A SCHWARZ LEMMA FOR MEROMORPHIC FUNCTIONS AND ESTIMATES FOR THE HYPERBOLIC METRIC

ALEXANDER YU. SOLYNIN

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ABSTRACT. We prove a generalization of the Schwarz lemma for meromorphic functions f mapping the unit disk $\mathbb{D}$ onto Riemann surfaces $\mathcal{R}$ with bounded in mean radial distances from $f(0)$ to the boundary of $\mathcal{R}$. A new variant of the Schwarz lemma is also proved for the Carathéodory class of analytic functions having positive real part in $\mathbb{D}$. Our results lead to several improved estimates for the hyperbolic metric.

1. INTRODUCTION AND RESULTS

Let f be analytic on the unit disk $\mathbb{D} = \{z : |z| < 1\}$. By the classical Schwarz lemma, the inequalities

$$(1.1) \quad (a) \quad |f'(0)| \leq 1 \quad \text{and} \quad (b) \quad |f(z) - f(0)| \leq |z| \quad \text{for every } z \in \mathbb{D}$$

hold for all f such that

$$(1.2) \quad \sup_{z \in \mathbb{D}} |f(z) - f(0)| \leq 1.$$

In Theorem 1 below, we show that (1.1)(a) remains valid when (1.2) is replaced by a much weaker condition that the geometric mean of the distance from $f(0)$ to the boundary of $f(\mathbb{D})$ is bounded by 1. Theorem 4 gives a similar improvement for the variant of Schwarz lemma for analytic functions with positive real part. Our results lead to several new estimates for the hyperbolic metric of planar domains.

Let f be meromorphic on $\mathbb{D}$ having an expansion

$$(1.3) \quad f(z) = a_p z^p + a_{p+1} z^{p+1} + \dots$$

at $z = 0$. Here p is an integer ≥ 1 and $a_p \neq 0$. Let $\mathcal{R}_f$ be the Riemann surface of f^{-1} and let P_0 be the point on $\mathcal{R}_f$ such that $f^{-1}(P_0) = 0$. We note that P_0 is a branch point of $\mathcal{R}_f$ of order $p - 1$ if $p \geq 2$. For a fixed $\delta > 0$ sufficiently small, the vector $\vec{v}(\varphi) = \delta e^{i\varphi}$, where $0 \leq \varphi < 2\pi p$, can be thought of as a direction vector on $\mathcal{R}_f$. Let $\mathcal{B}$ be the set (at most countable) of all branch points $P \neq P_0$ of $\mathcal{R}_f$.

Let $d_{P_0}(\varphi, f(\mathbb{D}))$ denote the Euclidean distance from P_0 to $\partial\mathcal{R}_f \cup \mathcal{B}$ measured on $\mathcal{R}_f$ in the direction defined by $\vec{v}(\varphi)$. We emphasize that for $1 \leq k < p$, the

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distances $d_{P_0}(\varphi, f(\mathbb{D}))$ and $d_{P_0}(\varphi + 2\pi k, f(\mathbb{D}))$ are measured on different sheets of $\mathcal{R}_f$ and therefore $d_{P_0}(\varphi, f(\mathbb{D})) \neq d_{P_0}(\varphi + 2\pi k, f(\mathbb{D}))$ for such k , in general.

Theorem 1. *Let f be meromorphic on $\mathbb{D}$ having expansion (1.3) at $z = 0$. Then*

$$(1.4) \quad |a_p| \leq \exp \left\{ \frac{1}{2\pi p} \int_0^{2\pi p} \log d_{P_0}(t, f(\mathbb{D})) dt \right\}.$$

Further, equality holds in (1.4) if and only if $f(z) = cz^p$ for some $c \in \mathbb{C} \setminus \{0\}$.

The proof of Theorem 1 is given in Section 3. Section 2 contains necessary results on the reduced modules of simply connected domains and triangles.

In 1907, Landau and Toeplitz proved that (1.1)(a) remains valid when (1.2) is replaced by the condition $\text{Diam } f(\mathbb{D}) \leq 2$. This result, along with many others, was discussed recently in [2]. Theorem 1 implies the following stronger form of the Landau-Toeplitz theorem.

Theorem 2. *Under the assumptions of Theorem 1 suppose, in addition, that the length of every straight line segment embedded into $\mathcal{R}_f$, which passes through P_0 , does not exceed 2. Then $|a_p| \leq 1$ with the sign of equality if and only if $f(z) = cz^p$ for some c such that $|c| = 1$.*

In particular, if $\text{Diam } f(\mathbb{D}) \leq 2$, then the previous conclusion holds.

Proof. By the arithmetic-geometric mean inequality and our assumption,

$$\sqrt{d_{P_0}(t, f(\mathbb{D}))d_{P_0}(t + \pi p, f(\mathbb{D}))} \leq (d_{P_0}(t, f(\mathbb{D})) + d_{P_0}(t + \pi p, f(\mathbb{D}))) / 2 \leq 1$$

for every t , $0 \leq t \leq \pi p$. This together with (1.4) implies that

$$(1.5) \quad |a_p| \leq \exp \left\{ \frac{1}{2\pi p} \int_0^{\pi p} \log (d_{P_0}(t, f(\mathbb{D}))d_{P_0}(t + \pi p, f(\mathbb{D}))) dt \right\} \leq 1.$$

By Theorem 1, equality holds in the first inequality in (1.5) if and only if $f(z) = cz^p$ for some $c \in \mathbb{C}$. Then equality occurs in the second of these inequalities if and only if $|c| = 1$. $\square$

In 1938, L. Ahlfors (see [1]) extended the Schwarz lemma by considering analytic maps from the disk to a Riemann surface. Ahlfors' result shows, in particular, that analytic maps are contractions in the hyperbolic metric. For planar domains, the hyperbolic metric can be defined as follows.

