FIRST EIGENVALUE OF A JACOBI OPERATOR OF HYPERSURFACES WITH A CONSTANT SCALAR CURVATURE

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ABSTRACT. Let M be an n -dimensional compact hypersurface with constant scalar curvature $n(n-1)r$, $r > 1$, in a unit sphere $S^{n+1}(1)$. We know that such hypersurfaces can be characterized as critical points for a variational problem of the integral $\int_M H dM$ of the mean curvature H . In this paper, we first study the eigenvalue of the Jacobi operator J_s of M . We derive an optimal upper bound for the first eigenvalue of J_s , and this bound is attained if and only if M is a totally umbilical and non-totally geodesic hypersurface or M is a Riemannian product $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-1$.

1. INTRODUCTION

Let M be an n -dimensional hypersurface in a unit sphere $S^{n+1}(1)$ of dimension $n+1$. We denote the second fundamental form of M and its squared norm by α and S , respectively. Then, a Schrödinger operator

$$J_m = -\Delta - S - n,$$

where Δ stands for the Laplace-Beltrami operator, arose naturally in the study of the stability of both minimal hypersurfaces in $S^{n+1}(1)$ and hypersurfaces with constant mean curvature in $S^{n+1}(1)$. The J_m is called a Jacobi operator or a stability operator, which represents the second variation of the volume. Its spectral behavior is directly related to the instability of such hypersurfaces (cf. [18] and [5]).

On the other hand, for any C^2 -function f , denoting its Hessian by (f_{ij}) , we define a differential operator

$$\square f = \sum_{i,j=1}^n (nH\delta_{ij} - h_{ij})f_{ij},$$

where H and h_{ij} denote the mean curvature and components of the second fundamental form of M . The differential operator $\square$ was introduced and used by S. Y. Cheng and Yau in [11] to study compact hypersurfaces with constant scalar

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curvature in $S^{n+1}(1)$. They proved that if M is an n -dimensional compact hypersurface with constant scalar curvature $n(n-1)r$, $r \geq 1$, and if the sectional curvature of M is non-negative, then M is a totally umbilical hypersurface $S^n(c)$ or a Riemannian product $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-1$, where $S^k(c)$ denotes a sphere of radius c . We should notice that the differential operator $\square$ is self-adjoint. By making use of the similar method which has been used by Nakagawa and the author in [9] and the differential operator $\square$ introduced by S.Y. Cheng and Yau, Li [14] has proved that if M is an n -dimensional compact hypersurface with constant scalar curvature $n(n-1)r$, $r \geq 1$, and if $S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$, then M is a totally umbilical hypersurface or a Riemannian product $S^{n-1}(c) \times S^1(\sqrt{1-c^2})$ with $c^2 = \frac{n-2}{nr} \leq \frac{n-2}{n}$. Furthermore, the Riemannian product $S^{n-1}(c) \times S^1(\sqrt{1-c^2})$ has been characterized in [6], [7], [8] and [10].

In [1], Alencar, do Carmo and Colares have studied the stability of hypersurfaces with constant scalar curvature in $S^{n+1}(1)$. In this case, the Jacobi operator J_s is given by

$$J_s = -\square - \{n(n-1)H + nHS - f_3\},$$

which is associated to the variational characterization of hypersurfaces with constant scalar curvature in $S^{n+1}(1)$, where $f_3 = \sum_{j=1}^n k_j^3$ and k_j 's are the principal curvatures of M (cf. [16] and [17]). The spectral behavior of J_s is also directly related to the instability of hypersurfaces with constant scalar curvature.

The first eigenvalue of the Jacobi operator J_m of minimal hypersurfaces in $S^{n+1}(1)$ was studied by Simons [18], Wu [19] and Perdomo [15]. A characterization of the Clifford torus is given by the first eigenvalue of the Jacobi operator J_m , that is, they proved that if M is an n -dimensional compact orientable minimal hypersurface in $S^{n+1}(1)$, then the first eigenvalue $\lambda_1^{J_m}$ of the Jacobi operator J_m satisfies

- (1) $\lambda_1^{J_m} = -n$ and M is totally geodesic.
- (2) $\lambda_1^{J_m} \leq -2n$ and $\lambda_1^{J_m} = -2n$ if and only if M is a Clifford torus $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$, $1 \leq m \leq n-1$.

