ON STRICT INCLUSIONS IN HIERARCHIES OF CONVEX BODIES

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ABSTRACT. Let $\mathcal{I}_k$ be the class of convex k -intersection bodies in $\mathbb{R}^n$ (in the sense of Koldobsky) and $\mathcal{I}_k^m$ be the class of convex origin-symmetric bodies all of whose m -dimensional central sections are k -intersection bodies. We show that 1) $\mathcal{I}_k^m \not\subset \mathcal{I}_k^{m+1}$, $k+3 \leq m < n$, and 2) $\mathcal{I}_l \not\subset \mathcal{I}_k$, $1 \leq k < l < n-3$.

1. INTRODUCTION

Let K and L be origin-symmetric star bodies in $\mathbb{R}^n$. Following Lutwak [L] we say that K is the *intersection body* of L if the radius of K in every direction is equal to the volume of the central hyperplane section of L perpendicular to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\rho_K(\xi) = \text{vol}_{n-1}(L \cap \xi^\perp).$$

The closure in the radial metric of the class of intersection bodies of star bodies gives the class of *intersection bodies*.

A generalization of the concept of an intersection body was introduced by Koldobsky in [K3]. Let $1 \leq k < n$ and let K and L be origin-symmetric star bodies in $\mathbb{R}^n$. We say that K is a *k -intersection body* of L if for every $(n-k)$ -dimensional subspace $H \subset \mathbb{R}^n$

$$\text{Vol}_k(K \cap H^\perp) = \text{Vol}_{n-k}(L \cap H).$$

The closure in the radial metric gives the class of *k -intersection bodies*, which will be denoted by $\mathcal{I}_k$. Note that $\mathcal{I}_1$ is the class of intersection bodies.

Koldobsky [K2] introduced the concept of embedding of a normed spaces in L_p , $p < 0$, and in [K3] he proved that k -intersection bodies are the unit balls of spaces that embed in L_{-k} .

A well-known property of L_p -spaces, proved in [BDK], is that, for any $0 < p < q \leq 2$, the space L_q embeds isometrically in L_p , so L_p -spaces become larger when p decreases from 2. Koldobsky [K2] extended this result to negative p : Every n -dimensional subspace of L_q , $0 < q \leq 2$, embeds in L_p for every $-n < p < 0$.

However, it is an open problem as to whether a normed space $X = (\mathbb{R}^n, \|\cdot\|)$ being embedded in L_{-p} for some $0 < p < n-3$ implies that X embeds in L_{-q} for

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all $p < q < n$. In particular, is it true that every k -intersection body is also an m -intersection body for $1 < k < m < n - 3$? Note that in some cases the above statement is known to be true. Since the product of positive definite distributions is also positive definite, one immediately obtains that if X embeds in L_{-p} , $0 < p < n$, and p divides q , $p < q < n$, then X also embeds in L_{-q} ; see [M1].

Related to these are the questions of showing that these classes of bodies are different for different values of p and q .

It was shown by Koldobsky [K1] that there is an n -dimensional ($n \geq 3$) Banach subspace of $L_{1/2}$ that does not embed in L_1 (also a subspace of $L_{1/4}$ that does not embed in $L_{1/2}$). Later Borwein and his colleagues at the Center for Computational Mathematics at Simon Fraser University showed (by computer methods) that there is a Banach space that embeds in $L_{a/64}$ but not in $L_{(a+1)/64}$ for $a = 1, 2, \dots, 63$. Another construction was given by Kalton and Koldobsky in [KK] and it allowed them to extend these results to all $0 < p < q \leq 1$. Schlieper [S] used the construction from [K1] to show that there is a normed space that embeds in L_{-4} but does not embed in L_{-2} , and also a normed space that embeds in $L_{-1/3}$ but not in $L_{-1/6}$.

In this paper we extend Schlieper's result to arbitrary integers, namely, we construct examples of origin-symmetric convex bodies which are k -intersection bodies, but not l -intersection bodies for $1 \leq l < k < n - 3$. We should remark that all origin-symmetric convex bodies are k -intersection bodies for $k = n - 1, n - 2$ and $n - 3$; see [K4, p. 78].

Another result that we present here is motivated by papers of Weil [W], Neyman [N] and Yaskina [Y]. Weil constructed a convex body in $\mathbb{R}^n$ ($n \geq 3$) that is not a zonoid but all its projections onto hyperplanes are zonoids. Neyman showed that there are n -dimensional normed spaces that do not embed in L_p , but all their $(n - 1)$ -dimensional subspaces embed in L_p for $p > 0$. Yaskina constructed a body in $\mathbb{R}^n$ ($n \geq 5$) which is not an intersection body, but all of its central hyperplane sections are intersection bodies. (Note that all central sections of an intersection body are necessarily intersection bodies; see Fallert, Goodey and Weil [FGW]).

