ENTROPY-COST INEQUALITIES FOR DIFFUSION SEMIGROUPS WITH CURVATURE UNBOUNDED BELOW

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ABSTRACT. The weighted log-Sobolev inequality and the entropy-cost inequality are established for a class of diffusion semigroups with curvature unbounded below. Concrete examples are presented to illustrate the main results.

1. INTRODUCTION

Let (M, g) be a d -dimensional connected complete Riemannian manifold either with convex boundary or without boundary, and let $L := \Delta + \nabla V$ for some second differentiable function V on M . Let $K \in C(M)$ be such that

$$(1.1) \quad (\text{Ric} - \text{Hess}_V)(X, X) \geq -K(x), \quad X \in T_x M, x \in M.$$

When K is constant (i.e. the curvature of L is bounded below), many properties of the corresponding diffusion semigroup P_t have been well described in the literature; see e.g. [2, 14] and references therein for gradient estimates and semigroup functional inequalities, [12, 13, 11] for a dimension-free Harnack inequality and its applications to various contractivity properties, [10] for estimates of probability distances, and [4] for an HWI (entropy-cost-information) inequality.

Recently, the dimension-free Harnack inequality was extended in [1] to the case that $K \leq c(1 + \rho_o^2)$ for some constant $c > 0$ and $o \in M$, where ρ_o is the Riemannian distance function to o . The main purpose of this paper is to study the log-Sobolev inequality and the entropy-cost inequality for unbounded K . To this end, we assume that $\mu(dx) := e^{V(x)} dx$ is a probability measure so that the L -diffusion process is nonexplosive. Let ρ be a distance on M . For any probability measure ν on M , define the L^2 -Wasserstein distance (or the L^2 -transportation cost) between μ and ν w.r.t. ρ :

$$W_2^\rho(\mu, \nu) := \inf_{J \in \mathcal{C}(\mu, \nu)} \left\{ \int_{M \times M} \rho(x, y)^2 J(dx, dy) \right\}^{1/2},$$

where $\mathcal{C}(\mu, \nu)$ is the set of all couplings for μ and ν .

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To derive the entropy-cost inequality, we shall make use of the following generalized gradient estimate:

$$(1.2) \quad \tilde{g}(\tilde{\nabla} P_t f, \tilde{\nabla} P_t f) \leq \alpha(t) P_t |\nabla f|^2, \quad t \geq 0, f \in C_b^1(M),$$

where $\tilde{g}$ is another metric on M and $\tilde{\nabla}$ is the corresponding gradient operator. When $\tilde{g}$ is much smaller, this condition is much weaker than known ones for $\tilde{g} = g$ in the literature (cf. [2]).

Theorem 1.1. *If there exist measurable $\alpha : [0, \infty) \rightarrow (0, \infty)$ and a metric $\tilde{g}$ such that (1.2) holds, then*

$$(1.3) \quad \mu((P_t f) \log P_t f) \leq \frac{W_2^{\tilde{\rho}}(f\mu, \mu)^2}{4 \int_0^t \alpha(s)^{-1} ds}, \quad t > 0, f \geq 0, \mu(f) = 1,$$

where $\tilde{\rho}$ is the Riemannian distance induced by the metric $\tilde{g}$.

Next, to derive the log-Sobolev type inequality, we need the following condition on curvature:

$$(1.4) \quad x \mapsto P_t^K 1(x) := \mathbb{E}^x \exp \left[2 \int_0^t K(x_s) ds \right] \text{ is locally bounded,}$$

where x_s is the L -diffusion process. This condition is satisfied by many unbounded above functions K (cf. examples below).

Theorem 1.2. *If (1.4) holds, then*

$$(1.5) \quad \mu(f^2 \log f^2) \leq \mu((P_t f^2) \log P_t f^2) + 4 \int_0^t \mu((P_s^K 1) |\nabla f|^2) ds, \quad f \in C_0^1(M), t \geq 0.$$

If, in particular,

$$A(x) := \int_0^\infty P_t^K 1(x) dt < \infty, \quad x \in M,$$

is locally bounded, then the following weighted log-Sobolev inequality holds:

$$(1.6) \quad \mu(f^2 \log f^2) \leq 4\mu(A |\nabla f|^2), \quad f \in C_0^1(M), \mu(f^2) = 1.$$

Remark. (a) If K is constant, then (1.4) holds for $\tilde{g} = g$ and $\alpha(t) = e^{2Kt}$ (cf. [2]). Thus, (1.3) reduces to

$$\mu((P_t f) \log P_t f) \leq \frac{2KW_2^\rho(f\mu, \mu)^2}{1 - \exp[-2Kt]}, \quad t > 0, f \geq 0, \mu(f) = 1,$$

where ρ is the Riemannian distance induced by g . This coincides with an inequality in [4], which, together with (1.5) for $P_s^K 1 = e^{2sK}$, implies the so-called HWI inequality therein.