Very recently, Alías, Barros and Brasil [3] have extended the above results to compact hypersurfaces with constant mean curvature in $S^{n+1}(1)$. They have obtained that if M is an n -dimensional compact orientable hypersurface with constant mean curvature in $S^{n+1}(1)$, then the first eigenvalue $\lambda_1^{J_m}$ of the Jacobi operator J_m satisfies

- (1) $\lambda_1^{J_m} = -n(1+H^2)$ and M is totally umbilical or
- (2) $\lambda_1^{J_m} \leq -2n(1+H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}}|H| \max \sqrt{S-nH^2}$ and equality holds if and only if M is a Riemannian product $S^{n-1}(c) \times S^1(\sqrt{1-c^2})$.

We should notice that the first eigenvalue of the Jacobi operator J_m of the totally umbilical hypersurface $S^{n+1}(1)$ is different from the one of the Riemannian product $S^{n-1}(c) \times S^1(\sqrt{1-c^2})$ in $S^{n+1}(1)$.

Since the Laplace-Beltrami operator is always elliptic, the Jacobi operator J_m is always elliptic. But, in general, the operator $\square$, and hence the Jacobi operator J_s , are not elliptic. When $r > 1$, the differential operator $\square$ is elliptic. In fact, from the Gauss equation (2.4) in section 2, we have that the mean curvature H satisfies $n^2H^2 > S$. Hence, we can assume $H > 0$. Thus, the differential operator $\square$ is

elliptic if and only if $nH - k_j > 0$ for $j = 1, 2, \dots, n$, where k_j 's are the principal curvatures of M . If, for some j , $nH \leq k_j$ holds, then $n^2H^2 \leq k_j^2 \leq S$. This is impossible.

When $r = 1$, let k_i , $i = 1, 2, \dots, n$, denote the principal curvatures of M . We consider the elementary symmetric functions S_r of the principal curvatures

$$S_0 = 1, \quad S_r = \sum_{i_1 < \dots < i_r} k_{i_1} \cdots k_{i_r} \quad (1 \leq r \leq n).$$

$\square$ is elliptic if and only if $n \geq 3$ and the $S_3 \neq 0$ on M (cf. [4], [2] and [12]).

In [4], Alías, Brasil and Sousa have studied the first eigenvalue of the Jacobi operator J_s of an n -dimensional hypersurface with constant scalar curvature $n(n-1)$. They have proved that if M is an n -dimensional compact orientable hypersurface with constant scalar curvature $n(n-1)$ in $S^{n+1}(1)$ and if $n \geq 3$ and S_3 does not vanish on M , then the first eigenvalue $\lambda_1^{J_s}$ of the Jacobi operator J_s satisfies

$$\lambda_1^{J_s} \leq -2n(n-1) \min |H|.$$

Further, the equality holds if and only if M is a Riemannian product $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-1$, with scalar curvature $n(n-1)$.

In this paper, we investigate the first eigenvalue of the Jacobi operator J_s of an n -dimensional hypersurface with constant scalar curvature $n(n-1)r$, $r > 1$, and give a characterization of the totally umbilical and non-totally geodesic hypersurface and the Riemannian product $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-1$, by the first eigenvalue of J_s . Namely, we will prove the following:

Theorem 1.1. *Let M be an n -dimensional compact orientable hypersurface with constant scalar curvature $n(n-1)r$, $r > 1$, in $S^{n+1}(1)$. Then, the Jacobi operator J_s is elliptic, the mean curvature H does not vanish on M and the first eigenvalue $\lambda_1^{J_s}$ of the Jacobi operator J_s satisfies*

$$\lambda_1^{J_s} \leq -\{2n(n-1) + n^2(n-1)(r-1)\} \min |H| + n(n-1)(r-1)\{(n-1)(r-1) + 1\} \frac{1}{\min |H|}$$

and the equality holds if and only if either M is totally umbilical and non-totally geodesic or M is a Riemannian product $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-1$, with $r > 1$.

Corollary 1.2. *Let M be an n -dimensional compact orientable hypersurface with constant scalar curvature $n(n-1)r$, $r > 1$, in $S^{n+1}(1)$. Then, the Jacobi operator J_s is elliptic and the first eigenvalue $\lambda_1^{J_s}$ of the Jacobi operator J_s satisfies*

$$\lambda_1^{J_s} \leq -n(n-1)r\sqrt{r-1}$$

and the equality holds if and only if M is totally umbilical and non-totally geodesic.

