L^p BOUNDS FOR OSCILLATORY HYPER-HILBERT TRANSFORM ALONG CURVES

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ABSTRACT. We study the oscillatory hyper-Hilbert transform

$$(1) \quad H_{n,\alpha,\beta}f(x) = \int_0^1 f(x - \Gamma(t))e^{it^{-\beta}}t^{-1-\alpha}dt$$

along the curve $\Gamma(t) = (t^{p_1}, t^{p_2}, \dots, t^{p_n})$, where $p_1, p_2, \dots, p_n, \alpha, \beta$ are some real positive numbers. We prove that if $\beta > (n+1)\alpha$, then $H_{n,\alpha,\beta}$ is bounded on L^p whenever $p \in (\frac{2\beta}{2\beta-(n+1)\alpha}, \frac{2\beta}{(n+1)\alpha})$. Furthermore, we also prove that $H_{n,\alpha,\beta}$ is bounded on L^2 when $\beta = (n+1)\alpha$. Our work improves and extends some known results by Chandarana in 1996 and in a preprint. As an application, we obtain an L^p boundedness result for some strongly parabolic singular integrals with rough kernels.

1. INTRODUCTION

The study of the Hilbert transform

$$H_{\Gamma}f(x) = P.V. \int_{-\infty}^{+\infty} f(x - \Gamma(t))\frac{dt}{t}$$

along an appropriate curve $\Gamma(t)$ in R^n is a quite interesting topic that has attracted many mathematicians. Fabes and Rivire [3] (see also [18]) were led to the study of H_{Γ} in their attempt to generalize the method of rotation by Caldern and Zygmund. The main contribution was made by Wainger and his colleagues. Their research has lasted for more than two decades and continues to impact the direction of the modern harmonic analysis. Readers can see [7, 9, 10, 16, 17] among numerous references; in particular, the good survey papers [14] by Stein and Wainger, and [18] by Wainger.

Another interesting operator in harmonic analysis is the hyper-Hilbert transform

$$H_{\alpha}f(x) = P.V. \int_{-1}^1 f(x - t)\frac{dt}{t|t|^{\alpha}}, \quad 0 < \alpha < 1.$$

One can prove that the operator H_{α} is bounded from the Sobolev space L^p_{α} to the Lebesgue space L^p , $1 < p < \infty$, because of the mean zero of the kernel of

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H_α . On the other hand, one naturally expects that, without the assumption of the mean zero on the kernel, the worsened singularity of H_α near the origin can be counterbalanced by an oscillating factor $e^{i|t|^{-\beta}}$ ($\beta > 0$) as t approaches zero. Such thought motivated the study of the oscillatory hyper-Hilbert transform (see [6]) and the strongly singular integral operators in high dimensional spaces. For more details of the strongly singular integral operators, the reader can see Fefferman [4], Fefferman and Stein [5], Wainger [15], and Stein [12].

The above mentioned operators now lead us to consider the operator

$$H_{2,\alpha,\beta}f(x,y) = \int_{-1}^1 f(x-t, y-\gamma(t))e^{i|t|^{-\beta}} \frac{dt}{t|t|^\alpha}, \quad \alpha, \beta > 0,$$

where $\gamma(t) = |t|^{p_2}$ or $\text{sgn}(t)|t|^{p_2}$, $p_2 > 0$, initially on $f \in C_0^\infty$. Actually such an operator is defined as an improper integral so that the denominator $t|t|^\alpha$ can be replaced by $|t|^{1+\alpha}$, if $\beta > \alpha$. Zielinski [19] studied the L^2 boundedness of $H_{2,\alpha,\beta}$ along the parabola (t, t^2) , and proved that

$$\|H_{2,\alpha,\beta}f\|_2 \leq C\|f\|_2 \Leftrightarrow \beta \geq 3\alpha.$$

Recently Chandarana proved the following theorem.