(b) If, in particular, $K < 0$, then (1.6) reduces to

$$\mu(f^2 \log f^2) \leq -\frac{2}{K} \mu(|\nabla f|^2), \quad \mu(f^2) = 1,$$

which follows from the well known Bakry-Emery criterion ([3]).

To see that Theorems 1.1 and 1.2 apply to many unbounded below curvature situations, we introduce the following consequences with more explicit conditions on curvature. Let ρ_o be the Riemannian distance to a fixed point $o \in M$. By the

Laplacian comparison theorem, there exists a nonnegative function $\gamma \in C([0, \infty))$ such that

$$(1.7) \quad L\rho_o(x) \leq \gamma \circ \rho_o(x), \quad \rho_o(x) \geq 1, x \notin \text{cut}(o),$$

where $\text{cut}(o)$ is the cut-locus of o (see (1.12) below for a concrete choice of γ). We note that since $L\rho_o \rightarrow \infty$ as $\rho_o \rightarrow 0$, we exclude in (1.7) a neighborhood of o .

Corollary 1.3. *Let $\gamma \geq 0$ satisfy (1.7). If there exists $C > 0$ such that*

$$(1.8) \quad 2\gamma(r)^2 + \gamma'(r) \leq C + C \int_0^r \gamma(s)ds, \quad r \geq 0,$$

and

$$(1.9) \quad \lim_{\rho_o \rightarrow \infty} \frac{K^+}{\int_0^{\rho_o} \gamma(s)ds} = 0,$$

then there exist constants $\delta, c > 0$ such that

$$(1.10) \quad P_t^K 1 \leq \exp \left[\int_0^{\rho_o} \gamma(s)ds + c(\delta, t) + ct \right], \quad t \geq 0,$$

where $c(\delta, t) := \sup_M \{2Kt - e^{-\delta t} \int_0^{\rho_o} \gamma(s)ds\} < \infty$ due to (1.9). Consequently, (1.4) is fulfilled and (1.2) is satisfied with

$$\alpha(s) := e^{c(\delta, s) + cs}$$

and

$$\tilde{g} := \exp \left[\int_0^{\rho_o} \gamma(s)ds \right] g,$$

so that (1.3) holds for this α and the Riemannian distance induced by $\tilde{g}$.

Next, we present a result on the weighted log-Sobolev inequality (1.6) for operators with curvature unbounded below.

Corollary 1.4. *Let $W \in C^2(M)$ be nonnegative with $\|\nabla V\|_\infty < \infty$, and $\gamma \in C(M)$ with $\gamma \rightarrow \infty$ as $\rho_o \rightarrow \infty$, such that*

$$(1.11) \quad LW \leq \theta - \gamma, \quad K \leq c_0 + c_1 \gamma$$

for some constants $\theta, c_0, c_1 > 0$. Then for any $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that (1.6) holds for

$$A(x) := c(\varepsilon)e^{2(c_1 + \varepsilon)W(x)}, \quad x \in M.$$

To illustrate our results, we present some examples below. To this end, we first observe that

$$(1.12) \quad L\rho_o(x) \leq C + \int_0^{\rho_o(x)} K(\eta_s)ds, \quad \rho_o(x) \geq 1, x \notin \text{cut}(o)$$

holds for $C := d - 1 + \sup_{\rho_o \leq 1} |\nabla V|$ and $\eta : [0, \rho_o(x)] \rightarrow M$ the minimal geodesic from o to x . Indeed, let $\{U_i : 1 \leq i \leq d - 1\}$ be constant vector fields along η such that $\{\dot{\eta}, U_1, \dots, U_{d-1}\}$ is an orthonormal basis. For any differentiable function h with $h(0) = 0, h(\rho_o(x)) = 1$, using $\{h(s)U_i\}$ to replace the Jacobi fields in the second variational formula for $\Delta\rho_o$, and applying the index lemma (see e.g. [5]), we obtain

$$(1.13) \quad \Delta\rho_o \leq \int_0^{\rho_o(x)} \{(d-1)\dot{h}(s)^2 - h(s)^2 \text{Ric}(\dot{\eta}(s), \dot{\eta}(s))\}ds.$$