Here we generalize Yaskina's construction to prove the following. Let $\mathcal{I}_k^m$ be the class of convex bodies all of whose m -dimensional central sections are k -intersection bodies. There exists an origin-symmetric convex body $K \subset \mathbb{R}^n$, such that $K \in \mathcal{I}_k^m$ but $K \notin \mathcal{I}_k^{m+1}$, $k + 3 \leq m < n$. One should compare this result with the fact (see e.g. [M1]) that all central sections of a k -intersection body are also k -intersection bodies provided that the dimension of the sections is greater than k . Therefore, $\mathcal{I}_k^{m+1} \subset \mathcal{I}_k^m$.

Finally let us remark that another generalization of intersection bodies was introduced by Zhang [Z]. These are called *generalized k -intersection bodies* by Koldobsky and *k -Busemann-Petty bodies* by E. Milman. See [K3], [K4, Section 4.5], [M1], and [M2] for many interesting results explaining the relation between these different generalizations and their connection to the lower-dimensional Busemann-Petty problem.

2. $\mathcal{I}_k^m \not\subset \mathcal{I}_k^{m+1}$

Let us start with the following criterion for k -intersection bodies.

Theorem 2.1 (Koldobsky [K3]). *Let K be an origin-symmetric star body in $\mathbb{R}^n$, $1 \leq k < n$. K is a k -intersection body if and only if the Fourier transform of $\|x\|_K^{-k}$ is a positive distribution.*

The main result of this section is the following theorem.

Theorem 2.2. *Let $k + 3 \leq m < n$. There exists an origin-symmetric convex body K that belongs to $\mathcal{I}_k^m$, but not to $\mathcal{I}_k^{m+1}$.*

Proof. For a small $\epsilon > 0$ define a body K by

$$\|x\|_K^{-k} = |x|_2^{-k} - 2\epsilon^{m-k}\|x\|_E^{-k}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

where $|x|_2$ is the Euclidean norm and E is the ellipsoid given by

$$\|x\|_E = \left(x_1^2 + \cdots + x_m^2 + \frac{x_{m+1}^2 + \cdots + x_n^2}{\epsilon^2} \right)^{1/2}.$$

Since $\|x\|_E^{-1} \leq |x|_2^{-1}$, we have that $\|x\|_K^{-1}$ is positive for a small ϵ , and so the body K is well defined.

The proof of the theorem follows from the following three lemmas.

Lemma 2.3. *The body K is convex for small enough ϵ .*

Proof. This is a standard perturbation argument; cf. [K4, p.96]. By construction, the body K is obtained by perturbing the Euclidean ball. Since the latter has strictly positive curvature, it is enough to control the first and second derivatives of the function $\epsilon^{m-k}\|x\|_E^{-k}$. One can see that their order is $O(\epsilon^{m-k-2})$, which is small for small enough ϵ . Therefore K also has positive curvature. $\square$

Recall that the Fourier transform of $|x|_2^{-k}$, $0 < k < n$, equals (see [GS, p. 363])

$$(|x|_2^{-k})^\wedge(y) = C_{n,k}|y|_2^{-n+k},$$

where

$$C_{n,k} = \frac{2^{n-k}\pi^{n/2}\Gamma((n-k)/2)}{\Gamma(k/2)}.$$

In order to compute the Fourier transform for the norms of ellipsoids, note that if T is an invertible linear transformation on $\mathbb{R}^n$, then

$$(|Tx|_2^{-k})^\wedge(y) = C_{n,k}|\det T|^{-1}|(T^*)^{-1}y|_2^{-n+k}.$$

Lemma 2.4. *For every m -dimensional subspace H of $\mathbb{R}^n$, the body $K \cap H$ is a k -intersection body.*

Proof. We have

$$\|x\|_{K \cap H}^{-k} = |x|_{B_2 \cap H}^{-k} - 2\epsilon^{m-k}\|x\|_{E \cap H}^{-k}.$$

Since E is an ellipsoid with semiaxes ϵ and 1, $E \cap H$ is also an ellipsoid with semiaxes $a_1, \dots, a_m$ such that $\epsilon \leq a_i \leq 1$, $\forall i = 1, \dots, m$. There is a coordinate system in H such that

$$\|y\|_{K \cap H}^{-k} = (y_1^2 + \cdots + y_m^2)^{-k/2} - 2\epsilon^{m-k} \left(\frac{y_1^2}{a_1^2} + \cdots + \frac{y_m^2}{a_m^2} \right)^{-k/2}.$$