Let Ω be a domain on $\overline{\mathbb{C}}$. If the complement $\overline{\mathbb{C}} \setminus \Omega$ contains at least three points, then Ω has the unit disk $\mathbb{D}$ as its universal covering surface. If $\Pi : \mathbb{D} \rightarrow \Omega$ is a universal covering map, the hyperbolic (or Poincaré) density λ_Ω for Ω is then given by

$$(1.6) \quad \lambda_\Omega(\Pi(z))|\Pi'(z)| = 1/(1 - |z|^2), \quad \text{where } z \in \mathbb{D}, \quad w = \Pi(z).$$

For further properties and applications of the hyperbolic metric we refer to [1] and [4, Ch. 9].

Theorem 3. *Let Ω be a domain in $\overline{\mathbb{C}}$ endowed with the hyperbolic metric λ_Ω . If $z_0 \in \Omega$ is finite, then*

$$(1.7) \quad \lambda_\Omega(z_0) \geq \exp \left\{ -\frac{1}{2\pi} \int_0^{2\pi} \log d_{z_0}(t, \Omega) dt \right\}$$

with the sign of equality if and only if Ω is a disk centered at z_0 .

In particular, if $\frac{1}{2\pi} \int_0^{2\pi} \log d_z(t, \Omega) dt \leq 0$ for all finite $z \in \Omega$, then

$$(1.8) \quad \inf_{z \in \Omega} \lambda_\Omega(z) \geq 1.$$

If, in addition, Ω is bounded, then $\lambda_\Omega(z)$ achieves its minimal value $m \geq 1$ at some point $z_0 \in \Omega$. If $m = 1$, then Ω is a disk of radius 1 centered at z_0 .

Proof. Let $\Pi : \mathbb{D} \rightarrow \Omega$ be a universal covering map such that $\Pi(0) = z_0$. By (1.6), $\lambda_\Omega(z_0) = |\Pi'(0)|^{-1}$. Now (1.7) together with the assertion on the cases of equality follows from Theorem 1 applied to $\Pi(z)$.

If $\frac{1}{2\pi} \int_0^{2\pi} \log d_{z_0}(t, \Omega) dt \leq 0$ for all $z_0 \in \Omega$, then (1.7) also holds for every $z_0 \in \Omega$, which implies (1.8). If, in addition, Ω is bounded, then $\lambda_\Omega(z)$ is continuous in Ω and $\lambda_\Omega(z) \rightarrow \infty$ as $z \rightarrow \partial\Omega$; see [4, Ch. 9]. Therefore $\lambda_\Omega(z)$ achieves its minimum $m \geq 1$ at some point $z_0 \in \Omega$; i.e. $\lambda_\Omega(z_0) = m$. Finally, if $m = 1$, then, for such z_0 , equation (1.7) holds with the sign of equality, which in its turn implies that Ω is a disk of radius 1 centered at z_0 . $\square$

Corollary 1. *If $\lambda_\Omega(z) \leq 1$ for at least one point $z \in \Omega$, then Ω either contains a closed segment of length 2 or it is a disk of radius 1.*

Proof. If Ω contains no closed segments of length ≥ 2 , then $d_z(t, \Omega) + d_z(t + \pi, \Omega) \leq 2$ for all $z \in \Omega$ and every t , $0 \leq t < 2\pi$. Applying (1.7) and the arithmetic-geometric mean inequality as in the proof of Theorem 2, we conclude that $\lambda_\Omega(z) \geq \exp \left\{ -\frac{1}{2\pi} \int_0^{2\pi} \log d_z(t, \Omega) dt \right\} \geq 1$ for every $z \in \Omega$. Moreover, if $\lambda_\Omega(z_0) = 1$ for some $z_0 \in \Omega$, then Ω must be the disk of radius 1 centered at z_0 . $\square$

Next we consider the set $\mathcal{P}$ of all functions f analytic in $\mathbb{D}$ such that $f(0) = 1$ and $\Re f(z) > 0$ for $z \in \mathbb{D}$. Thus, $\mathcal{P}$ is a well-known Carathéodory class of analytic functions having positive real part in $\mathbb{D}$. If $f \in \mathcal{P}$, then

$$(1.9) \quad (a) \quad |f'(0)| \leq 2 \quad \text{and} \quad (b) \quad |f(z) - 1| \leq \frac{2|z|}{1 - |z|} \quad \text{for every } z \in \mathbb{D}.$$

The inequalities in (1.9) play the role of a Schwarz lemma for the class $\mathcal{P}$. Our next theorem, the proof of which is given in Section 5, generalizes this for a much broader class of functions.

Theorem 4. *Let f be analytic in $\mathbb{D}$ with $f(0) = 1$ such that for every $v \in \mathbb{R}$ the set $\{w = t + iv : 0 \leq t \leq 2\} \setminus f(\mathbb{D})$ is not empty. Then inequalities (1.9) remain valid.*

Further, equality holds in (1.9)(a) or in (1.9)(b) for some $z \in \mathbb{D} \setminus \{0\}$ if and only if $f(z) = f_\alpha(z)$ or $f(z) = 2 - f_\alpha(z)$ for some $\alpha \in \mathbb{R}$, where $f_\alpha(z) = (1 - e^{i\alpha}z)/(1 + e^{i\alpha}z)$.

