocycle $f : C_2 \times C_2 \rightarrow K^*$:

$$f(1, 1) = f(1, \bar{\tau}) = f(\bar{\tau}, 1) = 1, \quad f(\bar{\tau}, \bar{\tau}) = b.$$

This is a central simple algebra over the subfield of invariants $k((t))$ [9, Theorem 4.4.1]. Evidently, this simple algebra is split by $\overline{k((t))}$, i.e.

$$(6.2) \quad [kC_4]_t[y; \eta_t]e_1 / \langle q_t(y)e_1 \rangle \otimes_{k((t))} \overline{k((t))} \simeq M_2(\overline{k((t))}).$$

(In fact, $K *^f C_2$ splits already over $k((t))$, since b is a C_2 -norm of a root of the irreducible polynomial $\pi(x)$ and therefore f is cohomologically trivial.)

Next, since η_t is trivial on $[kC_4]_t e_2$, the skew polynomial ring $[kC_4]_t e_2[y; \eta_t]$ is actually an ordinary polynomial ring $k((t))[y]$. Again by (5.2),

$$[kC_4]_t[y; \eta_t]e_2 / \langle q_t(y)e_2 \rangle \simeq k((t))[y] / \langle y^2 + zay + c(c+a) \rangle.$$

Similarly,

$$[kC_4]_t[y; \eta_t]e_3 / \langle q_t(y)e_3 \rangle \simeq k((t))[y] / \langle y^2 + zay + d(d+a) \rangle.$$

The polynomials $y^2 + zay + c(c+a)$ and $y^2 + zay + d(d+a)$ are separable (since za is non-zero). Thus, both $[kC_4]_t[y; \eta_t]e_2 / \langle q_t(y)e_2 \rangle$ and $[kC_4]_t[y; \eta_t]e_3 / \langle q_t(y)e_3 \rangle$ are separable $k((t))$ -algebras, and for $i = 2, 3$,

$$(6.3) \quad [kC_4]_t[y; \eta_t]e_i / \langle q_t(y)e_i \rangle \otimes_{k((t))} \overline{k((t))} \simeq \overline{k((t))} \oplus \overline{k((t))}.$$

Equations (6.1), (6.2) and (6.3) yield

$$[kQ_8]_t \otimes_{k((t))} \overline{k((t))} \simeq \bigoplus_{i=1}^4 \overline{k((t))} \oplus M_2(\overline{k((t))}),$$

as required.

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We wish to thank M. Schaps for pointing out to us that there is an error in the attempted proof in [5] that the quaternion group is a counterexample to the DF conjecture. Here is her explanation: The given relations for the group algebra are incorrect. Using the notation in pages 166-7 of [5], if $a = 1 + i$, $b = 1 + j$ and $z = i^2 = j^2$, then $ab + ba = ij(1 + z)$, while $a^2 = b^2 = 1 + z$. There is a further error later on when the matrix algebra is deformed to four copies of the field, since a non-commutative algebra can never have a flat deformation to a commutative algebra.

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