Taking the Fourier transform of $\|y\|_{K \cap H}^{-k}$ in the plane H we get

$$(\|y\|_{K \cap H}^{-k})^\wedge(\xi) = C_{m,k} \left(|\xi|_2^{-m+k} - 2\epsilon^{m-k} \prod_{i=1}^m a_i \cdot (a_1^2 \xi_1^2 + \cdots + a_m^2 \xi_m^2)^{(-m+k)/2} \right).$$

Let a_j be the smallest semiaxis. Then for some $\lambda \geq 1$ we have $a_j = \lambda\epsilon$. Therefore,

$$\prod_{i=1}^m a_i \leq \lambda\epsilon.$$

On the other hand if $\xi \in S^{m-1} \subset H$, then

$$(a_1^2 \xi_1^2 + \dots + a_m^2 \xi_m^2)^{(-m+k)/2} \leq a_j^{-m+k} = (\lambda\epsilon)^{-m+k}.$$

Therefore,

$$2\epsilon^{m-k} \prod_{i=1}^m a_i \cdot (a_1^2 \xi_1^2 + \dots + a_m^2 \xi_m^2)^{(-m+k)/2} \leq 2\epsilon^{m-k} \lambda\epsilon (\lambda\epsilon)^{-m+k} \leq 2\epsilon.$$

So, if $\epsilon \leq 1/2$, then $(\|y\|_{K \cap H}^{-k})^\wedge(\xi) \geq 0$ for all $\xi \in S^{n-1} \cap H$ and all H . Therefore all m -dimensional sections of K are k -intersection bodies. $\square$

Lemma 2.5. *There exists an $(m+1)$ -dimensional section of K which is not a k -intersection body.*

Proof. Let $H = \{x \in \mathbb{R}^n : x_{m+2} = \dots = x_n = 0\}$. Then

$$\|x\|_{K \cap H}^{-k} = (x_1^2 + \dots + x_{m+1}^2)^{-k/2} - 2\epsilon^{m-k} \left(x_1^2 + \dots + x_m^2 + \frac{x_{m+1}^2}{\epsilon^2} \right)^{-k/2}.$$

The Fourier transform in the variables $x_1, \dots, x_{m+1}$ equals

$$\begin{aligned} (\|x\|_{K \cap H}^{-k})^\wedge(\xi) &= C_{m+1,k} \left((\xi_1^2 + \dots + \xi_{m+1}^2)^{(-m+k-1)/2} \right. \\ &\quad \left. - 2\epsilon^{m-k} \epsilon (\xi_1^2 + \dots + \xi_m^2 + \epsilon^2 \xi_{m+1}^2)^{(-m+k-1)/2} \right). \end{aligned}$$

If $\xi = (0, \dots, 0, 1) \in S^m \subset H$, then

$$(\|x\|_{K \cap H}^{-k})^\wedge(\xi) = C_{m+1,k} (1 - 2\epsilon^{m-k} \epsilon \epsilon^{-m+k-1}) = -C_{m+1,k} < 0.$$

Therefore $K \cap H$ is not a k -intersection body. $\square$

3. $\mathcal{I}_l \not\subset \mathcal{I}_k$, $l > k$

We will need a few auxiliary lemmas.

Lemma 3.1. *Let $k \in \mathbb{N} \cup \{0\}$ and $f \in C^\infty(S^{n-1})$, f being even. Let $x = (r, \theta)$ be polar coordinates in $\mathbb{R}^n$, so that $f(\theta)r^{-p} = f\left(\frac{x}{|x|_2}\right)|x|_2^{-p}$. Then the Fourier transform of the distribution $f(\theta)r^{-p}$, $0 < p < n$, is a homogeneous degree $-n+p$ continuous on $\mathbb{R}^n \setminus \{0\}$ function, whose values on the unit sphere can be computed as follows:*

(i) *If $q < 2k$, q is not an odd integer, then $\forall x \in S^{n-1}$,*

$$\begin{aligned} (f(\theta)r^{-n+q+1})^\wedge(x) &= \frac{(-1)^{k+1}\pi}{2\Gamma(2k-q)\sin(\pi(2k-q-1)/2)} \\ &\quad \times \int_{S^{n-1}} |(x, \xi)|^{2k-q-1} \Delta^k (f(\theta)r^{-n+q+1})(\xi) d\xi, \end{aligned}$$

where Δ is the Laplace operator on $\mathbb{R}^n$.