Remark 1.3. We should notice that the totally umbilical hypersurfaces do not appear in the result of [4]. From the proof of Theorem 1.1 in section 3, we shall see that our results do hold for the case where $r = 1$, if we assume that the Jacobi operator J_s is elliptic. Hence, the result in [4] can be seen as a direct consequence of Theorem 1.1.

2. PRELIMINARY

Throughout this paper, all manifolds are assumed to be smooth and connected without boundary. Let M be an n -dimensional hypersurface in a unit sphere $S^{n+1}(1)$. We choose a local orthonormal frame $\{\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{e}_{n+1}\}$ and the dual coframe $\{\omega_1, \dots, \omega_n, \omega_{n+1}\}$ in such a way that $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a local orthonormal frame on M . Hence, we have

$$\omega_{n+1} = 0$$

on M . From Cartan's lemma, we have

$$(2.1) \quad \omega_{in+1} = \sum_{j=1}^n h_{ij} \omega_j, \quad h_{ij} = h_{ji}.$$

The mean curvature H and the second fundamental form α of M are defined, respectively, by

$$H = \frac{1}{n} \sum_{i=1}^n h_{ii}, \quad \alpha = \sum_{i,j=1}^n h_{ij} \omega_i \otimes \omega_j \mathbf{e}_{n+1}.$$

When the mean curvature H of M is identically zero, we recall that M is by definition a minimal hypersurface.

From the structure equations of M , the Gauss equation and the Codazzi equation are given by

$$(2.2) \quad R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}),$$

$$(2.3) \quad h_{ijk} = h_{ikj}.$$

From (2.2), we have

$$(2.4) \quad n(n-1)r = n(n-1) + n^2 H^2 - S,$$

where $n(n-1)r$ and S denote the scalar curvature and the squared norm of the second fundamental form of M , respectively.

For any C^2 -function f on M , we define its gradient and Hessian by

$$\begin{aligned} df &= \sum_{i=1}^n f_i \omega_i, \\ \sum_{j=1}^n f_{ij} \omega_j &= df_i + \sum_{j=1}^n f_j \omega_{ji}. \end{aligned}$$

Thus, the differential operator $\square$ is defined by

$$\square f = \sum_{i,j=1}^n (nH\delta_{ij} - h_{ij}) f_{ij}.$$

3. PROOFS OF RESULTS

First of all, we will consider the first eigenvalue of the Jacobi operator J_s of both the totally umbilical and non-totally geodesic hypersurface and the Riemannian product $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-1$.

Example 3.1. Let M be a totally umbilical and non-totally geodesic hypersurface in $S^{n+1}(1)$. In this case, $\square = (n-1)H\Delta$ and

$$J_s = -\square - \{n(n-1)H + nHS - f_3\} = -\{(n-1)H\Delta + n(n-1)H(1+H^2)\}.$$

Hence,

$$\lambda_1^{J_s} = -n(n-1)H(1+H^2) = -n(n-1)r\sqrt{r-1},$$

from the Gauss equation (2.4). Since

$$\begin{aligned} & n(n-1)r\sqrt{r-1} \\ &= \{2n(n-1) + n^2(n-1)(r-1)\}H - n(n-1)(r-1)\{(n-1)(r-1) + 1\}\frac{1}{H}, \end{aligned}$$

we know that

$$\begin{aligned} \lambda_1^{J_s} &= n(n-1)r\sqrt{r-1} \\ &= -\{2n(n-1) + n^2(n-1)(r-1)\}H \\ &\quad + n(n-1)(r-1)\{(n-1)(r-1) + 1\}\frac{1}{H}. \end{aligned}$$

Example 3.2. Let $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-1$, be a hypersurface with $r > 1$ in $S^{n+1}(1)$. In this case, the position vector is $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in S^m(c) \times S^{n-m}(\sqrt{1-c^2})$ and the unit normal vector at the point $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in S^m(c) \times S^{n-m}(\sqrt{1-c^2})$ is given by $\mathbf{e}_{n+1} = (-\frac{\sqrt{1-c^2}}{c}\mathbf{x}_1, \frac{c}{\sqrt{1-c^2}}\mathbf{x}_2)$. Its principal curvatures are given by

$$k_1 = \cdots = k_m = \frac{\sqrt{1-c^2}}{c}, \quad k_{m+1} = \cdots = k_n = -\frac{c}{\sqrt{1-c^2}}.$$