Theorem A ([1]). *Suppose that $\gamma(t) = |t|^{p_2}$ or $\text{sgn}(t)|t|^{p_2}$, $p_2 \geq 2$, and*

$$H_{2,\alpha,\beta}f(x,y) = \int_{-1}^1 f(x-t, y-\gamma(t))e^{i|t|^{-\beta}} \frac{dt}{t|t|^\alpha}, \quad \alpha, \beta > 0.$$

Then

- (i) $\|H_{2,\alpha,\beta}f\|_2 \leq A\|f\|_2$ if and only if $\beta \geq 3\alpha$;
- (ii) $\|H_{2,\alpha,\beta}f\|_p \leq B\|f\|_p$ whenever $\beta > 3\alpha$ and

$$1 + \frac{3\alpha(\beta+1)}{\beta(\beta+1) + \beta - 3\alpha} < p < 1 + \frac{\beta(\beta+1) + \beta - 3\alpha}{3\alpha(\beta+1)}.$$

In the case R^3 , he considered the following operator:

$$H_{3,\alpha,\beta}f(x,y,z) = \int_{-1}^1 f(x-t, y-\gamma_1(t), z-\gamma_2(t))e^{i|t|^{-\beta}} \frac{dt}{t|t|^\alpha},$$

where $\gamma_1(t) = |t|^{p_2}$ or $\text{sgn}(t)|t|^{p_2}$ and $\gamma_2(t) = |t|^{p_3}$ or $\text{sgn}(t)|t|^{p_3}$.

He obtained the following theorem.

Theorem B ([2]). *Suppose that $p_2 \geq 2$, $p_3 \geq p_2 + 1$. Then*

- (i) $\|H_{3,\alpha,\beta}f\|_2 \leq A\|f\|_2$ if and only if $\beta \geq 4\alpha$;
- (ii) $\|H_{3,\alpha,\beta}f\|_p \leq B\|f\|_p$ whenever $\beta > 4\alpha$ and

$$1 + \frac{4\alpha(\beta+1)}{\beta(\beta+1) + \beta - 4\alpha} < p < 1 + \frac{\beta(\beta+1) + \beta - 4\alpha}{4\alpha(\beta+1)}.$$

The purpose of this paper is multiform. First, we present a straight elementary proof on L^p boundedness of $H_{n,\alpha,\beta}$ for any $n \geq 2$. Second, our result not only covers Theorem A and Theorem B (except the “only if” part of the L^2 boundedness), but also removes all restrictions on $\{p_i\}$. Third, we obtain a large range of p for the L^p boundedness (see Remarks 1 and 2).

Let $\theta = (\theta_1, \theta_2, \dots, \theta_n) \in R^n$ and

$$\Gamma_\theta(t) = (\theta_1|t|^{p_1}, \theta_2|t|^{p_2}, \dots, \theta_n|t|^{p_n}) \text{ or } \text{sgn}(t)(\theta_1|t|^{p_1}, \theta_2|t|^{p_2}, \dots, \theta_n|t|^{p_n}).$$

The operator $H_{n,\alpha,\beta}$ we are going to study is defined as

$$(2) \quad H_{n,\alpha,\beta}f(x) = \int_{-1}^1 f(x - \Gamma_\theta(t)) e^{i|t|^{-\beta}} |t|^{-1-\alpha} dt, \quad \beta > \alpha.$$

Our main result is the following theorem.

Theorem 1. *Suppose $p_1, p_2, \dots, p_n, \alpha, \beta$ are positive in the definition of $\Gamma_\theta(t)$. Then*

- (i) $\|H_{n,\alpha,\beta}f\|_p \leq C\|f\|_p$ whenever $\beta > (n+1)\alpha$ and $\frac{2\beta}{2\beta-(n+1)\alpha} < p < \frac{2\beta}{(n+1)\alpha}$.
- (ii) $\|H_{n,\alpha,\beta}f\|_2 \leq C\|f\|_2$ when $\beta = (n+1)\alpha$.