(ii) If q is an even integer, $q = 2k$, then $\forall x \in S^{n-1}$,

$$(f(\theta)r^{-n+2k+1})^\wedge(x) = (-1)^k \pi \int_{S^{n-1} \cap x^\perp} \Delta^k (f(\theta)r^{-n+2k+1})(\xi) d\xi.$$

(iii) If q is an odd integer, $q = 2k - 1 \geq 1$, then $\forall x \in S^{n-1}$,

$$\begin{aligned} (f(\theta)r^{-n+2k})^\wedge(x) &= (-1)^k \int_{S^{n-1}} \ln |(x, \xi)| \Delta^k (f(\theta)r^{-n+2k})(\xi) d\xi \\ &+ (-1)^k (2-n) \int_{S^{n-1}} \Delta^{k-1} (f(\theta)r^{-n+2k})(\xi) d\xi. \end{aligned}$$

Proof. (i) and (ii) are proved in [K4, Lemma 3.16].

(iii) is essentially from [Y, Lemma 2.4]. For completeness we include a proof. For q close to 1 we use part (i) with $k = 1$ to get

$$(1) \quad (f(\theta)r^{-n+q+1})^\wedge(x) = \frac{\pi \int_{S^{n-1}} |(x, \xi)|^{1-q} \Delta(f(\theta)r^{-n+q+1})(\xi) d\xi}{2\Gamma(2-q) \sin \frac{\pi(1-q)}{2}}.$$

When q approaches 1, both the numerator and denominator in the right hand side tend to zero. Indeed, let us show that the limit of the numerator is zero:

$$\lim_{q \rightarrow 1} \int_{S^{n-1}} |(x, \xi)|^{1-q} \Delta(f(\theta)r^{-n+q+1})(\xi) d\xi = \int_{S^{n-1}} \Delta(f(\theta)r^{-n+2})(\xi) d\xi.$$

Recall the relation between the spherical Laplacian Δ_S and Euclidean Laplacian Δ (see e.g. [G, p. 7]). If f is a homogeneous function of degree m , then on the sphere

$$\Delta_S f = \Delta f - m(m+n-2)f.$$

Since $f(\theta)r^{-n+2}$ is a homogeneous function of degree $-n+2$, the previous formula implies $\Delta(f(\theta)r^{-n+2})(\xi) = \Delta_S(f(\theta)r^{-n+2})(\xi)$. Due to the fact that Δ_S is a self-adjoint operator, [G, p. 7], we have

$$\int_{S^{n-1}} \Delta_S(f(\theta)r^{-n+2})(\xi) d\xi = 0.$$

In order to compute the limit of (1) as $q \rightarrow 0$, apply l'Hopital's rule:

$$\begin{aligned} (f(\theta)r^{-n+2})^\wedge(x) &= - \int_{S^{n-1}} \ln |(x, \xi)| \Delta(f(\theta)r^{-n+2})(\xi) d\xi \\ &- \int_{S^{n-1}} \Delta(f(\theta)r^{-n+2} \ln r)(\xi) d\xi. \end{aligned}$$

Computing the Laplacian in the latter integral and using Euler's formula for derivatives of homogeneous functions, we get

$$\begin{aligned} (f(\theta)r^{-n+2})^\wedge(x) &= - \int_{S^{n-1}} \ln |(x, \xi)| \Delta(f(\theta)r^{-n+2})(\xi) d\xi \\ &- (2-n) \int_{S^{n-1}} f(\xi) d\xi. \end{aligned}$$

Using the relation between the Fourier transform and differentiation, and applying the latter formula to the function $\Delta^{k-1}(f(\theta)r^{-n+2k})$ which is homogeneous of

degree $-n + 2$, we have

$$\begin{aligned} (f(\theta)r^{-n+2k})^\wedge(x) &= (-1)^{k-1} (\Delta^{k-1}(f(\theta)r^{-n+2k}))^\wedge(x) \\ &= (-1)^k \int_{S^{n-1}} \ln |(x, \xi)| \Delta^k(f(\theta)r^{-n+2k})(\xi) d\xi \\ &\quad + (-1)^k (2-n) \int_{S^{n-1}} \Delta^{k-1}(f(\theta)r^{-n+2k})(\xi) d\xi. \end{aligned} \quad \square$$

We will need the following spherical version of Parseval's formula; for the proof see [K4, Section 3.4].

