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PRODUCTS OF BOREL SUBGROUPS

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Abstract. We investigate the Borelness of the product of two Borel subgroups in Polish groups. While the intersection of these two subgroups is Polishable, the Borelness of their product is confirmed. On the other hand, we construct two $\mathbf{\Delta}^0_3$ subgroups whose product is not Borel in every uncountable abelian Polish group.

1. Introduction

A *Polish group* is a topological group whose topology is separable and completely metrizable. We are interested in the following question on Polish groups:

> Let G be a Polish group and H, K two Borel subgroups of G. When is their product HK Borel?

The question arises from a fact: if both H, K are closed subgroups, then HK is Borel (see [2], the proof of Theorem 3.1). This fact is surprising to us, compared with a well-known fact that the product of two Borel subsets of a Polish group is analytic, but not necessarily Borel, even if both subsets are closed. Furthermore, there is another easy fact about products of Borel subgroups. If $H \cap K = \{e\}$, then HK is Borel. So there must be a result surpassing these facts. In fact we establish the following theorem.

Theorem 1.1. *Let G be a Polish group and H, K two Borel subgroups of G. If $H \cap K$ is Polishable, then HK is Borel.*

On the other hand, we also prove that the product HK is not necessarily Borel in general cases.

Theorem 1.2. *Let G be an uncountable abelian Polish group. Then there are two $\mathbf{\Delta}^0_3$ subgroups of G such that their product is not Borel.*

We also considered extending this theorem to non-abelian Polish groups. It turned out to be related to a question proposed in [3].

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2. Borelness of products of Borel subgroups

We recall some basic notation before proving Theorem 1.1.

Let G be a Polish group and H a Borel subgroup of G. H is called *Polishable* if it admits a Polish group topology generating the Borel structure inherited from G. As noted in [1], §1.6, a Borel subgroup H of a Polish group G is Polishable if and only if there is a Polish group H^* and a continuous homomorphism $\phi : H^* \to G$ with $\phi(H^*) = H$.

If G is a Polish group and X is a Polish space, an *action* of G on X is a map $a : G \times X \to X$ satisfying, for all $g, h \in G$ and $x \in X$, $a(g, a(h, x)) = a(gh, x)$ and $a(e, x) = x$, where e is the identity element of G. If the action a is continuous, X is called a *Polish G-Space*. When there is no danger of confusion we usually write $g \cdot x$ for $a(g, x)$. A subset $Y \subseteq X$ is *invariant* if $g \cdot y \in Y$ for any $g \in G$ and $y \in Y$. Given an invariant Borel set $B \subseteq X$, Becker and Kechris considered enlarging the topology of X to a Polish topology in which B is open.

Theorem 2.1 (Becker-Kechris [1], §5.1.5)**.** *Let X be a Polish G-space and let $B \subseteq X$ be an invariant Borel set. There exists a Polish topology finer than the original topology of X (and thus having the same Borel structure) in which B is now open and the action is still continuous.*

Let X be a Polish space. The *Effros Borel space* $\mathcal{F}(X)$ is defined by the space of all closed subsets of X with the Borel structure generated by sets of the form

$$\{F \in \mathcal{F}(X) \,:\, F \cap U \neq \emptyset\}$$

for U an open subset of X. We recall that $\mathcal{F}(X)$ is a standard Borel space and the following selection theorem due to Kuratowski-Ryll-Nardzewski (see [4], §12.C).

Theorem 2.2 (Kuratowski-Ryll-Nardzewski)**.** *Let X be a Polish space. Then there is a Borel function $s : \mathcal{F}(X) \to X$ such that for non-empty $F \in \mathcal{F}(X)$, $s(F) \in F$.*

We call the above function s a *Borel selector for $\mathcal{F}(X)$*.

Now we are ready to prove the main theorem in this section.

Theorem 2.3. *Let G be a Polish group and H, K Borel subgroups of G. If $H \cap K$ is a Polishable subgroup of G, then HK is Borel.*

Proof. Fix a Polish group topology on $H \cap K$. Define an action of $H \cap K$ on G^2 as:

$$g \cdot (h, k) = (hg^{-1}, gk).$$

Note that $\mathrm{id}_{H\cap K} : H \cap K \to G$ is a Borel homomorphism, thus continuous; we can see that G^2 is a Polish $(H \cap K)$-space.