Taking $h(s) = s \wedge 1$, we arrive at

$$\begin{aligned} L\rho_o(x) &\leq \int_0^{\rho_o(x)} \left\{ \frac{d}{ds} \langle \nabla V(\eta_s), h(s)\dot{\eta}(s) \rangle + (d-1)\dot{h}(s)^2 - h(s)^2 \text{Ric}(\dot{\eta}(s), \dot{\eta}(s)) \right\} ds \\ &\leq d-1 + \sup_{\rho_o \leq 1} |\nabla V| - \int_0^{\rho_o(x)} h(s)^2 \{ \text{Hess}_V(\dot{\eta}(s), \dot{\eta}(s)) - \text{Ric}(\dot{\eta}(s), \dot{\eta}(s)) \} ds. \end{aligned}$$

This implies (1.12) due to (1.1).

Example 1.1. Let M be a Cartan-Hadamard manifold so that ρ_o is smooth on $M \setminus \{o\}$. By (1.12), (1.11) holds for any $W \in C^2(M)$ with $W = \rho_o$ when $\rho_o \geq 1$, and $\gamma = -\int_0^{\rho_o} K(\eta_s) ds$, provided

$$(1.14) \quad \lim_{\rho_o(x) \rightarrow \infty} \int_0^{\rho_o(x)} K(\eta_s) ds = -\infty, \quad \limsup_{\rho_o(x) \rightarrow \infty} \frac{K(x)}{-\int_0^{\rho_o(x)} K(\eta_s) ds} < \infty.$$

In particular, assuming $\text{Ric} \geq -R \circ \rho_o$ for some nonnegative increasing function R with

$$\limsup_{s \rightarrow \infty} R(s)/s < \infty,$$

(1.14) holds for $V = -H \circ \rho_o$ when e.g. $H(0) = H'(0) = 0$ and

$$(1.15) \quad H''(s) := 2 - \sum_{n=1}^{\infty} n 1_{[n, n+n-1]}(s) \geq 1 - x, \quad x \geq 0.$$

This follows since taking

$$h(s) = (s - \rho_o + (R \circ \rho_o + 1)^{-1/2})^+ \mathfrak{B}R \circ \rho_o + 1$$

in (1.13) we obtain

$$(1.16) \quad \Delta \rho_o \leq c(1 + \mathfrak{B}R \circ \rho_o), \quad \rho_o \geq 1,$$

for some constant $c > 0$, and by (1.15) and the Hessian comparison theorem (cf. [5]),

$$\langle \nabla V, \nabla \rho_o \rangle = -H' \circ \rho_o \leq -\rho_o, \quad \text{Hess}_V \leq -H'' \circ \rho_o \leq \rho_o - 1.$$

Example 1.2. If there exists a constant $c_0 > 0$ such that

$$\text{Ric} \geq -c_0(1 + \rho_o^2), \quad \langle \nabla V, \nabla \rho_o \rangle \leq c_0(1 + \rho_o),$$

then (1.16) with $R(s) = c_0(1 + s^2)$ implies (1.7) for $\gamma(s) := c_1 s$ for some $c_1 > 0$, which satisfies (1.8). Therefore, assertions in Corollary 1.3 hold provided $K^+ \rho_o^{-2} \rightarrow 0$ as $\rho_o \rightarrow \infty$.

Example 1.3. Let $V \in C^2(\mathbb{R}^d)$ such that $\mu(dx) := e^{V(x)} dx$ is a probability measure. Consider $L := \sum_{i,j=1}^d a_{ij} \partial_i \partial_j + \sum_{i=1}^d b_i \partial_i$ on $\mathbb{R}^d$, where (a_{ij}) is non-degenerated and $b_i = \sum_{j=1}^d (\partial_j a_{ij} + a_{ij} \partial_j V)$. Then L is symmetric w.r.t. $\mu(dx) := e^{V(x)} dx$ (cf. [6]). Let $\sigma := \mathfrak{B}a$. If there exists $\lambda \in \mathbb{R}$ such that

$$(1.17) \quad \|\sigma(x) - \sigma(y)\|^2 + \langle x - y, b(x) - b(y) \rangle \leq \lambda |x - y|^2, \quad x, y \in \mathbb{R}^d,$$

where $\|\cdot\|$ is the Hilbert-Schmidt norm on $\mathbb{R}^d$, then according to the proof of Theorem 4.13 in [6]

$$|\nabla P_t f|^2 \leq e^{2\lambda t} P_t |\nabla f|^2, \quad t \geq 0.$$

Therefore, if $a \geq \lambda_0 I$ for some constant $\lambda_0 > 0$, then

$$|\nabla P_t f|^2 \leq \frac{e^{2\lambda t}}{\lambda_0} P_t \langle a \nabla f, \nabla f \rangle.$$