Thus, the mean curvature H of $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$ satisfies

$$(3.1) \quad nH = \frac{m - nc^2}{c\sqrt{1-c^2}}.$$

Since

$$\begin{aligned} (3.2) \quad & n(n-1)(r-1) = n^2H^2 - S \\ &= \frac{(m - nc^2)^2}{c^2(1-c^2)} - m\frac{1-c^2}{c^2} - (n-m)\frac{c^2}{1-c^2} \\ &= \frac{n(n-1)c^4 - 2m(n-1)c^2 + m(m-1)}{c^2(1-c^2)} \\ &= \frac{(n-1)(nc^2 + n - 2m)(c^2 - 1) + (n-m)(n-1-m)}{c^2(1-c^2)}, \end{aligned}$$

we know that $r > 1$ if and only if either

$$0 < c^2 < \frac{m}{n} - \frac{\sqrt{m(n-m)}}{n\sqrt{n-1}}$$

or

$$\frac{m}{n} + \frac{\sqrt{m(n-m)}}{n\sqrt{n-1}} < c^2 < 1.$$

Since the principal curvatures of $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$ are constant, we know that S and f_3 are constant. Because the differential operator $\square$ is self-adjoint and elliptic, we have that the first eigenvalue of the Jacobi operator J_s is given by

$$(3.3) \quad \lambda_1^{J_s} = -\{n(n-1)H + nHS - f_3\}.$$

Furthermore, from (3.1), we have

$$\begin{aligned} & n(n-1)H + nHS - f_3 \\ &= n(n-1)H + nH\left\{m\frac{1-c^2}{c^2} + (n-m)\frac{c^2}{1-c^2}\right\} \\ & \quad - m\frac{(1-c^2)^{\frac{3}{2}}}{c^3} + (n-m)\frac{c^3}{(1-c^2)^{\frac{3}{2}}} \\ (3.4) \quad &= nH\left\{n-1 + m\frac{1-c^2}{c^2} + (n-m)\frac{c^2}{1-c^2}\right. \\ & \quad \left. - \frac{m}{m-nc^2}\frac{(1-c^2)^2}{c^2} + \frac{(n-m)}{m-nc^2}\frac{c^4}{(1-c^2)}\right\} \\ &= \frac{nH}{m-nc^2}\left\{(n-1)(m-nc^2) + \frac{m(1-c^2)\{m-1-(n-1)c^2\}}{c^2}\right. \\ & \quad \left.+ \frac{(n-m)c^2\{m-(n-1)c^2\}}{1-c^2}\right\}. \end{aligned}$$

On the other hand, from (3.1) and (3.2), we have

$$(3.5) \quad \begin{aligned} & n(n-1)(r-1)\frac{1}{n^2H^2} \\ &= \frac{(n-1)(nc^2+n-2m)(c^2-1) + (n-m)(n-1-m)}{(m-nc^2)^2}, \end{aligned}$$

$$(3.6) \quad \begin{aligned} & \{2n(n-1) + n^2(n-1)(r-1)\}H \\ &= nH\frac{(n-1)\{(n-2)c^2+n-2m\}(c^2-1) + (n-m)(n-1-m)}{c^2(1-c^2)}. \end{aligned}$$

Therefore, we obtain, from (3.2), (3.5) and (3.6), by making use of a direct computation,