Denote by τ the original topology on G^2. Since $G^2 \setminus (H \times K)$ is an invariant Borel subset of G^2, by Becker-Kechris' Theorem, there is a Polish topology t on G^2 finer than τ such that (G^2, t) is still a Polish $(H \cap K)$-space and $G^2 \setminus (H \times K)$ is open; in other words, $H \times K$ is closed in (G^2, t).

For $(h, k) \in H \times K$, the orbit

$$[(h, k)] = \{(h', k') : h'k' = hk\} \cap (H \times K)$$

is closed in (G^2, t). Denote $\phi : H \times K \to \mathcal{F}(G^2)$ (where $\mathcal{F}(G^2)$ is the Effros Borel space defined from t), $\phi(h, k) = [(h, k)]$. We claim that ϕ is Borel. For $U \in t$, we have

$$\phi(h, k) \cap U \neq \emptyset \Leftrightarrow (h, k) \in [U] \cap (H \times K),$$

where $[U]$ stands for the saturation of U. Note that $[U]$ is open in t and t has the same Borel structure with τ, so $[U] \cap (H \times K)$ is Borel. Therefore ϕ is Borel.

Now let $s : \mathcal{F}(G^2) \to G^2$ be a Borel selector. Then

$$T = \{(h,k) : (h,k) = s(\phi(h,k))\}$$

is a Borel transversal for orbits in $H \times K$.

Let $f : T \to G$, $f(h,k) = hk$. Then f is a continuous injection with $f(T) = HK$, so HK is Borel. □

Note that the intersection of two Polishable subgroups is Polishable. Let H, K be two Polishable subgroups and d_1, d_2 compatible complete metrics on H, K respectively. Then $d_1 + d_2$ is a compatible complete metric on $H \cap K$. We have the following corollary.

Corollary 2.4. *Let G be a Polish group and H, K two Polishable subgroups of G. Then HK is Borel.*

3. Non-Borelness of products in abelian Polish groups

In this section, G is an uncountable abelian Polish group. We write the group operation on G as $+$ and 0 is the group identity in G. For $g \in G$, we use $o(g)$ to denote the *order* of g. Letting A be a subset of G, we denote by $\langle A \rangle$ the subgroup of G generated by A.

R. D. Mauldin [5] employed the concept of an independent set to study connected abelian groups with an element of infinite order. He proved that there are Borel subgroups of arbitrarily high Borel complexity in these kinds of groups. We will follow this method to prove Theorem 1.2. We will also extend Mauldin's results to general abelian groups.

Recall that a subset $M \subseteq G$ is *independent* provided that for distinct elements $g_1, \ldots, g_n \in M$ and $b_1, \ldots, b_n \in \mathbb{Z}$, if

$$\sum_{i=1}^{n} b_i g_i = 0,$$

then $b_1 = \cdots = b_n = 0$. Similarly, we say $M \subseteq G$ is *modulus m independent* for some positive integer m if

(i) $o(g) = m$ for any $g \in M$; and
(ii) for distinct elements $g_1, \ldots, g_n \in M$ and $b_1, \ldots, b_n \in \mathbb{Z}$, if $0 \le b_1, \ldots, b_n < m$ and

$$\sum_{i=1}^{n} b_i g_i = 0,$$

then $b_1 = \cdots = b_n = 0$.

For convenience we say that a subset C of a topological space is a *Cantor set* if it is homeomorphic to the Cantor space $2^{\mathbb{N}}$.

Theorem 3.1. *Let G be an uncountable abelian Polish group. Then there is a Cantor set $M \subseteq G$ which is independent or modulus m independent for some positive integer m.*

Proof. Since G is uncountable, we can find a sequence $(h_i)_{i\in\mathbb{N}}$ in G such that $h_i \neq 0$ for $i = 1, 2, \ldots$ and $\lim_i h_i = 0$. Without loss of generality, we can assume that either every h_i has infinite order, or $\lim_i o(h_i) = \infty$, or $o(h_i) = m$ for all $i \in \mathbb{N}$ where m is a positive integer.

Case 1. Every h_i has infinite order, or $\lim_i o(h_i) = \infty$. For any sequence of integers $a_1, a_2, \ldots, a_n$, if at least one $a_i \neq 0$, let

$$T(a_1, a_2, \ldots, a_n) = \{(x_1, x_2, \ldots, x_n) \in G^n : \sum_{i=1}^{n} a_i x_i = 0\}.$$

Without loss of generality, we may suppose $a_1 \neq 0$. Note that $o(h_i, 0, \ldots, 0) = o(h_i)$ in G^n and $(h_i, 0, \ldots, 0)$ is not in $T(a_1, a_2, \ldots, a_n)$ for i large enough. Clearly, $T(a_1, a_2, \ldots, a_n)$ is a closed subgroup of G^n. Since

$$\lim_{i\to\infty} (h_i, 0, \ldots, 0) = (0, 0, \ldots, 0),$$

$(0, 0, \ldots, 0)$ is not an interior point of $T(a_1, a_2, \ldots, a_n)$. It follows that $T(a_1, a_2, \ldots, a_n)$ has empty interior, thus is meager in G^n. Now the existence of a Cantor set of independent elements follows from Mycielski-Kuratowski's theorem (see [6]).