By this and Theorem 1.1 we arrive at

$$\mu((P_t f) \log P_t f) \leq \frac{\lambda W_2(f\mu, \mu)^2}{2\lambda_0(1 - e^{-2\lambda t})}, \quad t > 0, f \geq 0, \mu(f) = 1,$$

where W_2 is induced by the usual Euclidean distance. If moreover $\lambda < 0$, then P_t converges exponentially to μ in entropy w.r.t. W_2^2 . We remark that condition (1.17) is in general much simpler than the curvature condition induced by the metric $g := a^{-1}$.

2. PROOFS

Proof of Theorem 1.1. We follow the line of [4] but use the generalized gradient estimate (1.2) to replace the usual one. For any $t > 0$ and $x, y \in M$, let $\eta : [0, t] \rightarrow M$ be the minimal geodesic under $\tilde{g}$ from x to y . We have $\tilde{g}(\dot{\eta}_s, \dot{\eta}_s)^{1/2} = \tilde{\rho}(x, y)/t$, where $\dot{\eta}$ is the tangent vector field along η . Let

$$g(s) := \frac{t \int_0^s \alpha(r)^{-1} dr}{\int_0^t \alpha(r)^{-1} dr}, \quad s \in [0, t].$$

Then $g(0) = 0$ and $g(t) = t$. It follows from (1.2) that

$$\begin{aligned} \frac{d}{ds} (P_s \log P_{t-s} f)(\eta_{g(s)}) &= -P_s |\nabla \log P_{t-s} f|^2(\eta_{g(s)}) \\ &\quad + g'(s) \tilde{g}(\tilde{\nabla} P_s \log P_{t-s} f(\eta_{g(s)}), \dot{\eta}_{g(s)}) \\ &\leq \{ -P_s |\nabla \log P_{t-s} f|^2 + g'(s) t^{-1} \tilde{\rho}(x, y) \alpha(s)^{1/2} (P_s |\nabla \log P_{t-s} f|^2)^{1/2} \}(\eta_{g(s)}) \\ &\leq \frac{\alpha(s) g'(s)^2 \tilde{\rho}(x, y)^2}{4t^2} = \frac{\tilde{\rho}(x, y)^2}{4\alpha(s) (\int_0^t \alpha(r)^{-1} dr)^2}. \end{aligned}$$

Integrating w.r.t. ds over $[0, t]$, we arrive at

$$P_t \log f(y) \leq \log P_t f(x) + \frac{\tilde{\rho}(x, y)^2}{4 \int_0^t \alpha(s)^{-1} ds}.$$

Replacing f by $P_t f$ and noting that

$$\int_M \log P_{2t} f d\mu \leq \log \mu(f) = 0,$$

from the Monge-Kantorovich formula

$$W_2^{\tilde{\rho}}(\mu, f\mu)^2 = \sup \{ \mu(f g_2) - \mu(g_1) : g_1, g_2 \in C_b(M), g_2(y) \leq g_1(x) + \tilde{\rho}(x, y)^2 \},$$

we obtain

$$\mu(f P_t \log P_t f) \leq \frac{W_2^{\tilde{\rho}}(\mu, f\mu)^2}{4 \int_0^t \alpha(s)^{-1} ds}.$$