The constants C in (i) and (ii) are independent of f and $\theta \in R^n$.

Remark 1. An easy computation shows that the range of p here is strictly larger than those in Theorem A if $n = 2$ and Theorem B if $n = 3$.

Remark 2. The conditions on p_2, p_3 in Theorem A and Theorem B are all removed, only assuming they are positive.

Remark 3. By scaling, we can assume $\theta = (1, 1, \dots, 1)$ in the definition of $H_{n,\alpha,\beta}$.

Remark 4. By checking the proof, we can assume $\Gamma(t) = (t^{p_1}, t^{p_2}, \dots, t^{p_n})$ and consider the operator $\int_0^1 f(x - \Gamma(t)) e^{it^{-\beta}} t^{-1-\alpha} dt$. From now on we always assume

$$H_{n,\alpha,\beta}f(x) = \int_0^1 f(x - \Gamma(t)) e^{it^{-\beta}} t^{-1-\alpha} dt.$$

Remark 5. It is easy to check that if $p_i = p_j$ when $i \neq j$, then we can reduce $H_{n,\alpha,\beta}$ to $H_{n-1,\alpha,\beta}$. So from now on we assume $p_i \neq p_j$ when $i \neq j$.

Theorem 1 will be proved in next section. In the last section we will give an application of Theorem 1 on strongly parabolic singular integrals with rough kernels.

Throughout this note the letter C always denotes a positive constant which depends only on $p_1, p_2, \dots, p_n, \alpha, \beta$ and n . It may vary in different cases.

2. THE PROOF OF THEOREM 1

To prove Theorem 1, we need some lemmas.

Lemma 2. *Let $h_n(t) = a_0 + \sum_{j=1}^n a_j t^{p_j}$, $p_j \in R$. If $|a_0| + \sum_j |a_j| > 0$, then h_n has at most n zeros in $(0, +\infty)$.*

Proof. Clearly, the lemma is true for $n = 1$. Assume the lemma is true for $n - 1$; we will show that it is also true for n . Without loss of generality we assume $p_1 < p_2 < \dots < p_n$. Note that if $h_n(s_1) = h_n(s_2) = 0$ ($s_1 < s_2$), then there must be some $s \in (s_1, s_2)$ such that $h'_n(s) = \sum_{j=1}^n p_j a_j s^{p_j-1} = s^{p_1-1} (p_1 a_1 + \sum_{j=2}^n p_j a_j s^{p_j-p_1}) = 0$. By the assumption on $n - 1$, h'_n has at most $n - 1$ zeros in $(0, +\infty)$. Thus h_n has at most n zeros in $(0, +\infty)$.

We also need the following version of the Van der Corput Lemma.

Van der Corput Lemma ([13], p. 334). *Let ψ and ϕ be real valued smooth functions on the interval (a, b) and $k \in N$. If ψ satisfies $|\psi^{(k)}(t)| \geq 1$ for all $t \in (a, b)$ and (1) $k = 1, \psi'(t)$ is monotonic on (a, b) , or (2) $k \geq 2$, then we have*

$$\left| \int_a^b e^{i\lambda\psi(t)} \phi(t) dt \right| \leq C_k \lambda^{-1/k} (|\phi(b)| + \int_a^b |\phi'(t)| dt). \quad \square$$

Now we begin to prove Theorem 1.

At first choose a C^∞ function on the real line with support in $[1/2, 2]$ satisfying

$$\sum_{j=-\infty}^{\infty} \phi(2^j t) = 1, \forall t > 0.$$

We can decompose $H_{n,\alpha,\beta}$ as

$$H_{n,\alpha,\beta} f(x) = \sum_{j=0}^{\infty} \int_0^1 \phi(2^j t) f(x - \Gamma(t)) e^{it^{-\beta}} t^{-1-\alpha} dt = \sum_{j=0}^{\infty} H_j f(x)$$

with $\Gamma(t) = (t^{p_1}, t^{p_2}, \dots, t^{p_n})$. A direct computation shows that