Lemma 3.2. *Let K and L be origin-symmetric infinitely smooth star bodies in $\mathbb{R}^n$ and $0 < p < n$. Then*

$$(2) \quad \int_{S^{n-1}} (\|x\|_K^{-p})^\wedge(\xi) (\|x\|_L^{-n+p})^\wedge(\xi) d\xi = (2\pi)^n \int_{S^{n-1}} \|x\|_K^{-p} \|x\|_L^{-n+p} dx.$$

In what follows C will always be a non-zero constant, not necessarily the same in different lines. We also use the notation $a(\epsilon) \sim b(\epsilon)$, meaning that $\lim_{\epsilon \rightarrow 0} a(\epsilon)/b(\epsilon) = 1$.

Lemma 3.3. *Let $p, q > 0$ be integers, $p + q \leq n - 2$.*

(i) *If $n - p - q - 1$ is even, then for all $\xi \in S^{n-1}$*

$$(|x|_2^{-q} \|x\|_E^{-p})^\wedge(\xi) \leq C \epsilon^{-n+p+q+1}.$$

(ii) *If $n - p - q - 1$ is odd, then for every small $\alpha > 0$ there exists a constant C_α , such that for all $\xi \in S^{n-1}$,*

$$(|x|_2^{-q} \|x\|_E^{-p})^\wedge(\xi) \leq C_\alpha \epsilon^{-n+p+q+1/(1+\alpha)}.$$

(iii) *Moreover, in both cases,*

$$(|x|_2^{-q} \|x\|_E^{-p})^\wedge(e_n) \sim C \epsilon^{-n+p+q+1}.$$

Proof. (i) Let $n - p - q - 1 = 2k$. By Lemma 3.1 we have

$$(3) \quad (|x|_2^{-q} \|x\|_E^{-p})^\wedge(\xi) = \pi(-1)^k \int_{S^{n-1} \cap \xi^\perp} \Delta^k(|x|_2^{-q} \|x\|_E^{-p}) dx.$$

Applying Δ under the integral, each time we get a factor of $1/\epsilon^2$. This gives $\epsilon^{-2k} = \epsilon^{-n+p+q+1}$. Finally, use the fact that $\|x\|_E^{-1} \leq 1$ for $x \in S^{n-1}$.

To prove (iii) for this case, we use (3) again. Keeping only the terms of the highest order of ϵ , we get

$$(|x|_2^{-q} \|x\|_E^{-p})^\wedge(e_n) \sim C \int_{S^{n-1} \cap e_n^\perp} \frac{\partial^{2k}}{\partial x_n^{2k}} \|x\|_E^{-p} dx \sim C \frac{\partial^{2k}}{\partial x_n^{2k}} (1 + \frac{x_n^2}{\epsilon^2})^{-p} \Big|_{x_n=0} = C \epsilon^{-2k}.$$

(ii) Let $n - p - q - 1 = 2k - 1$. By Lemma 3.1,

$$\begin{aligned} (4) \quad (|x|_2^{-q} \|x\|_E^{-p})^\wedge(\xi) &= (-1)^k \int_{S^{n-1}} \ln |(x, \xi)| \Delta^k(|x|_2^{-q} \|x\|_E^{-p}) dx \\ &\quad + (-1)^k \int_{S^{n-1}} \Delta^{k-1}(|x|_2^{-q} \|x\|_E^{-p}) dx. \end{aligned}$$

Consider the first integral in (4). As before, applying the Laplacian k times under the integral, we get a factor of $\epsilon^{-2k} = \epsilon^{-n+p+q}$. Therefore we need to estimate terms of the following form:

$$\int_{S^{n-1}} \ln |(\theta, \xi)| \|\theta\|_E^{-n+q} d\theta.$$

By Hölder's inequality, for a small $\alpha > 0$,

$$\begin{aligned} & \left| \int_{S^{n-1}} \ln |(\theta, \xi)| \|\theta\|_E^{-n+q} d\theta \right| \\ & \leq \left(\int_{S^{n-1}} \left| \ln |(\theta, \xi)| \right|^{(1+\alpha)/\alpha} d\theta \right)^{\alpha/(1+\alpha)} \left(\int_{S^{n-1}} \|\theta\|_E^{(-n+q)(1+\alpha)} d\theta \right)^{1/(1+\alpha)} \\ & = C_\alpha \left(\int_{S^{n-1}} \|\theta\|_E^{(-n+q)(1+\alpha)} d\theta \right)^{1/(1+\alpha)}. \end{aligned}$$