$$\begin{aligned}
& \{2n(n-1) + n^2(n-1)(r-1)\}H \\
& - n(n-1)(r-1)\{(n-1)(r-1) + 1\}\frac{1}{H} \\
= & nH \frac{(n-1)\{(n-2)c^2 + n-2m\}(c^2-1) + (n-m)(n-1-m)}{c^2(1-c^2)} \\
& - nH \frac{(n-1)(nc^2 + n-2m)(c^2-1) + (n-m)(n-1-m)}{(m-nc^2)^2} \\
& \times \frac{\{n(n-2)c^2 + (n-1)(n-2m)\}(c^2-1) + (n-m)(n-1-m)}{c^2(1-c^2)} \\
= & \frac{nH}{(m-nc^2)^2c^2(1-c^2)} \left\{ 2(n-1)c^2(1-c^2)(m-nc^2)^2 \right. \\
& + \{(n-1)(nc^2 + n-2m)(c^2-1) + (n-m)(n-1-m)\} \\
& \times \left[(m-nc^2)^2 - \{n(n-2)c^2 + (n-1)(n-2m)\}(c^2-1) \right. \\
& \left. \left. + (n-m)(n-1-m) \right] \right\} \\
= & \frac{nH}{(m-nc^2)^2c^2(1-c^2)} \left\{ 2(n-1)c^2(1-c^2)(m-nc^2)^2 \right. \\
& \left. + [(n-1)(nc^2 + n-2m)(c^2-1) + (n-m)(n-1-m)](m-nc^2)(1-2c^2) \right\} \\
= & \frac{nH}{(m-nc^2)c^2(1-c^2)} \left\{ 2(n-1)c^2(1-c^2)(m-nc^2) \right. \\
& \left. + (n-1)(nc^2-m)(c^2-1)(1-2c^2) + (n-m)\{(n-1)c^2-m\}(1-2c^2) \right\} \\
= & \frac{nH}{(m-nc^2)} \left\{ \frac{(n-1)(m-nc^2)}{c^2} \right. \\
& \left. + \frac{(n-m)\{(n-1)c^2-m\}}{c^2} + \frac{(n-m)\{m-(n-1)c^2\}}{1-c^2} \right\} \\
= & \frac{nH}{m-nc^2} \left\{ (n-1)(m-nc^2) + \frac{m(1-c^2)\{m-1-(n-1)c^2\}}{c^2} \right. \\
& \left. + \frac{(n-m)c^2\{m-(n-1)c^2\}}{1-c^2} \right\}.
\end{aligned}$$

Hence, we infer, from (3.4) and the above equality, that

$$\begin{aligned}
\lambda_1^{J_s} &= -\{n(n-1)H + nHS - f_3\} \\
&= -\{2n(n-1) + n^2(n-1)(r-1)\}H \\
&\quad + n(n-1)(r-1)\{(n-1)(r-1) + 1\}\frac{1}{H}.
\end{aligned}$$

Remark 3.3. We must notice that the first eigenvalue of the Jacobi operator J_s of both the totally umbilical and non-totally geodesic hypersurfaces and the Riemannian product $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$ has the same representation formula. But, as we have seen in the introduction, the first eigenvalue of the Jacobi operator J_m of the totally umbilical hypersurface is different from the first eigenvalue of the Jacobi operator J_m of the Riemannian product $S^{n-1}(c) \times S^1(\sqrt{1-c^2})$. It is a very interesting fact.

Proof of Theorem 1.1. Since $r > 1$, from the assertion in the introduction, we know that $\square$ is elliptic and $n^2H^2 - S = n(n-1)(r-1) > 0$. Hence, $H \neq 0$. Thus, we can assume $H > 0$. We choose a local field of orthonormal frames $\mathbf{e}_1, \dots, \mathbf{e}_n$ on M such that, at the point that we consider,

$$h_{ij} = \begin{cases} k_i & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where the k_i 's are the principal curvatures of M . Thus,

$$\square(nH) = nH\Delta(nH) - \sum_{i=1}^n k_i(nH)_{ii}.$$

From the Gauss equation (2.4), we have

$$nH\Delta(nH) = \frac{1}{2}\Delta(nH)^2 - |\nabla(nH)|^2 = \frac{1}{2}\Delta S - |\nabla(nH)|^2.$$

By making use of a standard computation of Simons' type formula (cf. [11], [1] and [18]), we have

$$\frac{1}{2}\Delta S = \sum_{i,j,k=1}^n h_{ijk}^2 + \sum_{i=1}^n k_i(nH)_{ii} + nS - n^2H^2 + nHf_3 - S^2.$$

Hence, we infer

$$\square(nH) = \sum_{i,j,k=1}^n h_{ijk}^2 - n^2|\nabla H|^2 + nS - n^2H^2 + nHf_3 - S^2.$$