Corollary 2. *If $\Omega \cap \mathbb{R} \neq \emptyset$ and $\lambda_\Omega(x) \leq 1$ for at least one point $x \in \Omega \cap \mathbb{R}$, then Ω either contains a closed vertical segment of length 1, which is symmetric with respect to the real axis, or it is one of the following two half-planes: $H^+ = \{z : \Im z > -1/2\}$ or $H^- = \{z : \Im z < 1/2\}$.*

Proof. Suppose that $\lambda_\Omega(x_0) \leq 1$ for some $x_0 \in \mathbb{R}$. Let $\Pi : \mathbb{D} \rightarrow \Omega$ be a universal covering map such that $\Pi(0) = x_0$ and let $f(z) = 1 + 2i(\Pi(z) - x_0)$. Then by (1.6),

$$(1.10) \quad |f'(0)| = 2|\Pi'(0)| = 2\lambda_\Omega^{-1}(x_0) \geq 2.$$

Since f satisfies the assumptions of Theorem 4, the strict inequality in (1.10) will contradict (1.9)(a). If the last relation of (1.10) holds with the sign of equality,

then by Theorem 4, $f(z) = f_\alpha(z)$ or $f(z) = 2 - f_\alpha(z)$ for some $\alpha \in \mathbb{R}$. The latter shows that Ω must be one of the half-planes H^+ or H^- . $\square$

2. REDUCED MODULI OF CONFORMAL CONFIGURATIONS

Let D be a hyperbolic simply connected domain on a Riemann surface $\mathcal{R}$ and $P_0 \in D$. Let us fix a chart (U, φ) on $\mathcal{R}$ such that φ maps U onto the disk $\mathbb{D}_{\varepsilon_0} = \{\zeta : |\zeta| < \varepsilon_0\}$ for some $\varepsilon_0 > 0$ and $\varphi(P_0) = 0$. If $0 < \varepsilon < \varepsilon_0$, then $D(\varepsilon) = D \setminus \{P \in \mathcal{R} : |\varphi(P)| < \varepsilon\}$ is a doubly-connected domain on $\mathcal{R}$. Let $\text{mod}(D(\varepsilon))$ denote the modulus of $D(\varepsilon)$; i.e., $\text{mod}(D(\varepsilon)) = (1/2\pi) \log(r_2/r_1)$ if $D(\varepsilon)$ is conformally equivalent to an annulus $A(r_1, r_2) = \{z : r_1 < |z| < r_2\}$.

The reduced modulus $m_\zeta(D, P_0)$ of D at P_0 with respect to the uniformizing parameter $\zeta = \varphi(P)$ is then defined by

$$(2.1) \quad m_\zeta(D, P_0) = \lim_{\varepsilon \rightarrow 0} (\text{mod}(D(\varepsilon)) + (1/2\pi) \log \varepsilon),$$

where the finite limit exists; cf. [5, Theorem 2.8].

Let $T = T(v_0, v_1, v_2)$ be a triangle in D having the vertex v_0 at P_0 and the side e_0 opposite to v_0 on ∂D . Here by a triangle $T = T(v_0, v_1, v_2)$ we mean a configuration consisting of a simply connected domain T with three distinct boundary points v_0, v_1, v_2 (called vertices) marked on ∂T . Then the set $\partial T \setminus \{v_0, v_1, v_2\}$ consists of three sides, e_0, e_1 , and e_2 , where e_k is the side opposite to v_k .

Suppose further that for all sufficiently small $\varepsilon > 0$ the set $T \cap \{P : |\varphi(P)| = \varepsilon\}$ consists of a single arc γ_ε and the boundary of $\varphi(T \cap U)$ forms an interior angle ψ , $0 < \psi \leq 2\pi$, at $\zeta = 0$. Then the set $T(\varepsilon) = T \setminus \{P : |\varphi(P)| \leq \varepsilon\}$ may be considered as a quadrilateral having e_0 and γ_ε as a distinguished pair of its sides. Let $\text{mod}(T(\varepsilon))$ be the modulus of $T(\varepsilon)$ with respect to the family of curves separating e_0 from γ_ε inside $T(\varepsilon)$; see [5, Section 2.3].

The reduced modulus $m_\zeta(T; v_0|v_1, v_2)$ of the triangle T at its vertex v_0 with respect to the uniformizing parameter ζ is defined by

$$(2.2) \quad m_\zeta(T; v_0|v_1, v_2) = \lim_{\varepsilon \rightarrow 0} (\text{mod}(T(\varepsilon)) + (1/\psi) \log \varepsilon),$$

provided that the finite limit exists; see [10, § 1.2].

The definition (2.1) of the reduced modulus of a simply connected domain is classical due to O. Teichmüller. The definition (2.2) of the reduced modulus of a triangle is reminiscent of Teichmüller's definition (2.1). It was introduced by this author for planar triangles in [8] and for triangles on Riemann surfaces in [10]. We want to emphasize here that the limit in (2.2) may not exist even if the sides of T are analytic arcs; see [10, Example 1.1]. However, if for all sufficiently small $\varepsilon > 0$, $(\partial\varphi(T \cap U)) \cap \mathbb{D}_\varepsilon$ consists of two straight line segments, an easy symmetry and reflection argument shows that the finite reduced modulus $m_\zeta(T; v_0|v_1, v_2)$ exists.

