FLATNESS OF THE LINKED GRASSMANNIAN

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ABSTRACT. We show that the linked Grassmannian scheme, which arises in a functorial compactification of spaces of limit linear series, and in local models of certain Shimura varieties, is Cohen-Macaulay, reduced, and flat. We give an application to spaces of limit linear series.

1. INTRODUCTION

The linked Grassmannian was introduced in [9] as a tool for the construction of a moduli scheme of limit linear series, providing a functorial description which also compactified the spaces introduced by Eisenbud and Harris in [3]. It also arises in other contexts: for instance, the length 2 case over $\mathbb{Z}_p$ gives a local model for certain unitary Shimura varieties [4].

A few basic properties of the linked Grassmannian were proved in [9], but only as much as was necessary for the application to limit linear series. In particular, although the dimensions were computed, a substantial question that remained open (Question A.18 of *loc. cit.*) was the flatness of linked Grassmannian schemes over their base. In this paper, we address this question. We perform a detailed analysis in the case of linked Grassmannians of length 2, giving a local description at each point (Theorem 3.2 below). This description allows us to invoke the work of Zhang [11] to conclude that the linked Grassmannians are Cohen-Macaulay, from which we conclude flatness, as well as reducedness. Finally, we use an inductive argument to conclude the same statements for linked Grassmannians of arbitrary length (Theorem 4.2).

As consequences, we are able to conclude (Theorem 4.4) that when a limit linear series space has the expected dimension, it is flat over its base and Cohen-Macaulay. We also see that the linked Grassmannian itself provides a new class of flat degenerations of the Grassmannian; such degenerations have frequently been of interest in the combinatorial study of the Grassmannian, and the linked Grassmannian thus provides a potentially useful new tool for such analysis.

There is some substantial overlap of our results, in the special case of length 2 over $\mathbb{Z}_p$, with the reducedness and flatness results of Goertz [4], in particular section 4.4.5 and Theorem 4.12. However, because we are working over a more general base,

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and our local calculation is rather concise and gives a stronger Cohen-Macaulayness result, we have chosen to keep our presentation self-contained.

2. REVIEW OF THE LINKED GRASSMANNIAN

We briefly review the basic definitions and results of the linked Grassmannian.

Definition 2.1. Let S be an integral, locally Cohen-Macaulay scheme, and let $\mathcal{E}_1, \dots, \mathcal{E}_n$ be vector bundles on S , each of rank d . Given maps $f_i : \mathcal{E}_i \rightarrow \mathcal{E}_{i+1}$ and $g_i : \mathcal{E}_{i+1} \rightarrow \mathcal{E}_i$, for $1 \leq i \leq n-1$, and a positive integer $r < d$, we denote by $\mathcal{LG} := \mathcal{LG}(r, \{\mathcal{E}_i\}_i, \{f_i, g_i\}_i)$ the functor associating to each S -scheme T the set of sub-bundles $V_1, \dots, V_n$ of $\mathcal{E}_{1,T}, \dots, \mathcal{E}_{n,T}$ having rank r and satisfying $f_{i,T}(V_i) \subset V_{i+1}$, $g_{i,T}(V_{i+1}) \subset V_i$ for all i between 1 and $n-1$.

We say that $\mathcal{LG}$ is a **linked Grassmannian** functor if the following further conditions on the f_i and g_i are satisfied:

- (I) There exists some $s \in \mathcal{O}_S$ such that $f_i g_i = g_i f_i$ is scalar multiplication by s for all i .
- (II) Wherever s vanishes, the kernel of f_i is precisely equal to the image of g_i , and vice versa. More precisely, for any i and given any two integers r_1 and r_2 such that $r_1 + r_2 < d$, then the closed subscheme of S obtained as the locus where f_i has rank less than or equal to r_1 and g_i has rank less than or equal to r_2 is empty.
- (III) At any point of S , $\text{im } f_i \cap \ker f_{i+1} = 0$, and $\text{im } g_{i+1} \cap \ker g_i = 0$. More precisely, for any integer r_1 , and any i , we have locally closed subschemes of S corresponding to the locus where f_i has rank exactly r_1 , and $f_{i+1} f_i$ has rank less than or equal to $r_1 - 1$, and similarly for the g_i . Then we require simply that all of these subschemes be empty.