Note that Parseval's formula (2) gives

$$\begin{aligned} & \int_{S^{n-1}} \|\theta\|_E^{(-n+q)(1+\alpha)} d\theta \\ & = (2\pi)^{-n} \int_{S^{n-1}} \left(\|x\|_E^{(-n+q)(1+\alpha)} \right)^\wedge (\xi) \left(|x|_2^{n\alpha-q-q\alpha} \right)^\wedge (\xi) d\xi \\ (5) \quad & = C\epsilon \int_{S^{n-1}} \|\theta\|_{E^*}^{n\alpha-q-q\alpha} d\theta, \end{aligned}$$

where E^* is the ellipsoid given by

$$(6) \quad \|x\|_{E^*} = (x_1^2 + \cdots + x_{n-1}^2 + \epsilon^2 x_n^2)^{1/2}.$$

By the following elementary formula (see e.g. [G, p. 9])

$$(7) \quad \int_{S^{n-1}} f((x, \theta)) d\theta = |S^{n-2}| \int_{-1}^1 (1-t^2)^{(n-3)/2} f(t) dt, \quad x \in S^{n-1},$$

the integral in (5) equals

$$\begin{aligned} & = C\epsilon \int_{-1}^1 (1-t^2)^{(n-3)/2} (1-t^2 + \epsilon^2 t^2)^{(n\alpha-q-q\alpha)/2} dt \\ & \sim C\epsilon \int_{-1}^1 (1-t^2)^{(n-3+n\alpha-q-q\alpha)/2} dt = C\epsilon. \end{aligned}$$

The latter integral converges if $(n-q)(1+\alpha) > 1$, which is the case.

Therefore for all small $\alpha > 0$,

$$\left| \int_{S^{n-1}} \ln |(\theta, \xi)| \|\theta\|_E^{-n+q} d\theta \right| \leq C_\alpha \epsilon^{1/(1+\alpha)},$$

and hence the first integral in (4) can be bounded as follows:

$$\left| \int_{S^{n-1}} \ln |(x, \xi)| \Delta^k (|x|_2^{-q} \|x\|_E^{-p}) dx \right| \leq C_\alpha \epsilon^{-n+p+q+1/(1+\alpha)}.$$

We use similar ideas to estimate the second integral in (4). Applying Parseval's formula two times and using the relation between the Fourier transform and

differentiation, we get

$$\begin{aligned} \int_{S^{n-1}} \Delta^{k-1} (|x|_2^{-q} \|x\|_E^{-p}) dx &= \int_{S^{n-1}} \Delta^{\frac{n-p-q-2}{2}} (|x|_2^{-q} \|x\|_E^{-p}) |x|_2^{-2} dx \\ &= (2\pi)^{-n} \int_{S^{n-1}} \left(\Delta^{\frac{n-p-q-2}{2}} (|x|_2^{-q} \|x\|_E^{-p}) \right)^\wedge (\theta) (|x|_2^{-2})^\wedge (\theta) d\theta \\ &= C \int_{S^{n-1}} (|x|_2^{-q} \|x\|_E^{-p})^\wedge (\theta) d\theta = C \int_{S^{n-1}} \|x\|_E^{-p} d\theta = O(1), \end{aligned}$$

since $\|x\|_E^{-1} \leq 1$ on the sphere.

Combining the estimates for both terms in (4), one can see that

$$(|x|_2^{-q} \|x\|_E^{-q})^\wedge(\xi) \leq C_\alpha \epsilon^{-n+p+q+1/(1+\alpha)}.$$

Now we will show that almost the same degree of dependence is achieved when $\xi = e_n$. Indeed, using formula (4) with $\xi = e_n$ and dropping the second integral (which is small compared to the first integral), we get

$$(|x|_2^{-p} \|x\|_E^{-q})^\wedge(e_n) \sim (-1)^k \int_{S^{n-1}} \ln |x_n| \Delta^k (|x|_2^{-q} \|x\|_E^{-p}) dx.$$

Formula (7) applied to the last integral gives

$$= 2(-1)^k \int_0^1 \ln x_n \cdot (1 - x_n^2)^{\frac{n-3}{2}} \Delta^k (|x|_2^{-q} \|x\|_E^{-p}) \Big|_{x_1^2 + \dots + x_{n-1}^2 = 1 - x_n^2} dx_n.$$