This completes the proof by the symmetry of P_t w.r.t. μ . □

Proof of Theorem 1.2. By (1.4), the Feynman-Kac semigroup

$$P_t^K f(x) := \mathbb{E}^x f(x_t) e^{2 \int_0^t K(x_s) ds}, \quad t \geq 0, \quad x \in M,$$

is well-defined in $L^1(\mu)$ for all $f \in C_0(M)$. Indeed, letting

$$P_t^{K_n} f(x) := \mathbb{E}^x f(x_t) e^{2 \int_0^t K(x_s) \wedge n ds}, \quad n \geq 1,$$

we have

$$|\mu(g P_t^K f)| = \lim_{n \rightarrow \infty} |\mu(g P_t^{K_n} f)| = \lim_{n \rightarrow \infty} |\mu(f P_t^{K_n} g)| \leq \|g\|_\infty \left(\sup_{\text{supp } f} P_t^K 1 \right) \mu(|f|)$$

for all $f, g \in C_0(M)$. Thus, $P_t^K f \in L^1(\mu)$ and $\mu(|P_t^K f|) \leq \sup_{\text{supp } f} P_t^K 1$. Next, the Bismut formula (cf. [8, Proposition A6(i)]) implies

$$(2.1) \quad |\nabla P_t f(x)| \leq \mathbb{E}^x |\nabla f(x_t)| e^{\int_0^t K(x_s) ds}.$$

Then for any $f \in C_0^1(M)$,

$$\begin{aligned} & \frac{d}{ds} P_s(P_{t-s} f^2) \log(P_{t-s} f^2)(x) \\ & \leq P_s \frac{|\nabla P_{t-s} f^2|^2}{P_{t-s} f^2}(x) \leq \mathbb{E}^x \frac{1}{P_{t-s} f^2(x_s)} (\mathbb{E}^{x_s} |\nabla f^2|(x_{t-s}) e^{\int_0^{t-s} K(x_r) dr})^2 \\ & \leq 4 \mathbb{E}^x \mathbb{E}^{x_s} |\nabla f|^2(x_{t-s}) e^{2 \int_0^{t-s} K(x_r) dr} = 4 \mathbb{E}^x P_{t-s}^K |\nabla f|^2(x_s). \end{aligned}$$

Thus,

$$P_t f^2 \log f^2 \leq (P_t f^2) \log P_t f^2 + 4 \int_0^t P_s P_{t-s}^K |\nabla f|^2 ds.$$

Taking an integral w.r.t. μ and noting that P_t^K is symmetric w.r.t. μ , we prove (1.5). Finally, if A is locally bounded, (1.6) follows from (1.5) and the dominated convergence theorem by noting that $P_t f^2 \rightarrow \mu(f^2) = 1$ in $L^2(\mu)$ as $t \rightarrow \infty$ according to the weak Poincaré inequality (cf. [11]). $\square$

Proof of Corollary 1.3. Let $h \in C^\infty[0, \infty)$ such that $h|_{[0,1]} = 0$, $h' \geq 0$ and $h(r) = 1$ for $r \geq 2$. By (1.7) and the Itô formula due to Kendall [9], we have

$$(2.2) \quad dh(\rho_o(x_t)) \rho_o(x_t) \leq \beta 2 \{h'(\rho_o(x_t)) \rho_o(x_t) + h(\rho_o(x_t))\} db_t + (c_1 + \gamma(\rho_o(x_t))) dt,$$

where b_t is the one-dimensional Brownian motion and $c_1 > 0$ is a constant. For any $\delta > 0$, let

$$F_\delta(t) := \exp \left[e^{-\delta t} \int_0^{\rho_o(x_t)} h(s) \gamma(s) ds \right], \quad t \geq 0.$$

It follows from (2.2) that

$$dF_\delta(t) \leq dM_t + F_\delta(t) e^{-\delta t} \left\{ \gamma^2(\rho_o(x_t)) (1 + e^{-\delta t}) + \gamma'(\rho_o(x_t)) - \delta \int_0^{\rho_o(x_t)} \gamma(s) ds + c_0 \right\} dt$$

for some constant $c_0 > 0$ and a local martingale M_t . Taking $\delta = C$ and combining this with (1.8), we obtain

$$dF_\delta(t) \leq dM_t + (C + c_0) dt.$$

Thus, for $c := C + c_0$,

$$\mathbb{E}^x F_\delta(t) \leq F_\delta(0) e^{ct}, \quad t \geq 0.$$

Thus, it follows from (1.9) that

$$\mathbb{E}^x e^{2K(x_s)t} \leq \mathbb{E}^x F_\delta(s) e^{c(\delta,t)} \leq F_\delta(0) e^{cs+c(\delta,t)}, \quad t \geq s \geq 0.$$