$$(3) \quad \|H_j f\|_1 \leq C 2^{j\alpha} \|f\|_1.$$

By virtue of the Fourier transform, we have

$$\hat{H}_j f(\xi) = m_j(\xi) \hat{f}(\xi)$$

where

$$m_j(\xi) = m_j(\xi_1, \xi_2, \dots, \xi_n) = \int_0^1 \phi(2^j t) t^{-1-\alpha} e^{i(t^{-\beta} + \sum_{k=1}^n \xi_k t^{p_k})} dt.$$

The Plancherel theorem implies that

$$\|H_j\|_{L^2(R^n) \rightarrow L^2(R^n)} = \|m_j\|_\infty.$$

Now we estimate the norm of $\|m_j\|_\infty$. Note that using the coordinate transform we can replace $\alpha, \beta, p_1, p_2, \dots, p_n$ by $a\alpha, a\beta, ap_1, ap_2, \dots, ap_n$ for any real positive number a . So without loss of generality we assume $n < p_1 < p_2 < \dots < p_n$.

Set $\psi(t) = t^{-\beta} + \sum_{k=1}^n \xi_k t^{p_k}$; then $m_j(\xi)$ can be written as

$$(4) \quad m_j(\xi) = \int_0^1 \phi(2^j t) t^{-1-\alpha} e^{i\psi(t)} dt.$$

Take $A_m^0(t) = \sum_{k=1}^n P_{p_k}^m \xi_k t^{p_k}$, $m = 1, 2, \dots, n+1$, where $P_q^l = \prod_{i=1}^l (p+1-i)$. A simple computation shows that

$$\psi^{(m)}(t) = (-1)^m P_{\beta+m-1}^m t^{-m-\beta} + A_m^0(t) t^{-m}.$$

Set $I_m = \{t : (-1)^m A_m^0(t) \geq 0\}$. From Lemma 2 we know that I_m is the union of no more than n intervals. Furthermore I_1 can be divided into no more than $2n$ intervals on which $\psi'(t)$ is monotonic. These facts in the Van der Corput Lemma will be used.

Now we claim that

$$(0, 1) \subset \bigcup_{m=1}^{n+1} I_m.$$

If it isn't true, then there exists a $t \in (0, 1)$ such that

$$(-1)^m A_m^0(t) < 0$$

for every $m = 1, 2, \dots, n+1$.

When $h \leq m$, set $A_m^h(t) = \sum_{k=h+1}^n P_{p_k}^{m-h} [\prod_{i=1}^h (p_k - p_i)] \xi_k t^{p_k}$. A direct computation shows that when $h < m$ there holds

$$P_{p_k}^{m-h} - (p_{h+1} + h + 1 - m) P_{p_k}^{m-1-h} = P_{p_k}^{m-h-1} (p_k - p_{h+1}),$$

which means that

$$A_m^{h+1}(t) = A_m^h(t) - (p_{h+1} + h + 1 - m)A_{m-1}^h(t).$$

Thus using induction we get that $(-1)^m A_m^h(t) < 0$ for any $h \leq m$.

On the other hand, from the definition we have

$$\begin{aligned} A_n^{n-1}(t) &= P_{p_n}^1 \left[\prod_{i=1}^{n-1} (p_n - p_i) \right] \xi_n t^{p_n}, \\ A_{n+1}^{n-1}(t) &= P_{p_n}^2 \left[\prod_{i=1}^{n-1} (p_n - p_i) \right] \xi_n t^{p_n}. \end{aligned}$$

It is easy to see that

$$A_n^{n-1}(t) A_{n+1}^{n-1}(t) = P_{p_n}^1 P_{p_n}^2 (\xi_n t^{p_n} \prod_{i=1}^{n-1} (p_n - p_i))^2 > 0,$$

which is a contradiction to the fact that

$$(-1)^n A_n^{n-1}(t) < 0; \quad (-1)^{n+1} A_{n+1}^{n-1}(t) < 0.$$

So we get the desired conclusion, i.e.