Let $\mathcal{R}_1$ be a Riemann surface, $Q_0 \in \mathcal{R}_1$, and let (U_1, φ_1) be a chart on $\mathcal{R}_1$ such that $\varphi_1(Q_0) = 0$. If f maps D conformally onto a domain Ω , $Q_0 \in \Omega \subset \mathcal{R}_1$, such that $f(P_0) = Q_0$ and $\tau(\zeta) = \varphi_1(f(\varphi^{-1}(\zeta)))$, then it is well-known that

$$(2.3) \quad m_\tau(\Omega, Q_0) = m_\zeta(D, P_0) + (1/2\pi) \log |\tau'(0)|.$$

A similar formula holds for the change of the reduced modulus of a triangle:

$$(2.4) \quad m_\tau(f(T); Q_0|Q_1, Q_2) = m_\zeta(T; P_0|P_1, P_2) + (1/\psi_1) \log |A_1|;$$

see [10, Lemma 1.3]. In (2.4), we assume that f maps T conformally onto the triangle $f(T)$ such that $f(P_k) = Q_k$, $k = 0, 1, 2$, and that $\tau = \tau(\zeta)$ has the following expansion near $\zeta = 0$:

$$\tau(\zeta) = A_1 \zeta^{\psi_1/\psi} (1 + g(\zeta)),$$

where $A_1 \neq 0$ and g is analytic at $\zeta = 0$ such that $g(0) = 0$.

Next we consider the partitioning of a simply connected domain D or a triangle T on $\mathcal{R}$ into n nonoverlapping triangles $T_1, \dots, T_n$. In case of partitioning of a domain, we assume that $T_k \subset D$ and every $T_k = T_k(v_{0,k}, v_{1,k}, v_{2,k})$ has its vertex $v_{0,k}$ at $P_0 \in D$ and the opposite side $e_{0,k}$ on ∂D . We assume further that T_k has a nonzero angle $2\pi\alpha_k$ at $v_{0,k}$ and the reduced modulus $m_\zeta(T_k; v_{0,k}|v_{1,k}, v_{2,k})$ exists for every $1 \leq k \leq n$.

In the case of the partitioning of a triangle $T = T(v_0, v_1, v_2)$, we assume that $T_k \subset T$, T and T_k have their vertices v_0 and $v_{0,k}$ at the same point $P_0 \in \mathcal{R}$, and the side $e_{0,k}$ opposite to $v_{0,k}$ lies on the side e_0 of T . In addition, we assume that T has the angle $0 < \beta \leq 2\pi$ at v_0 and T_k has the angle $\beta\alpha_k$ at $v_{0,k}$, and the reduced moduli $m_\zeta(T; v_0|v_1, v_2)$ and $m_\zeta(T_k; v_{0,k}|v_{1,k}, v_{2,k})$ exist for all $1 \leq k \leq n$. In all cases under consideration we assume that $0 < \alpha_k \leq 1$ for $1 \leq k \leq n$ and $\sum_{k=1}^n \alpha_k = 1$.

Let f_D map D conformally onto $\mathbb{D}$ such that $f_0(P_0) = 0$ and let f_T map T conformally onto the circular sector $S_\beta(1)$ such that $f_T(v_0) = 0$, $f_T(v_1) = 1$, and $f_T(v_2) = e^{i\beta}$. Here $S_\beta(R) = \{z : 0 < |z| < R, 0 < \arg z < \beta\}$, $0 < R \leq \infty$.

Lemma 1 ([8], [10]). (1) Let $T_1, \dots, T_n$ be a partitioning of D defined above. Then

$$(2.5) \quad m_\zeta(D, P_0) \leq \sum_{k=1}^n \alpha_k^2 m_\zeta(T_k; v_{0,k}|v_{1,k}, v_{2,k}).$$

Further, equality holds in (2.5) if and only if for every k , $1 \leq k \leq n$, $f_D(T_k)$ is a circular sector of opening $2\pi\alpha_k$ and $f_D(e_{0,k})$ is a circular part of $\partial f_D(T_k)$.

(2) Let $T_1, \dots, T_n$ be a partitioning of a triangle T defined above. Then

$$(2.6) \quad m_\zeta(T; v_0|v_1, v_2) \leq \sum_{k=1}^n \alpha_k^2 m_\zeta(T_k; v_{0,k}|v_{1,k}, v_{2,k}).$$

Further, equality holds in (2.6) if and only if for every k , $1 \leq k \leq n$, $f_T(T_k)$ is a circular sector of opening $\beta\alpha_k$ and $f_T(e_{0,k})$ is a circular part of $\partial f_T(T_k)$.

For the proof of (2.5) and (2.6), which uses only basic facts of the theory of extremal metrics, we refer to [10, § 1].

Now we consider a planar triangle $T = T(v_0, v_1, v_2)$ in the z -plane lying in the wedge $S_\alpha(\infty)$, $0 < \alpha \leq 2\pi$. Let T have its vertices at the points $v_0 = 0$, $v_1 = r_1 > 0$, and $v_2 = r_2 e^{i\alpha}$, $r_2 > 0$, and let $e_1 = \{te^{i\alpha} : 0 < t < r_2\}$ and $e_2 = \{t : 0 < t < r_1\}$ be the sides of T issuing from v_0 . Suppose further that T is starlike with respect to its vertex v_0 . The latter means that the open interval $(0, z)$ joining 0 and z is in T whenever $z \in T$.

Lemma 2. Suppose that $T = T(v_0, v_1, v_2)$ is a starlike triangle defined above. Then

$$(2.7) \quad m_z(T; v_0|v_1, v_2) \leq \alpha^{-2} \int_0^\alpha \log d_0(t, T) dt.$$

Equality holds in (2.7) if and only if $T = S_\alpha(R)$ for some $0 < R < \infty$ and v_0, v_1, v_2 are the geometric vertices of $S_\alpha(R)$.