The main theorem of [9] on the linked Grassmannian is the following:

Theorem 2.2 ([9, Lem. A.3, Lem. A.12, Thm. A.15]). *$\mathcal{LG}$ is representable by a scheme LG ; this scheme is naturally a closed subscheme of the obvious product $G_1 \times \dots \times G_n$ of Grassmannian schemes over S , which is smooth of relative dimension $nr(d-r)$. Each component of LG has codimension $(n-1)r(d-r)$ in the product and maps surjectively to S ; in fact, the smooth points of LG are dense in every component of every fiber. If s is non-zero, then LG is also irreducible.*

3. THE MAIN CALCULATION

We now give an explicit local computation of the structure of a linked Grassmannian of length 2.

Notation 3.1. Let $M_{d,S}(2, s)$ be the affine scheme over S parametrizing pairs of $d \times d$ matrices M, N with coefficients in $\mathcal{O}_S$ and with $MN = NM = s$.

Theorem 3.2. *Fix vector bundles $\mathcal{E}_1, \mathcal{E}_2$ of rank d on S , maps $f : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ and $g : \mathcal{E}_2 \rightarrow \mathcal{E}_1$, $r \in \mathbb{N}$ and $s \in \mathcal{O}_S$ satisfying the conditions for a linked Grassmannian of length 2. Then given any field k and a k -valued point $x = (V_1, V_2) \in LG := LG(r, \{\mathcal{E}_1, \mathcal{E}_2\}, \{f, g\})$, there exists a Zariski neighborhood U of x in LG and an isomorphism*

$$U \xrightarrow{\sim} \mathbb{A}_{S'}^m \times M_{\ell, S'}(2, s)$$

sending x to the origin, where $S' \subseteq S$ is the image of U ,

$$\ell = \begin{cases} r - \dim(g(V_2)) - \dim(f(V_1)) : & s = 0 \text{ at } x, \\ 0 : & s \neq 0 \text{ at } x, \end{cases}$$

and $m = r(d - r) - \ell^2$.

Proof. The question is local on the base, so we may assume that $S = \operatorname{Spec} R$ and that $\mathcal{E}_1$ and $\mathcal{E}_2$ are trivial bundles. Furthermore, if $s \neq 0$ at x , then LG is locally isomorphic simply to the Grassmannian $G(r, d)$, so in a neighborhood of x we find that LG is isomorphic to $\mathbb{A}_S^{r(d-r)}$, as asserted. So we may suppose that $s = 0$ at x .

Let $\bar{x}$ be the image of x under the structure morphism from LG to S . Then the spaces (V_1, V_2) determining x are naturally subspaces of $\bar{x}^* \mathcal{E}_1$ and $\bar{x}^* \mathcal{E}_2$, and must satisfy $f(V_1) \subseteq V_2$ and $g(V_2) \subseteq V_1$. Let c be the dimension of the kernel of f on $\bar{x}^* \mathcal{E}_1$, and let d_1 and d_2 be the dimensions of $f(V_1)$ and $g(V_2)$, respectively.

We make the following choices:

- $\{\alpha_1^1, \dots, \alpha_{d_1}^1\}$ in V_1 so that $\{f(\alpha_1^1), \dots, f(\alpha_{d_1}^1)\}$ is a basis for $f(V_1)$;
- $\{\beta_1^1, \dots, \beta_{d_2}^1\}$ in V_2 so that $\{g(\beta_1^1), \dots, g(\beta_{d_2}^1)\}$ is a basis for $g(V_2)$;
- $\{\alpha_1^2, \dots, \alpha_{r-d_1-d_2}^2\}$ in $\bar{x}^* \mathcal{E}_1$ so that $\{f(\alpha_1^2), \dots, f(\alpha_{r-d_1-d_2}^2)\}$ extends the set $\{f(\alpha_1^1), \dots, f(\alpha_{d_1}^1)\}$ to a basis for $\ker g \cap V_2$;
- $\{\beta_1^2, \dots, \beta_{r-d_1-d_2}^2\}$ in $\bar{x}^* \mathcal{E}_2$ so that $\{g(\beta_1^2), \dots, g(\beta_{r-d_1-d_2}^2)\}$ extends the set $\{g(\beta_1^1), \dots, g(\beta_{d_2}^1)\}$ to a basis for $\ker f \cap V_1$;
- $\{\alpha_1^3, \dots, \alpha_{d-r-c+d_2}^3\}$ in $\bar{x}^* \mathcal{E}_1$ so that $\{f(\alpha_1^3), \dots, f(\alpha_{d-r-c+d_2}^3)\}$ extends the set $\{f(\alpha_1^1), \dots, f(\alpha_{d_1}^1), f(\alpha_1^2), \dots, f(\alpha_{r-d_1-d_2}^2)\}$ to a basis for $\ker g$, and
- $\{\beta_1^3, \dots, \beta_{d_1+c-r}^3\}$ in $\bar{x}^* \mathcal{E}_2$ so that $\{g(\beta_1^3), \dots, g(\beta_{d_1+c-r}^3)\}$ extends the set $\{g(\beta_1^1), \dots, g(\beta_{d_2}^1), g(\beta_1^2), \dots, g(\beta_{r-d_1-d_2}^2)\}$ to a basis for $\ker f$.