After the change of the variable $x_n = \epsilon \cdot z$, the latter equals

$$2(-1)^k \epsilon \int_0^{1/\epsilon} \ln(\epsilon z) \cdot (1 - \epsilon^2 z^2)^{\frac{n-3}{2}} \Delta_\epsilon^k (|x|_E^{-q} |x|_2^{-p}) \Big|_{x_1^2 + \dots + x_{n-1}^2 = 1 - \epsilon^2 z^2, x_n = z} dz,$$

where E^* is defined by (6), and

$$\Delta_\epsilon = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_{n-1}^2} + \frac{1}{\epsilon^2} \frac{\partial^2}{\partial x_n^2}.$$

Note that after writing $\ln(\epsilon z) = \ln \epsilon + \ln z$, we will have two integrals, the first being equal to

$$\begin{aligned} &2(-1)^k \epsilon \ln \epsilon \int_0^{1/\epsilon} (1 - \epsilon^2 z^2)^{\frac{n-3}{2}} \Delta_\epsilon^k (|x|_E^{-q} |x|_2^{-p}) \Big|_{x_1^2 + \dots + x_{n-1}^2 = 1 - \epsilon^2 z^2, x_n = z} dz \\ &= (-1)^k \ln \epsilon \int_{S^{n-1}} \Delta^k (|x|_2^{-q} \|x\|_E^{-p}) dx = (-1)^k \ln \epsilon \int_{S^{n-1}} \Delta (\Delta^{k-1} (|x|_2^{-q} \|x\|_E^{-p})) dx. \end{aligned}$$

Under the integral we have the Laplacian of a homogeneous function of degree $-n + 2$ which equals the spherical Laplacian. Since the spherical Laplacian is a self-adjoint operator, the latter integral is equal to zero.

Therefore we only need to compute the order of the integral

$$2(-1)^k \epsilon \int_0^{1/\epsilon} \ln z \cdot (1 - \epsilon^2 z^2)^{\frac{n-3}{2}} \Delta_\epsilon^k (|x|_E^{-q} |x|_2^{-p}) \Big|_{x_1^2 + \dots + x_{n-1}^2 = 1 - \epsilon^2 z^2, x_n = z} dz.$$

As before, the largest term is obtained when we apply $\frac{1}{\epsilon^2} \frac{\partial^2}{\partial x_n^2}$ to $|x|^{-p}$ successively k times:

$$\sim 2(-1)^k \epsilon^{-2k+1} \int_0^{1/\epsilon} \ln z \cdot (1 - \epsilon^2 z^2)^{\frac{n-3}{2}} \frac{\partial^{2k}}{\partial x_n^{2k}} (|x|^{-p}) \Big|_{x_1^2 + \dots + x_{n-1}^2 = 1 - \epsilon^2 z^2, x_n = z} dz.$$

It is enough to show that

$$\int_0^{1/\epsilon} \ln z \cdot (1 - \epsilon^2 z^2)^{\frac{n-3}{2}} \frac{\partial^{2k}}{\partial x_n^{2k}} (|x|^{-p}) \Big|_{x_1^2 + \dots + x_{n-1}^2 = 1 - \epsilon^2 z^2, x_n = z} dz$$

has a finite non-zero limit as $\epsilon \rightarrow 0$.

One can see that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_0^{1/\epsilon} \ln z \cdot (1 - \epsilon^2 z^2)^{\frac{n-3}{2}} \frac{\partial^{2k}}{\partial x_n^{2k}} (|x|^{-p}) \Big|_{x_1^2 + \dots + x_{n-1}^2 = 1 - \epsilon^2 z^2, x_n = z} dz \\ = \int_0^\infty \ln z \cdot \frac{\partial^{2k}}{\partial z^{2k}} (1 + z^2)^{-p/2} dz. \end{aligned}$$

To finish the proof, we need to show that the latter integral is not equal to zero. Let $P(z^2) = c_0 + c_1 z^2 + \dots + c_{k-1} z^{2k-2}$ be the Taylor polynomial of $(1 + z^2)^{-p/2}$ at zero of order $2k - 2$. Then clearly,

$$\int_0^\infty \ln z \frac{\partial^{2k}}{\partial z^{2k}} (1 + z^2)^{-p/2} dz = \int_0^\infty \ln z \frac{\partial^{2k}}{\partial z^{2k}} \left((1 + z^2)^{-p/2} - P(z^2) \right) dz.$$

After integration by parts $2k$ times and the change of the variable $t = z^2$ the integral becomes

$$\begin{aligned} &= -(2k - 1)! \int_0^\infty z^{-2k} \left((1 + z^2)^{-p/2} - P(z^2) \right) dz \\ &= -\frac{1}{2} (2k - 1)! \int_0^\infty t^{-k-1/2} \left((1 + t)^{-p/2} - P(t) \right) dt. \end{aligned}$$