Proof. We may assume that $L_0 = \int_0^\alpha \log d_0(t, T) dt < \infty$. First, we approximate T by a sequence of triangles $T^{(n)} = T^{(n)}(v_0^n, v_1^n, v_2^n)$ such that the side e_0^n of $T^{(n)}$ opposite to $v_0^n = 0$ consists of n circular arcs and $n - 1$ radial segments (possibly degenerate) joining these arcs. The precise construction is as follows.

For a fixed $n \geq 1$ and $1 \leq k \leq n$, let $d_{n,k} = \min d_0(t, T)$, where the minimum is taken over all t such that $\alpha(k-1)/n \leq t \leq \alpha k/n$. The circular sectors

$$T^{n,k} = \{z : |z| < d_{n,k}, \alpha(k-1)/n < \arg z < \alpha k/n\}, \quad 1 \leq k \leq n,$$

will be considered as triangles $T^{n,k}(v_0^{n,k}, v_1^{n,k}, v_2^{n,k})$ with vertices

$$v_0^{n,k} = 0, \quad v_1^{n,k} = d_{n,k} e^{i\alpha(k-1)/n}, \quad v_2^{n,k} = d_{n,k} e^{i\alpha k/n}.$$

Now let $T^{(n)}$ be the interior of the closure of the union $\bigcup_{k=1}^n T^{n,k}$. Then $T^{(n)}$ can be considered as a triangle with vertices $v_0^n = 0$, $v_1^n = d_{n,k}$, and $v_2^n = d_{n,k} e^{i\alpha}$. Since $d_0(t, T^{(n)}) \leq d_0(t, T^{(n+1)}) \leq d_0(t, T)$ for all $0 \leq t \leq \alpha$ and all $n \geq 1$, it follows from the Levi's theorem that

$$(2.8) \quad \frac{\alpha}{n} \sum_{k=1}^n \log d_{n,k} = \int_0^\alpha \log d_0(t, T^{(n)}) dt \rightarrow \int_0^\alpha \log d_0(t, T) dt$$

monotonically as $n \rightarrow \infty$.

Since $T^{(n)} \subset T^{(n+1)}$ and $\bigcup_{n=1}^\infty T^{(n)} = T$, we conclude that the sequence of domains $T^{(n)}$ converges to the kernel T . Then, using the Schwarz reflection principle and Carathéodory's theorem on the convergence to the kernel, we obtain that

$$(2.9) \quad m_z(T^{(n)}; v_0^n | v_1^n, v_2^n) \rightarrow m_z(T; v_0 | v_1, v_2) \quad \text{as } n \rightarrow \infty.$$

Applying Lemma 1(2) with $\alpha_k = 1/n$ to the partitioning of $T^{(n)}$ into the triangles $T^{n,k}$, $k = 1, \dots, n$, we obtain the inequality

$$m_z(T^{(n)}; v_0^n | v_1^n, v_2^n) \leq \frac{1}{n^2} \sum_{k=1}^n m_z(T^{n,k}; v_0^{n,k} | v_1^{n,k}, v_2^{n,k}) = \frac{1}{\alpha n} \sum_{k=1}^n \log d_{n,k},$$

which, when combined with (2.8) and (2.9), implies (2.7).

Suppose now that (2.7) holds with the sign of equality. Let $r(\theta) = \{z : \arg z = \theta\}$. For $0 < \theta < \alpha$, the ray $r(\theta)$ divides T into two triangles, say T_1 and T_2 . Here T_1 has vertices v_0 , v_1 , and $v' = d_0(\theta, T) e^{i\theta}$ and T_2 has vertices v_0 , v' , and v_2 . Let $\beta_1 = \theta/\alpha$, $\beta_2 = (\alpha - \theta)/\alpha$. By Lemma 1(2), we have

$$(2.10) \quad m_z(T; v_0 | v_1, v_2) \leq \beta_1^2 m_z(T_1; v_0 | v_1, v') + \beta_2^2 m_z(T_2; v_0 | v', v_2).$$

Moreover, equality holds in (2.10) if and only if f_T maps the interval $(0, d_0(\theta, T) e^{i\theta})$ onto the radius $(0, e^{i\theta})$. Here f_T is a conformal mapping from T onto the sector $S_\alpha(1)$ such that $f_0(v_0) = 0$, $f_0(v_1) = 1$, and $f_0(v_2) = e^{i\alpha}$.

Applying (2.7) to each of the triangles T_1 and T_2 separately and then combining the results, we obtain

$$(2.11) \quad \beta_1^2 m_z(T_1; v_0 | v_1, v') + \beta_2^2 m_z(T_2; v_0 | v', v_2) \leq \alpha^{-2} \int_0^\alpha \log d_0(t, T) dt.$$

It follows from (2.10) and (2.11) that if (2.7) holds with the sign of equality, we must have equality in (2.10) for every θ , $0 < \theta < \alpha$. Then the equality assertion of Lemma 1(2) implies that f_T maps every radial interval $(0, d_0(\theta, T) e^{i\theta})$ onto the radius $(0, e^{i\theta})$ of the sector $S_\alpha(1)$. Then $\arg(f(z)/z) = 0$ for all $z \in T$. As is well

known, the latter implies that $f(z)/z$ is constant on T . Hence, $f(z) = cz$ for some nonzero $c \in \mathbb{C}$. Therefore, T must be a circular sector of opening α . $\square$

Lemma 3. (1) Let $\mathcal{R}$ be a Riemann surface over the z -sphere $\overline{\mathbb{C}}$, D a simply connected domain on $\mathcal{R}$, and P_0 a point of D over $z = 0$. Assume further that P_0 is a branch point of $\mathcal{R}$ of order $p - 1 \geq 0$. In particular, if $p = 1$, then P_0 is a regular point of $\mathcal{R}$. Let $\zeta = z^{1/p}$ be a uniformizing parameter near P_0 . Then

$$(2.12) \quad m_\zeta(D, P_0) \leq 1/(2\pi p) \int_0^{2\pi p} \log d_{P_0}(t, D) dt.$$

Equality holds in (2.12) if and only if D is a p -sheeted disk on $\mathcal{R}$ centered at P_0 .