$$(5) \quad (0, 1) \subset \bigcup_{m=1}^{n+1} I_m.$$

From the definition, we know that when $t \in I_m$ there holds

$$(6) \quad |\psi^{(m)}(t)| = |(-1)^m P_{\beta+m-1}^m t^{-m-\beta} + A_m^0(t) t^{-m}| \geq C t^{-\beta-m}.$$

Set $J_m = I_m - \bigcup_{i=m+1}^{n+1} I_i$; then $m_1 \neq m_2$ implies $J_{m_1} \cap J_{m_2} = \emptyset$. From (5) we have

$$(0, 1) \subset \bigcup_{m=1}^{n+1} J_m.$$

Now using the Van der Corput Lemma and (6), it can be shown that

$$\begin{aligned} |m_j(\xi)| &\leq \sum_{m=1}^{n+1} \left| \int_{J_m} \phi(2^j t) t^{-1-\alpha} e^{i\psi(t)} dt \right| \\ &\leq C_n \sum_{m=1}^{n+1} 2^{-j \frac{\beta+m}{m}} (1 + \int_{J_m} 2^j |\phi'(2^j t)| t^{-1-\alpha} dt) \\ (7) \quad &\leq C \sum_{m=1}^{n+1} 2^{j(\alpha - \frac{\beta}{m})} \\ &\leq C 2^{j(\alpha - \frac{\beta}{n+1})}. \end{aligned}$$

From the Plancherel theorem we have

$$(8) \quad \|H_j f\|_2 \leq C_n 2^{j(\alpha - \frac{\beta}{n+1})} \|f\|_2.$$

Interpolating between (3) and (8) for $p \in (1, 2)$ we obtain that

$$(9) \quad \|H_j f\|_p \leq C 2^{j[\alpha\theta + (\alpha - \frac{\beta}{n+1})(1-\theta)]} \|f\|_p = C 2^{j(\frac{\beta(\theta-1)}{n+1} + \alpha)} \|f\|_p$$

where $\theta = \frac{2}{p} - 1$. When $\frac{2\beta}{2\beta - (n+1)\alpha} < p \leq 2$ there holds

$$\frac{\beta(\theta - 1)}{n + 1} + \alpha = \frac{2\beta}{n + 1} \left(\frac{1}{p} - 1 \right) + \alpha < 0.$$

So we can get that

$$\begin{aligned} \|H_{n,\alpha,\beta} f\|_p &\leq \sum_{j=0}^{\infty} \|H_j f\|_p \\ (10) \quad &\leq C \sum_{j=0}^{\infty} 2^{j(\frac{2\beta}{n+1}(\frac{1}{p}-1)+\alpha)} \|f\|_p \\ &\leq C \|f\|_p. \end{aligned}$$

When $2 < p < \frac{2\beta}{(n+1)\alpha}$, the L^p boundedness of $H_{n,\alpha,\beta}$ is the direct consequence of the dual argument.

Now we study the operator $H_{n,\alpha,\beta}$ when $\beta = (n+1)\alpha$.

By the Plancherel theorem, to prove the L^2 boundedness of $H_{n,\alpha,\beta}$ we only need to show that

$$(11) \quad |m(\xi)| = \left| \sum_{j=0}^{\infty} m_j(\xi) \right| = \left| \sum_{j=0}^{\infty} \int_0^1 \phi(2^j t) t^{-1-\alpha} e^{i\psi(t)} dt \right| \leq C \text{ for any } \xi \in R^n$$

where $\psi(t) = t^{-\beta} + \sum_{k=1}^n \xi_k t^{p_k}$.

Consider

$$\psi'(t) = -\beta t^{-\beta-1} + \sum_{k=1}^n p_k \xi_k t^{p_k-1}.$$

From Lemma 2 we know that $(0, 1)$ can be divided into no more than $n+2$ intervals on which ψ' is monotonic, so for convenience we assume ψ' is monotonic on $(0, 1)$.