Case 2. $o(h_i) = m$ for all $i \in \mathbb{N}$. Then

$$T(m) = \{x \in G : mx = 0\}$$

is a closed subgroup of G. Since $h_i \in T(m)$ and $\lim_i h_i = 0$, 0 is a limit point in $T(m)$. Then every point in $T(m)$ is a limit point; i.e., $T(m)$ is perfect. Thus $T(m)$ is uncountable.

For convenience, we assume that $T(m) = G$. Then for any sequence of integers $a_1, a_2, \ldots, a_n$, if $0 \leq a_1, a_2, \ldots, a_n < m$ and at least one $a_i \neq 0$, $T(a_1, a_2, \ldots, a_n)$ is a nowhere dense closed subgroup of G^n. By the same argument, there exists a Cantor set of modulus m independent elements. □

Lemma 3.2. *Let X be a K_σ space, Y a Hausdorff space and $f : X \to Y$ a continuous injection. For any $A \subseteq X$ and $2 \leq \alpha < \omega_1$, if A is $\mathbf{\Sigma}^0_\alpha$ ($\mathbf{\Pi}^0_\alpha$ or $\mathbf{\Delta}^0_\alpha$), then $f(A)$ is $\mathbf{\Sigma}^0_\alpha(f(X))$ ($\mathbf{\Pi}^0_\alpha(f(X))$ or $\mathbf{\Delta}^0_\alpha(f(X))$ respectively).*

Proof. First, we assume that A is $\mathbf{\Sigma}^0_2$. Since X is K_σ, so is A. Then $f(A)$ is also K_σ. It is $\mathbf{\Sigma}^0_2$ because Y is Hausdorff. On the other hand, if A is $\mathbf{\Pi}^0_2$, then $X \setminus A$ is $\mathbf{\Sigma}^0_2$. This implies $f(X) \setminus f(A) = f(X \setminus A)$ is $\mathbf{\Sigma}^0_2$ and $f(A)$ is $\mathbf{\Pi}^0_2(f(X))$. Now a routine induction on α will finish the proof. □

Theorem 3.3. *Let G be an uncountable abelian Polish group. Then there exist two $\mathbf{\Delta}^0_3$ subgroups H, K such that $H + K$ is not Borel.*

Proof. Let M be a Cantor set of independent or modulus m independent elements. We split M into two Cantor sets M_1 and M_2 so that $M_1 \cap M_2 = \emptyset$ and $M_1 \cup M_2 = M$. Fix a G_δ subset $A \subseteq M_1$ such that A is homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$, an analytic subset $B \subseteq M_2$ such that B is not Borel and a continuous surjection $\phi : A \to B$. Denote

$$C = \{x + \phi(x) : x \in A\}.$$

Now we define $H = \langle A \rangle$ and $K = \langle C \rangle$. It is easy to see that $H + K = \langle A \cup B \rangle$.

Fix a linear ordering $<$ on M_1 such that $\{(x, y) \in M_1^2 : x < y\}$ is an open subset of M_1^2. We denote

$$[M_1]^n = \{(x_1, \ldots, x_n) \in M_1^n : x_1 < x_2 < \cdots < x_n\}.$$

Then $[M_1]^n$ is an open subset of M_1^n. Since M_1^n is compact, $[M_1]^n$ is K_σ.