(2) Let $\mathcal{R}$ be a Riemann surface over $\overline{\mathbb{C}}$, v_0 a point of $\mathcal{R}$ over $z = 0$, and $T = T(v_0, v_1, v_2)$ a triangle on $\mathcal{R}$ such that its sides e_1 and e_2 are straight line intervals on $\mathcal{R}$ forming an angle $\alpha > 0$ at the vertex v_0 . Let v_0 be a branch point of $\mathcal{R}$ of order $p - 1 \geq 0$ and $\zeta = z^{1/p}$ a uniformizing parameter near v_0 . Then $0 < \alpha \leq 2\pi p$ and

$$(2.13) \quad m_\zeta(T; v_0 | v_1, v_2) \leq \alpha^{-2} \int_0^\alpha \log d_{v_0}(t, T) dt.$$

Equality holds in (2.13) if and only if T is a circular sector on $\mathcal{R}$ of opening α .

Proof. (a) Let g_D be a conformal map from $\mathbb{D}$ onto D such that $g_D(0) = P_0$. Let $r_n \rightarrow 1$ as $n \rightarrow \infty$ be an increasing sequence such that g_D has no critical points on the circle C_{r_n} , where $C_r = \{z : |z| = r\}$. Let $D_n = g_D(\mathbb{D}_{r_n})$ and $\gamma_n = g_D(C_{r_n})$. Let $E_n = \{z_P : P \in \hat{E}_n\}$, where z_P is the affix of $P \in \mathcal{R}$ and $\hat{E}_n$ is the set consisting of all branch points of $\mathcal{R}$ lying in D_n and all points P of γ_n such that the tangent line to γ_n at P passes through $z = 0$. Since γ_n is an analytic curve, E_n is finite. Let $\mathcal{L} = \mathbb{R}_0 \cup \mathcal{L}'$, where $\mathcal{L}' = \bigcup L$ and the union is taken over all closed rays L on the z -sphere, which issue from the origin and pass through z_P for some $P \in E_n$. Let $\hat{\mathcal{L}}$ be the lift of $\mathcal{L}$ onto $\mathcal{R}$. Then the set $D'_n = D \setminus \hat{\mathcal{L}}$ contains a finite number of connected components having P_0 on the boundary. Let $T_1, \dots, T_N$ be the set of all such connected components enumerated in such a way that T_k lies over the sector $S_k = \{z : \theta_{k-1} < \theta < \theta_k\}$, where $0 = \theta_0 < \theta_2 < \dots < \theta_N = 2\pi p$.

Then every T_k can be considered as a planar triangle $T_k = T_k(v_{0,k}, v_{1,k}, v_{2,k})$ in S_k having its vertex $v_{0,k}$ at $z = 0$ and the opposite side $e_{0,k}$ joining the boundary rays of S_k . In addition, T_k is starlike with respect to $v_{0,k}$. Let $2\pi p \alpha_k = \theta_k - \theta_{k-1}$ be the angle of T_k at the vertex $v_{0,k}$. Then $0 < \alpha_k \leq 1$ and $\sum_{k=1}^N \alpha_k = 1$. Therefore by Lemma 1(1), we have

$$(2.14) \quad m_\zeta(D_n, P_0) \leq \sum_{k=1}^N \alpha_k^2 m_\zeta(T_k; v_{0,k} | v_{1,k}, v_{2,k}).$$

Using (2.4) and applying Lemma 2, we obtain

$$(2.15) \quad m_\zeta(T_k; v_{0,k} | v_{1,k}, v_{2,k}) = m_z(T_k; v_{0,k} | v_{1,k}, v_{2,k}) \leq \alpha_k^{-2} \int_{\theta_{k-1}}^{\theta_k} \log d_{P_0}(t, T_k) dt$$

for all $1 \leq k \leq N$. This together with (2.14) implies that

$$(2.16) \quad m_\zeta(D_n, P_0) \leq \frac{1}{2\pi p} \int_0^{2\pi p} \log d_{P_0}(t, D_n) dt.$$

By Levi's theorem, $\int_0^{2\pi p} \log d_{P_0}(t, D_n) dt \rightarrow \int_0^{2\pi p} \log d_{P_0}(t, D) dt$ as $n \rightarrow \infty$. Using (2.3), we obtain that $m_\zeta(D_n, P_0) = (1/2\pi) \log(r_n |g'(0)|) \rightarrow m_\zeta(D, P_0)$ as $n \rightarrow \infty$. Therefore, (2.12) follows from (2.16). The proof of inequality (2.13) follows the same lines as above and therefore is omitted.