Set

$$\begin{aligned} P_0 &= \{t : \beta t^{-\beta} > 2 \sum_{i=1}^n p_i |\xi_i| t^{p_i}\}; \\ P_k &= \{t : p_k |\xi_k| t^{p_k} > 2(\beta t^{-\beta} + \sum_{i \neq k} p_i |\xi_i| t^{p_i})\}. \end{aligned}$$

From Lemma 2 it can be shown that $P = P_0 \cup \bigcup_{k=1}^n P_k$ is the union of no more than $(n+2)(n+1)$ intervals. On the other hand, from definition when $t \in P$ we can obtain that

$$\psi'(t) > \frac{\beta}{2} t^{-\beta-1}.$$

By the Van der Corput Lemma, we have

$$(12) \quad \left| \int_P \phi(2^j t) t^{-1-\alpha} e^{i\psi(t)} dt \right| \leq C 2^{-j\beta} \left(1 + \int_P 2^j |\phi'(2^j t)| t^{-1-\alpha} dt \right) \leq C 2^{j(\alpha-\beta)}.$$

Now we claim that $(2^{-j-1}, 2^{-j+1})$ is contained in P except for no more than C integers j . If $t \in (0, 1) - P$, then there must exist k and l ($k \neq l$) satisfying one of the following three conditions:

$$\begin{aligned} p_k |\xi_k| t^{p_k} &\leq \beta t^{-\beta} \leq 2np_k |\xi_k| t^{p_k}, \\ \beta t^{-\beta} &\leq p_k |\xi_k| t^{p_k} \leq 2n\beta t^{-\beta}, \\ p_l |\xi_l| t^{p_l} &\leq p_k |\xi_k| t^{p_k} \leq 2np_l |\xi_l| t^{p_l}. \end{aligned}$$

If $(2^{-j-1}, 2^{-j+1}) \cap \{t : p_k |\xi_k| t^{p_k} \leq \beta t^{-\beta} \leq 2np_k |\xi_k| t^{p_k}\} \neq \emptyset$, then we have

$$(13) \quad \frac{p_k |\xi_k|}{\beta} \leq 2^{(j+1)(\beta+p_k)} \text{ and } \frac{2np_k |\xi_k|}{\beta} \geq 2^{(j-1)(\beta+p_k)}.$$

It can be shown that there are no more than C integers j satisfying (13), where C doesn't depend on $|\xi_k|$. A similar argument shows that

$$(14) \quad |\{j : (2^{-j-1}, 2^{-j+1}) \cap ((0, 1) - P) \neq \emptyset\}| \leq C.$$

Using (14) and a similar estimate to (7) we can get that

$$(15) \quad \begin{aligned} & \sum_j \left| \int_{(0,1)-P} \phi(2^j t) t^{-1-\alpha} e^{i\psi(t)} dt \right| \\ & \leq C \sum_{j: (2^{-j-1}, 2^{-j+1}) \cap ((0,1)-P) \neq \emptyset} 2^{j(\alpha - \frac{\beta}{n+1})} \\ & \leq C. \end{aligned}$$

Now from (12) and (15) it can be derived that

$$(16) \quad \begin{aligned} |m(\xi)| & \leq \left| \int_P t^{-1-\alpha} e^{i\psi(t)} dt \right| + \left| \int_{(0,1)-P} t^{-1-\alpha} e^{i\psi(t)} dt \right| \\ & \leq \sum_{j=0}^{\infty} \left| \int_P \phi(2^j t) t^{-1-\alpha} e^{i\psi(t)} dt \right| + \sum_{j=0}^{\infty} \left| \int_{(0,1)-P} \phi(2^j t) t^{-1-\alpha} e^{i\psi(t)} dt \right| \\ & \leq C \sum_{j=0}^{\infty} 2^{j(\alpha-\beta)} + C \\ & \leq C. \end{aligned}$$

So $H_{n,\alpha,\beta}$ is bounded on L^2 when $\beta = (n+1)\alpha$.