We lift each α_j^i defined above to a section $\tilde{\alpha}_j^i$ of $\mathcal{E}_1$, and each β_j^i to a section $\tilde{\beta}_j^i$ of $\mathcal{E}_2$. These lifts allow us to define nice bases for these bundles in a neighborhood of $\bar{x}$. Specifically, for i between 1 and d , we define e_i^1 and e_i^2 as follows:

- $e_1^1, \dots, e_{d_1}^1 := \tilde{\alpha}_1^1, \dots, \tilde{\alpha}_{d_1}^1$;
- $e_{d_1+1}^1, \dots, e_{r-d_2}^1 := g(\tilde{\beta}_1^2), \dots, g(\tilde{\beta}_{r-d_1-d_2}^2)$;
- $e_{r-d_2+1}^1, \dots, e_r^1 := g(\tilde{\beta}_1^1), \dots, g(\tilde{\beta}_{d_2}^1)$;
- $e_{r+1}^1, \dots, e_{2r-d_1-d_2}^1 := \tilde{\alpha}_1^2, \dots, \tilde{\alpha}_{r-d_1-d_2}^2$;
- $e_{2r-d_1-d_2+1}^1, \dots, e_{d+r-d_1-c}^1 := \tilde{\alpha}_1^3, \dots, \tilde{\alpha}_{d-r-c+d_2}^3$;
- $e_{d+r-d_1-c+1}^1, \dots, e_d^1 := g(\tilde{\beta}_1^3), \dots, g(\tilde{\beta}_{d_1+c-r}^3)$;
- $e_1^2, \dots, e_{d_1}^2 := f(\tilde{\alpha}_1^1), \dots, f(\tilde{\alpha}_{d_1}^1)$;
- $e_{d_1+1}^2, \dots, e_{r-d_2}^2 := f(\tilde{\alpha}_1^2), \dots, f(\tilde{\alpha}_{r-d_1-d_2}^2)$;
- $e_{r-d_2+1}^2, \dots, e_r^2 := \tilde{\beta}_1^1, \dots, \tilde{\beta}_{d_2}^1$;
- $e_{r+1}^2, \dots, e_{2r-d_1-d_2}^2 := \tilde{\beta}_1^2, \dots, \tilde{\beta}_{r-d_1-d_2}^2$;
- $e_{2r-d_1-d_2+1}^2, \dots, e_{d+r-d_1-c}^2 := f(\tilde{\alpha}_1^3), \dots, f(\tilde{\alpha}_{d-r-c+d_2}^3)$;
- $e_{d+r-d_1-c+1}^2, \dots, e_d^2 := \tilde{\beta}_1^3, \dots, \tilde{\beta}_{d_1+c-r}^3$.

Then $e_1^1, \dots, e_d^1$ and $e_1^2, \dots, e_d^2$ restrict to bases for $\bar{x}^* \mathcal{E}_1$ and $\bar{x}^* \mathcal{E}_2$, respectively. Shrinking S to a smaller neighborhood of $\bar{x}$, we can thus assume that $e_1^1, \dots, e_d^1$ is a basis for $\mathcal{E}_1$ and $e_1^2, \dots, e_d^2$ is a basis for $\mathcal{E}_2$.