(b) To prove the equality assertion for (2.12), we slit D along a radial segment $I(\varphi) \subset D$ joining P_0 and ∂D in the direction defined by the vector $\vec{v}(\varphi)$. We assume here that $I(\varphi)$ does not contain branch points of $\mathcal{R}$. Then $I(\varphi)$ exists for a.e. φ , $0 \leq \varphi \leq 2\pi p$. The slit domain $D \setminus I(\varphi)$ may be considered as a triangle $T = T(v_0, v_1, v_2)$ on $\mathcal{R}$ with the vertex v_0 at P_0 , whose sides e_1 and e_2 are the two sides of the segment $I(\varphi)$. Applying (2.5) and (2.13), we obtain

$$m_\zeta(D, P_0) \leq m_\zeta(T; v_0 | v_1, v_2) \leq (1/2\pi p) \int_0^{2\pi p} \log d_{P_0}(t, D) dt.$$

Equality holds in the first of these inequalities if and only if f_D^{-1} maps $I(\varphi)$ onto a radius of $\mathbb{D}$. Since this holds for a.e. $0 \leq \varphi \leq 2\pi p$, it follows that $\arg(f_D(z)/z)$ is constant for all $z \in \mathbb{D} \setminus \{0\}$. Since P_0 is a branch point of $\mathcal{R}$ of order $p \geq 0$, the latter implies that $f(z) = cz^p$ for some $c \neq 0$. Then, of course, D is a p -sheeted disk centered at P_0 . The proof of the equality statement for (2.13) is similar. $\square$

3. PROOF OF THEOREM 1

Let f satisfy the assumptions of the theorem and let $\zeta = w^{1/p}$ be a uniformizing parameter near P_0 . By (2.3), $\log |a_p| = 2\pi m_\zeta(f(\mathbb{D}), P_0)$, which when combined with inequality (2.12) implies (1.4). Using the equality statement of Lemma 3(1), we conclude that equality occurs in (1.4) if and only if $f(\mathbb{D})$ is a p -sheeted disk having a single branch point at P_0 . Then, of course, $f(z) = cz^p$ for some $c \in \mathbb{C} \setminus \{0\}$.

4. THE HYPERBOLIC METRIC AND POLARIZATION

To prove Theorem 4, we will use a polarization comparison theorem for the hyperbolic metric [9]. Let L be a directed straight line on $\mathbb{C}$ and let H^+ and H^- be the left half-plane and the right half-plane with respect to L , respectively. For $z \in \overline{\mathbb{C}}$, let z^* denote the reflection of z in L . The polarization E_L of a given set $E \subset \overline{\mathbb{C}}$ with respect to L is then defined by

$$(4.1) \quad E_L = ((E \cup E^*) \cap \overline{H}^+) \cup ((E \cap E^*) \cap H^-),$$

where $E^* = \{z : z^* \in E\}$.

Theorem 5 ([9]). *Let D be a domain on $\overline{\mathbb{C}}$ carrying the hyperbolic metric $\lambda_D(z)$. Let D_L be the polarization of D with respect to L . Then D_L is an open (not necessarily connected) set on $\overline{\mathbb{C}}$ having a hyperbolic metric λ_{D_L} , and the following inequalities hold:*

$$(4.2) \quad \lambda_{D_L}(z) \lambda_{D_L}(z^*) \leq \lambda_D(z) \lambda_D(z^*),$$

$$(4.3) \quad \min\{\lambda_{D_L}(z), \lambda_{D_L}(z^*)\} \leq \min\{\lambda_D(z), \lambda_D(z^*)\}.$$

Further, equality holds in (4.2) for some $z \in D$ such that $z^* \in D$ or in (4.3) for some $z \in D$ if and only if D_L coincides with D up to symmetry with respect to L .

Theorem 5 implies earlier comparison and symmetrization results on the hyperbolic metric due to V. Jørgensen, D. Minda, and A. Weitsman; see a recent survey [1]. Combining Theorem 5 with appropriate conformal mappings, we easily obtain

modifications of inequalities (4.2) and (4.3) for polarizations with respect to circles, ellipses, and some other curves.

5. PROOF OF THEOREM 4

Let $\mathcal{F}$ be the set of all functions f analytic in $\mathbb{D}$ satisfying the assumptions of the theorem. Then $\mathcal{F}$ is a normal family. Indeed, if $f \in \mathcal{F}$, then f omits points $w_0 = \infty$, $w_1 = r_1 + i$, and $w_2 = r_2 - i$ for some $0 \leq r_1, r_2 \leq 2$. Let $\mathcal{G} = \{g = \varphi_f \circ f : f \in \mathcal{F}\}$, where $\varphi_f(w) = (w_1 - w_2)^{-1}(2w - (w_1 + w_2))$. Then every $g \in \mathcal{G}$ omits in $\mathbb{D}$ three points $w = \infty$, $w = 1$, and $w = -1$. Hence, $\mathcal{G}$ is normal by Montel's theorem. Then it is clear that the set $\mathcal{F} = \{\varphi_f^{-1} \circ g\}$ is also normal.

(a) To prove (1.9)(a), we consider the following maximization problem: Find

$$(5.1) \quad \sup |f'(0)| \quad \text{over all } f \in \mathcal{F}.$$

Since $\mathcal{F}$ is normal and $f(0) = 1$ for all $f \in \mathcal{F}$, it follows that the supremum in (5.1) is finite and there is an $F \in \mathcal{F}$ such that $|F'(0)| = \sup_{f \in \mathcal{F}} |f'(0)|$. We claim that $F = f_\alpha$ or $F = 2 - f_\alpha$ for some $\alpha \in \mathbb{R}$.