Case 1. M is independent. For any sequence of non-zero integers $s = (a_1, \dots, a_n) \in (\mathbb{Z} \setminus \{0\})^{<\mathbb{N}}$, let $f_s : [M_1]^n \to G$ by

$$f_s(x_1, \dots, x_n) = \sum_{i=1}^{n} a_i x_i.$$

By the independentness of M, f_s is a continuous injection and the $f_s([M_1]^n)$'s are pairwise disjoint. Let N_s be a homeomorphic copy of $[M_1]^n$ so that all these N_s's ($s \in (\mathbb{Z}\setminus\{0\})^{<\mathbb{N}}$) are pairwise disjoint. Let $N = \bigcup_s N_s$ be the sum space of the N_s's such that every N_s is a clopen subset of N. Let τ_s be a homeomorphism from N_s to $[M_1]^n$. For $w \in N_s$, define $f(w) = f_s(\tau_s(w))$. Then $f : N \to G$ is a continuous injection. N is K_σ, since every N_s is K_σ. It is easy to see that

$$\langle M_1 \rangle = \bigcup_{s \in (\mathbb{Z}\setminus\{0\})^{<\mathbb{N}}} f_s([M_1]^n) = f(N),$$

so it is an F_σ subgroup. By the same argument, we can see that $\langle M_2 \rangle$ is K_σ.

We denote $A_s = \tau_s^{-1}([A]^n)$. Since $A \subseteq M_1$ is G_δ, note that $[A]^n = [M_1]^n \cap A^n$, we have $A_s = \tau_s^{-1}(A^n)$ is a G_δ subset of N_s. This implies that

$$N \setminus \bigcup_{s \in (\mathbb{Z}\setminus\{0\})^{<\mathbb{N}}} A_s = \bigcup_{s \in (\mathbb{Z}\setminus\{0\})^{<\mathbb{N}}} (N_s \setminus A_s)$$

is F_σ, so $\bigcup_s A_s$ is $\mathbf{\Pi}^0_2$. By Lemma 3.2,

$$H = \langle A \rangle = \bigcup_{s \in (\mathbb{Z}\setminus\{0\})^{<\mathbb{N}}} f_s([A]^n) = f(\bigcup_{s \in (\mathbb{Z}\setminus\{0\})^{<\mathbb{N}}} A_s)$$

is $\mathbf{\Pi}^0_2(\langle M_1 \rangle)$, thus is a $\mathbf{\Delta}^0_3$ subgroup.

Now we turn to study $K = \langle C \rangle$. For $w \in A_s$, let $s = (a_1, \dots, a_n), \tau(w) = (x_1, \dots, x_n) \in A^n$, and define

$$\Phi(w) = \sum_{i=1}^{n} a_i \phi(x_i).$$

Then $\Phi : \bigcup_s A_s \to \langle M_2 \rangle$ is continuous. The graph of Φ,

$$\mathrm{graph}(\Phi) = \{(w, y) \in N \times \langle M_2 \rangle : w \in \bigcup_{s \in (\mathbb{Z}\setminus\{0\})^{<\mathbb{N}}} A_s, y = \Phi(w)\},$$

is a closed subset of $(\bigcup_s A_s) \times \langle M_2 \rangle$, thus is a G_δ subset of $N \times \langle M_2 \rangle$.

Now we define a continuous injection $g : N \times \langle M_2 \rangle \to G$ by $g(w, y) = f(w) + y$. Then $g(N \times \langle M_2 \rangle) = \langle M \rangle$ is F_σ. Note that $K = \langle C \rangle = g(\mathrm{graph}(\Phi))$. By Lemma 3.2, K is $\mathbf{\Pi}^0_2(\langle M \rangle)$, thus is a $\mathbf{\Delta}^0_3$ subgroup.

In the end, since $H + K = \langle A \cup B \rangle$ and $\langle A \cup B \rangle \cap M_2 = B$ is not Borel, $H + K$ is not Borel.

Case 2. M is modulus m independent for some positive integer m. The proof is similar to that in Case 1 but the range of s is $\{1, \dots, m-1\}^{<\mathbb{N}}$. □

Remark 3.4. Notice that both H and K are $\mathbf{\Pi}^0_2$ subsets of some F_σ spaces. So they are intersections of an F_σ set and a G_δ set. We can see that in some sense this is optimal. We know that G_δ subgroups of a Polish group G are closed, and if G is locally compact, F_σ subgroups of G are K_σ. Products of these two kinds of subgroups are always Borel.

Remark 3.5. If we take B in the preceding proof to be $\mathbf{\Sigma}^0_\alpha(\alpha \geq 2)$ or $\mathbf{\Pi}^0_\beta(\beta \geq 3)$, then $H + K = \langle A \cup B \rangle$ is of the same point class as B. In other words, sums of two $\mathbf{\Delta}^0_3$ subgroups can be of arbitrarily high Borel complexity.

4. Further remarks

In this section we pose some problems on products of Borel subgroups.