Therefore, we have

$$\begin{aligned} J_s H &= -\square H - \{n(n-1)H + nHS - f_3\}H \\ &= -\left\{ \frac{1}{n} \sum_{i,j,k=1}^n h_{ijk}^2 - n|\nabla H|^2 + S - nH^2 + Hf_3 \right. \\ &\quad \left. - \frac{S^2}{n} + n(n-1)H^2 + nH^2S - Hf_3 \right\} \\ &= -\left\{ \frac{1}{n} \sum_{i,j,k=1}^n h_{ijk}^2 - n|\nabla H|^2 + S + n(n-2)H^2 + nH^2S - \frac{S^2}{n} \right\} \\ &= -\left[\frac{1}{n} \sum_{i,j,k=1}^n h_{ijk}^2 - n|\nabla H|^2 + 2n(n-1)H^2 \right. \\ &\quad \left. + n^2(n-1)(r-1)H^2 - n(n-1)(r-1)\{(n-1)(r-1) + 1\} \right]. \end{aligned}$$

From the min-max principle, we have

$$\begin{aligned}
\lambda_1^{J_s} &\leq \frac{\int_M H J_s H dM}{\int_M H^2 dM} \\
&= - \frac{\int_M H (\frac{1}{n} \sum_{i,j,k=1}^n h_{ijk}^2 - n |\nabla H|^2) dM}{\int_M H^2 dM} \\
&\quad - \frac{\int_M H \{2n(n-1) + n^2(n-1)(r-1)\} H^2 dM}{\int_M H^2 dM} \\
&\quad + \frac{\int_M H n(n-1)(r-1) \{(n-1)(r-1) + 1\} dM}{\int_M H^2 dM} \\
&\leq - \left\{ 2n(n-1) + n^2(n-1)(r-1) \right\} \min |H| \\
&\quad + n(n-1)(r-1) \left\{ (n-1)(r-1) + 1 \right\} \frac{1}{\min |H|}.
\end{aligned}$$

Here we have used the following inequality (cf. [1] and [14]):

$$\sum_{i,j,k=1}^n h_{ijk}^2 \geq n^2 |\nabla H|^2,$$

which can be proved by using the Gauss equation (2.4) and $r > 1$. When the equality holds, we know that H is constant and $\sum_{i,j,k=1}^n h_{ijk}^2 = 0$ on M , that is, the second fundamental form of M is parallel. Hence, M is an isoparametric hypersurface with at most two distinct principal curvatures (cf. [13]). Thus, M is totally umbilical and non-totally geodesic or a Riemannian product $S^m(c) \times S^{n-m}(\sqrt{1-c^2})$, $1 \leq m \leq n-1$. $\square$

Proof of Corollary 1.2. From the Gauss equation (2.4) and $r > 1$, we have $n(n-1)H^2 \geq n^2H^2 - S = n(n-1)(r-1) > 0$. Hence, $|H| \geq \sqrt{r-1}$ and the equality holds at umbilical points. Since $|H| \geq \sqrt{r-1} > 0$, we have $H \neq 0$. From $|H| \geq \sqrt{r-1}$, we infer

$$\begin{aligned}
&- \{2n(n-1) + n^2(n-1)(r-1)\} \min |H| \\
&+ n(n-1)(r-1) \{(n-1)(r-1) + 1\} \frac{1}{\min |H|} \\
&\leq -n(n-1)r \min |H| \leq -n(n-1)r\sqrt{r-1}.
\end{aligned}$$

Therefore, we know that the first eigenvalue of the Jacobi operator J_s satisfies $\lambda_1^{J_s} \leq -n(n-1)r\sqrt{r-1}$. When equality holds, we know

$$\begin{aligned}
\lambda_1^{J_s} &= - \{2n(n-1) + n^2(n-1)(r-1)\} \min |H| \\
&\quad + n(n-1)(r-1) \{(n-1)(r-1) + 1\} \frac{1}{\min |H|} \\
&= -n(n-1)r\sqrt{r-1}.
\end{aligned}$$

Hence, H is constant from Theorem 1.1 and $S = nH^2$. Namely, M is totally umbilical. If M is totally umbilical, from Example 3.1, we know that $\lambda_1^{J_s} = -n(n-1)r\sqrt{r-1}$. $\square$

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