Using integration by parts in the opposite order and observing that $P(t)$ is the Taylor polynomial of $(1 + t)^{-p/2}$, we get

$$\begin{aligned} &= -\frac{1}{2} \frac{(2k - 1)!}{(1/2)(3/2) \dots (k - 1/2)} \int_0^\infty t^{-1/2} \frac{\partial^k}{\partial t^k} \left((1 + t)^{-p/2} - P(t) \right) dt \\ &= -\frac{1}{2} \frac{(2k - 1)!}{(1/2)(3/2) \dots (k - 1/2)} \int_0^\infty t^{-1/2} \frac{\partial^k}{\partial t^k} \left((1 + t)^{-p/2} \right) dt. \end{aligned}$$

The latter is clearly a non-zero constant. $\square$

Now we are ready to prove the main result of this section.

Theorem 3.4. *For every $1 \leq k < l < n - 3$ there exists an origin-symmetric convex body $K \in \mathbb{R}^n$ that does not belong to $\mathcal{I}_k$, but belongs to $\mathcal{I}_l$.*

Proof. For a small $\epsilon > 0$ define a body K by

$$\|x\|_K^{-1} = |x|_2^{-1} - \epsilon^{n-k-3/2} \|x\|_E^{-1}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

where E is the ellipsoid with the norm

$$\|x\|_E^{-1} = \left(x_1^2 + \dots + x_{n-1}^2 + \frac{x_n^2}{\epsilon^2} \right)^{-1/2}.$$

Convexity of K follows along the lines of Lemma 2.3. (One will need $\epsilon^{n-k-3/2-2}$ to be small, which is the case since $n - 3 > k$.)

Consider the $-l$ th power of the norm of K ,

$$\|x\|_K^{-l} = |x|_2^{-l} - \epsilon^{n-k-3/2} l |x|_2^{-l+1} \|x\|_E^{-1} + R(x),$$

where $R(x)$ is the sum of the terms of the following form:

$$\epsilon^{i(n-k-3/2)}|x|_2^{-l+i}\|x\|_E^{-i}, \quad i \geq 2.$$

Applying the Fourier transform, we have for all $\xi \in S^{n-1}$,

$$(\|x\|_K^{-l})^\wedge(\xi) = C(l, n) - \epsilon^{n-k-3/2}l(|x|_2^{-l+1}\|x\|_E^{-1})^\wedge(\xi) + \widehat{R}(\xi).$$

By Lemma 3.3 the order of the second term is at most

$$\epsilon^{n-k-3/2}\epsilon^{-n+l+1/(1+\alpha)} = \epsilon^{l-k-1/2-\alpha/(1+\alpha)} \rightarrow 0, \text{ as } \epsilon \rightarrow 0,$$

if α is small enough. $\widehat{R}$ is even smaller, since it contains terms of the order $\epsilon^{i(n-k-3/2)}\epsilon^{-n+l+1/(1+\alpha)}$, $i \geq 2$.

Therefore if ϵ is small, then $(\|x\|_K^{-l})^\wedge(\xi) \geq 0$, and so $K \in \mathcal{I}_l$.

Now consider the $-k$ th power of the norm of K ,

$$\|x\|_K^{-k} = |x|_2^{-k} - \epsilon^{n-k-3/2}k|x|_2^{-k+1}\|x\|_E^{-1} + Q(x).$$

Computing the Fourier transform in the direction of $\xi = e_n$, we have

$$(8) \quad (\|x\|_K^{-k})^\wedge(e_n) = C(k, n) - \epsilon^{n-k-3/2}k(|x|_2^{-k+1}\|x\|_E^{-1})^\wedge(e_n) + \widehat{Q}(e_n).$$

$\widehat{Q}(e_n)$ is small, since it has terms of order at most $\epsilon^{i(n-k-3/2)}\epsilon^{-n+k+1}$, $i \geq 2$.

Therefore, we will pay attention only to the second term in (8). By Lemma 3.3,

$$\epsilon^{n-k-3/2}(|x|_2^{-k+1}\|x\|_E^{-1})^\wedge(e_n) \sim C\epsilon^{n-k-3/2}\epsilon^{-n+k+1} = C\epsilon^{-1/2}.$$

If we choose $\epsilon > 0$ small enough so that the latter is greater than $C(k, n)$, then $(\|x\|_K^{-k})^\wedge(e_n) < 0$. So $K \notin \mathcal{I}_k$. $\square$

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