We first consider products of more than two Borel subgroups. If G is an abelian Polish group and H, K are two Polishable subgroups of G, then there is a continuous homomorphism from the direct product $H \oplus K$ onto HK. So HK is still Polishable. Then Corollary 2.4 can be extended to the product of finitely or even countably many Polishable subgroups. But when G is a non-abelian Polish group, we only have a weaker version of the result.

Lemma 4.1. *Let G be a Polish group. Let H be a closed normal subgroup and K a Polishable subgroup. Then HK is Polishable.*

Proof. This lemma is an easy corollary of Theorem 5.1 in [3]. However, we would like to give a different proof here. Let us consider a group operation on $H \times K$. For $h_1, h_2 \in H$ and $k_1, k_2 \in K$, let

$$(h_1, k_1) \circ (h_2, k_2) = (h_1 k_1 h_2 k_1^{-1}, k_1 k_2).$$

We know that $(H \times K, \circ)$ is a standard semi-direct product of H and K. Given a Polish group topology on K, since $\mathrm{id}_K : K \to G$ is continuous as a Borel homomorphism, we can see that $\circ$ is continuous under the product topology on $H \times K$. So $(H \times K, \circ)$ is a Polish group.

Let $f : H \times K \to G$, $f(h, k) = hk$. Then f is a continuous homomorphism with $f(H \times K) = HK$. Hence HK is Polishable. □

If H, K are two Polishable subgroups of G and H is normal (not necessarily closed), then is HK still Polishable? We do not know the answer.

Corollary 4.2. *Let G be a Polish group and $H_1, H_2, \ldots, H_n$ Polishable subgroups of G. If at least $(n-2)$ many H_i's are closed and normal, then the product $H_1 H_2 \cdots H_n$ is Borel.*

A curious question at this point is, without the assumption of the normality, whether this corollary is still true? Even in an extreme case as follows, the question is left unanswered.

Question 4.3. If H_1, H_2, H_3 are three closed subgroups of a Polish group, is their product $H_1 H_2 H_3$ necessarily Borel?

Another problem we are concerned with is whether every uncountable Polish group contains two Borel subgroups whose product is not Borel. We can extend Theorem 3.3 to some more general cases.

Theorem 4.4. *Let G, H be two Polish groups and $\phi : G \to H$ a continuous homomorphism onto H. If H contains two Borel subgroups whose product is not Borel, so does G.*

Proof. Let K_1, K_2 be two Borel subgroups of H such that K_1K_2 is not Borel. Then $\phi^{-1}(K_1)$ and $\phi^{-1}(K_2)$ are two Borel subgroups of G. We need only prove that $\phi^{-1}(K_1)\phi^{-1}(K_2)$ is not Borel.

Let $N = \ker(\phi)$. Then N is a closed normal subgroup of G. The induced map $\phi^* : G/N \to H$ is a homeomorphism (see [1], Theorem 1.2.6). Let $s_N : G/N \to G$ be a Borel selector for the cosets of N (Dixmier; cf. [1], Theorem 1.2.4); that is, $s_N(gN) \in gN$ for all $g \in G$. We denote $D = \phi^{-1}(K_1)\phi^{-1}(K_2)$. Then $D = \phi^{-1}(K_1K_2)$ and $\phi^* s_N^{-1}(D) = K_1K_2$. Since K_1K_2 is not Borel, neither is D. $\square$

We say that a Polish group G *involves* a Polish group H if there is a closed subgroup G' of G and a continuous homomorphism from G' onto H.

Corollary 4.5. *Let G be a Polish group such that G involves an uncountable abelian Polish group. Then G contains two $\mathbf{\Delta}^0_3$ subgroups whose product is not Borel.*

Remark 4.6. Every uncountable solvable Polish group involves an uncountable abelian Polish group (see [3], Corollary 5.5), so contains two $\mathbf{\Delta}^0_3$ subgroups whose product is not Borel. For an arbitrary uncountable Polish group, this leads to the following question proposed in [3], §6, again.

Question 4.7. Does every uncountable Polish group involve an uncountable abelian Polish group?

As a final remark, we do not know whether the converse of Theorem 1.1 remains true. In other words, does Theorem 1.1 characterize the Polishability of the intersection of two Borel subgroups?

Question 4.8. Let G be an uncountable Polish group and H a non-Polishable Borel subgroup of G. (1) Are there always two Borel subgroups K_1, K_2 of G such that $H = K_1 \cap K_2$ and K_1K_2 is not Borel? Or (2) are there always a Polish group G' and two Borel subgroups K_1, K_2 of G' such that H is Borel isomorphic to $K_1 \cap K_2$ and K_1K_2 is not Borel?

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