Let $\Omega = F(\mathbb{D})$. Since F is extremal for (5.1), Littlewood's principle of subordination (see [3, Theorem 2.21]) implies that F is a universal covering of Ω by $\mathbb{D}$. Hence, $|F'(0)| = \lambda_\Omega^{-1}(1)$. Let $M = \{w : 0 \leq \Re w \leq 2\} \setminus F(\mathbb{D})$, $\alpha = \inf\{\Re w : w \in M\}$, and $\beta = \sup\{\Re w : w \in M\}$. If $\alpha = \beta$, then $\alpha \neq 1$ and Ω must be the half-plane $\{w : \Re w > \alpha\}$ if $\alpha < 1$ or the half-plane $\{w : \Re w < \alpha\}$ if $\alpha > 1$. In this case, the theorem easily follows.

Assume that $\alpha < \beta$. Let $u_0 = 1$ if $\alpha < 1 < \beta$ and $u_0 = (\alpha + \beta)/2$ otherwise. Let $L(u_0)$ be the vertical line through $w = u_0$ directed upward if $0 < u_0 \leq 1$ and downward if $1 < u_0 < 2$. Let Ω_1 be the polarization of Ω with respect to $L(u_0)$ and let F_1 be the universal covering of Ω_1 by $\mathbb{D}$ such that $F_1(0) = 1$. From (4.1), it is clear that $F_1 \in \mathcal{F}$. Using (1.6) and (4.3), we obtain that $|F_1'(0)| = \lambda_{\Omega_1}^{-1}(1) \geq \lambda_\Omega^{-1}(1) = |F'(0)|$. Hence, F_1 is also an extremal for (5.1) and we must have $\lambda_{\Omega_1}(1) = \lambda_\Omega(1)$. Then Ω_1 coincides with Ω up to symmetry with respect to $L(u_0)$ by the equality statement of Theorem 5.

Let $\Omega_2 = \Omega_1 \cup \{w : \Re w > u_0\}$ if $u_0 \leq 1$ and $\Omega_2 = \Omega_1 \cup \{w : \Re w < u_0\}$ if $u_0 > 1$. Then, of course, $\Omega_2 \supset \Omega_1$, and it follows from our assumptions that $\Omega_2 \neq \Omega_1$. In particular, $\lambda_{\Omega_2}(1) < \lambda_{\Omega_1}(1)$.

Since Ω coincides with its polarization Ω_1 up to symmetry with respect to $L(u_0)$, it follows from the definition of polarization that $\{w = t + iv : \alpha \leq t \leq u_0\} \setminus \Omega_2 \neq \emptyset$ for every v if $u_0 \leq 1$ and $\{w = t + iv : u_0 \leq t \leq \beta\} \setminus \Omega_2 \neq \emptyset$ for every v if $u_0 > 1$. This implies that $F_2 \in \mathcal{F}$, where F_2 is the universal covering of Ω_2 by $\mathbb{D}$ such that $F_2(0) = 1$. Since F_1 is an extremal for (5.1), we have $\lambda_{\Omega_2}^{-1}(1) = |F_2'(0)| \leq |F_1'(0)| = \lambda_{\Omega_1}^{-1}(1)$ contradicting the inequality $\lambda_{\Omega_2}(1) < \lambda_{\Omega_1}(1)$. This completes the proof of (1.9)(a) together with the cases of equality.

(b) For $0 < r < 1$, let $M_f(r) = \max_{0 \leq \theta \leq 2\pi} |f(re^{i\theta}) - 1|$. To prove (1.9)(b), we consider, for any given $0 < r < 1$, the problem of finding maximum

$$(5.2) \quad M(r) := \max M_f(r) \quad \text{over all } f \in \mathcal{F}.$$

Since $\mathcal{F}$ is normal there is an $F \in \mathcal{F}$ extremal for (5.2). Let $\Omega = F(\mathbb{D})$ and let α , β , Ω_1 , Ω_2 , and F_1 , F_2 be the bounds, domains, and functions defined as in part (a) for our new F . If $\alpha = \beta$, then the theorem easily follows.

Assume that $\alpha < \beta$. It follows from Theorem 14 in [9] that $M_{F_1}(r) > M_F(r)$ unless Ω_1 coincides with Ω up to symmetry with respect to $L(u_0)$. Hence, F_1 is also an extremal for the problem (5.2). Then using the subordination principle (see [4, Theorem 2.21]), we conclude that $M_{F_2}(r) > M_{F_1}(r)$ unless $\Omega_2 = \Omega_1$. Since $\Omega_2 \neq \Omega_1$ in the case under consideration, the latter contradicts the extremality of F_1 for (5.2). Now the desired conclusion follows.

6. REMARKS

The method discussed in Section 2 was developed in [8], [10]. A variant of this technique was used in the preprint [7] to prove the following analog of (1.4) for the inner radius:

$$(6.1) \quad \log r(D, z_0) \leq (1/2\pi) \int_0^{2\pi} \log d_{z_0}(t, D) dt.$$

We remind the reader that the inner radius $r(D, z_0)$ of a domain D having Green's function $g(z, z_0)$ with a pole at $z_0 \in D$ is defined by

$$\log r(D, z_0) = \lim_{z \rightarrow z_0} (g(z, z_0) + \log |z - z_0|).$$

In the case when D is simply connected and starlike with respect to z_0 , inequality (6.1) is due to G. Pólya and G. Szegő [6, p. 225].

The standard way to extend the Schwarz lemma for functions with a multiple point at $z = 0$ is to factor $f(z) - f(0)$ as $z^p g(z)$ and then apply the basic case of the lemma to $g(z)$. We note that this approach does not work in the context of Theorem 1 since the factor $g(z)$ does not satisfy the requirements of this theorem in general.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, TEXAS TECH UNIVERSITY, BOX 41042,
LUBBOCK, TEXAS 79409

E-mail address: alex.solynin@ttu.edu