Therefore,

$$P_t^K 1(x) := \mathbb{E}^x e^{\int_0^t 2K(x_s) ds} \leq \frac{1}{t} \int_0^t \mathbb{E}^x e^{2K(x_s)t} ds \leq \exp \left[\int_0^{\rho_o(x)} \gamma(s) ds + ct + c(\delta, t) \right].$$

This implies (1.10). Hence, (1.2) follows from (2.1) and the Schwartz inequality, and (1.3) follows from Theorem 1.1. $\square$

Proof of Corollary 1.4. By (1.11),

$$dW(x_t) \leq \mathbb{B}2 \langle \nabla W(x_t), u_t^{-1} dB_t \rangle + (\theta - \gamma \circ \rho_o(x_t)) dt,$$

where B_t is the Brownian motion on $\mathbb{R}^d$ and u_t is the horizontal L -diffusion process on the space of frame bundles. This implies

$$\begin{aligned} \mathbb{E}^x e^{\lambda \int_0^t \gamma \circ \rho_o(x_s) ds} &\leq \mathbb{E}^x e^{\lambda \theta t + \lambda W(x) + \mathbb{B}2 \int_0^t \langle \nabla W(x_s), u_s^{-1} dB_s \rangle} \\ &\leq e^{\lambda \theta t + \lambda W(x) + \lambda^2 \|\nabla W\|_\infty^2 t}, \quad \lambda, t > 0. \end{aligned}$$

Thus, letting $c_2 := 2\|\nabla W\|_\infty^2 (c_1 + \varepsilon)^2$, we obtain

$$(2.3) \quad \mathbb{E}^x e^{2(c_1 + \varepsilon) \int_0^t \gamma \circ \rho_o(x_s) ds} \leq e^{2(c_1 + \varepsilon)\theta t + 2c_2 t + 2(c_1 + \varepsilon)W(x)}, \quad t \geq 0.$$

We will complete the proof in two steps.

(a) Assume, instead of the second inequality of (1.11), that

$$(2.4) \quad K \leq (c_1 + \varepsilon)\gamma \circ \rho_o - N$$

for some constant $N > c_2 + \theta(c_1 + \varepsilon)$. Then, it follows from (2.3) that

$$\begin{aligned} P_t^K 1(x) &:= \mathbb{E}^x e^{\int_0^t 2K(x_s) ds} \leq \mathbb{E}^x e^{2(c_1 + \varepsilon) \int_0^t \gamma \circ \rho_o(x_s) ds - 2Nt} \\ &\leq e^{2(c_1 + \varepsilon)W(x) - 2(N - (c_1 + \varepsilon)\theta - c_2)t}. \end{aligned}$$

This implies the desired assertion.

(b) In general, for fixed $N > c_2 + \theta(c_1 + \varepsilon)$, we have

$$u_N := \inf \left\{ s \geq 0 : \int_0^s (N + c_0 - \varepsilon\gamma(u)) du \geq 0 \right\} < \infty$$

since $\gamma(u) \rightarrow \infty$ as $u \rightarrow \infty$. Define

$$V_N := - \int_0^{\rho_o} ds \int_0^{s \wedge u_N} (N + c_0 - \varepsilon\gamma(u)) du.$$

Then V_N is bounded and

$$\langle \nabla V_N, \nabla \rho_o \rangle = - \int_0^{\rho_o \wedge u_N} (N + c_0 - \varepsilon\gamma(u)) du \leq 0.$$

So, (1.11) holds for $L_N := L + \nabla V_N$. Next, since M is a Cartan-Hadamard, by the Hessian comparison theorem we have $\text{Hess}_{V_N} \leq -(N + c_0 - \varepsilon\gamma(\rho_o))1_{[0, u_N]}(\rho_o)$. Thus, by (1.11)

$$(2.5) \quad \text{Ric} - \text{Hess}_{V+V_N} \geq N + c_0 - K - \varepsilon\gamma(\rho_o) \geq N - (c_1 + \varepsilon)\gamma(\rho_o).$$

Therefore, (2.4) holds for L_N in place of L . That is, by (a), (1.6) holds for $\mu_N := e^{V_N} \mu / \mu(e^{V_N})$ in place of μ . Then the proof is completed since the log-Sobolev inequality is stable under the bounded perturbation V_N . More precisely, if (1.6) holds for μ_N in place of μ , it also holds for μ itself with constant $c(\varepsilon)$ in the definition of $A(x)$ replaced by $c(\varepsilon)e^{2\|V_N\|_\infty}$; see e.g. [7, Remark 1.20]. $\square$

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