3. STRONGLY PARABOLIC SINGULAR INTEGRALS

Let δ_μ be the non-isotropic dilation on R^n defined by

$$\delta_\mu(x) = (\mu^{p_1} x_1, \mu^{p_2} x_2, \dots, \mu^{p_n} x_n) \text{ for } x = (x_1, x_2, \dots, x_n) \in R^n,$$

where $p_1, p_2, \dots, p_n, \mu$ are some real positive numbers. It was known that there is a unique quasi-norm $\rho = \rho(x)$ satisfying

$$\rho(\delta_\mu(x)) = \mu \rho(x), \text{ for all } \mu > 0.$$

In order to study a more general parabolic differential operator with constant coefficients, Fabes and Rivire studied the parabolic singular integral operator

$$(17) \quad Tf(x) = P.V. \int k(y) f(x-y) dy,$$

where the kernel k satisfies

$$\begin{aligned} (1) \quad & k(\delta_\mu(x)) = \mu^{-N} k(x), \quad N = \sum_{i=1}^n p_i; \\ (2) \quad & k \in C^\infty(R^n \setminus \{0\}); \\ (3) \quad & \int_{S^{n-1}} k(x') J(x') d\sigma(x') = 0 \end{aligned}$$

where S^{n-1} is the unit sphere and $J(x') = \sum_{i=1}^n p_i x_i'^2$.

Furthermore, Tf can be written as

$$(18) \quad Tf(x) = P.V. \int_{R^n} f(x-y)\Omega(y')\rho(y)^{-N}dy$$

where $\Omega(y') = k(y')$, $y' = \frac{y}{|y|}$ for any $y \neq 0$.

The following theorem can be found in [8].

Theorem C. *If $\Omega \in L^n L(S^{n-1})$ and satisfies*

$$\int_{S^{n-1}} \Omega(y')J(y')d\sigma(y') = 0,$$

then T is bounded on L^p for $1 < p < \infty$.

In this section we study a strongly rough singular integral $T_{\alpha,\beta}(\alpha, \beta > 0)$ defined by

$$T_{\alpha,\beta}f(x) = \int_{R^n} f(x-y)\Omega(y')e^{i\rho(y)^{-\beta}} \frac{dy}{\rho(y)^{N+\alpha}}.$$

As an application of our main theorem, we have

Theorem 3. *If $\Omega \in L^1(S^{n-1})$, then*

- (i) $\|T_{\alpha,\beta}f\|_p \leq C\|f\|_p$ whenever $\beta > (n+1)\alpha$ and $\frac{2\beta}{2\beta-(n+1)\alpha} < p < \frac{2\beta}{(n+1)\alpha}$.
- (ii) $\|T_{\alpha,\beta}f\|_2 \leq C\|f\|_2$ when $\beta = (n+1)\alpha$.

Proof. By a change of variables in polar type, we have

$$T_{\alpha,\beta}f(x) = \int_{S^{n-1}} \Omega(y')J(y') \int_0^\infty f(x - A_\rho y')e^{i\rho^{-\beta}} \frac{d\rho}{\rho^{N+\alpha}} d\sigma(y'),$$

where $A_\rho y' = (y'_1 \rho^{p_1}, y'_2 \rho^{p_2}, \dots, y'_n \rho^{p_n})$. So

$$\|T_{\alpha,\beta}f\|_p \leq \int_{S^{n-1}} |\Omega(y')J(y')| \left\| \int_0^\infty f(\cdot - \Gamma_{y'}(\rho))e^{i\rho^{-\beta}} \frac{d\rho}{\rho^{N+\alpha}} \right\|_p d\sigma(y').$$

Then this theorem is the direct consequence of Theorem 1 and the Minkowski inequality. $\square$

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