With respect to these bases, the maps f and g have block forms

$$f = \begin{pmatrix} \text{Id}_{d_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{Id}_{r-d_1-d_2} & 0 & 0 \\ 0 & 0 & s \text{Id}_{d_2} & 0 & 0 & 0 \\ 0 & s \text{Id}_{r-d_1-d_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \text{Id}_{d-r-c+d_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & s \text{Id}_{d_1+c-r} \end{pmatrix},$$

$$g = \begin{pmatrix} s \text{Id}_{d_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{Id}_{r-d_1-d_2} & 0 & 0 \\ 0 & 0 & \text{Id}_{d_2} & 0 & 0 & 0 \\ 0 & s \text{Id}_{r-d_1-d_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s \text{Id}_{d-r-c+d_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \text{Id}_{d_1+c-r} \end{pmatrix}.$$

If we consider LG as a subscheme of $G(r, d)_R \times_R G(r, d)_R$ in the obvious way, x has an affine neighborhood in this product given by a pair of $d-r$ by r matrices A^1, A^2 , where this pair corresponds to the pair of subspaces (V'_1, V'_2) satisfying

$$V'_1 = \text{colspan} \begin{pmatrix} \text{Id}_r \\ A^1 \end{pmatrix},$$

$$V'_2 = \text{colspan} \begin{pmatrix} \text{Id}_r \\ A^2 \end{pmatrix}.$$

We wish to find equations for the intersection of this neighborhood with LG . A pair (A^1, A^2) will lie in LG if and only if $f(V'_1)$ is contained in V'_2 and $g(V'_2)$ is contained in V'_1 . To write these conditions explicitly, we break the matrices defining V'_1 and V'_2 into block form as follows:

$$V'_1 = \text{colspan} \begin{pmatrix} \text{Id}_{d_1} & 0 & 0 \\ 0 & \text{Id}_{r-d_1-d_2} & 0 \\ 0 & 0 & \text{Id}_{d_2} \\ A^1_{11} & A^1_{12} & A^1_{13} \\ A^1_{21} & A^1_{22} & A^1_{23} \\ A^1_{31} & A^1_{32} & A^1_{33} \end{pmatrix}$$

and similarly for A^2 . Here the horizontal divisions of A are chosen so that the block division above corresponds to the block division for the matrices for f and g .

Now we have:

$$f(V'_1) = \text{colspan} \begin{pmatrix} \text{Id}_{d_1} & 0 & 0 \\ A^1_{11} & A^1_{12} & A^1_{13} \\ 0 & 0 & s \text{Id}_{d_2} \\ 0 & s \text{Id}_{r-d_1-d_2} & 0 \\ A^1_{21} & A^1_{22} & A^1_{23} \\ sA^1_{31} & sA^1_{32} & sA^1_{33} \end{pmatrix},$$

and it is easy to see that this is contained in V'_2 if and only if the matrix identity

$$\begin{pmatrix} A^2_{11} & A^2_{12} & A^2_{13} \\ A^2_{21} & A^2_{22} & A^2_{23} \\ A^2_{31} & A^2_{32} & A^2_{33} \end{pmatrix} \begin{pmatrix} \text{Id}_{d_1} & 0 & 0 \\ A^1_{11} & A^1_{12} & A^1_{13} \\ 0 & 0 & s \text{Id}_{d_2} \end{pmatrix} = \begin{pmatrix} 0 & s \text{Id}_{r-d_1-d_2} & 0 \\ A^1_{21} & A^1_{22} & A^1_{23} \\ sA^1_{31} & sA^1_{32} & sA^1_{33} \end{pmatrix}$$

is satisfied. Similarly, $g(V'_2)$ is contained in V'_1 if and only if we have

$$\begin{pmatrix} A_{11}^1 & A_{12}^1 & A_{13}^1 \\ A_{21}^1 & A_{22}^1 & A_{23}^1 \\ A_{31}^1 & A_{32}^1 & A_{33}^1 \end{pmatrix} \begin{pmatrix} s \operatorname{Id}_{d_1} & 0 & 0 \\ A_{11}^2 & A_{12}^2 & A_{13}^2 \\ 0 & 0 & \operatorname{Id}_{d_2} \end{pmatrix} = \begin{pmatrix} 0 & s \operatorname{Id}_{r-d_1-d_2} & 0 \\ s A_{21}^2 & s A_{22}^2 & s A_{23}^2 \\ A_{31}^2 & A_{32}^2 & A_{33}^2 \end{pmatrix}.$$

If we solve for the matrix variables $A_{13}^1, A_{23}^1, A_{31}^2, A_{32}^2, A_{33}^2, A_{11}^2, A_{21}^2$, and A_{22}^1 in terms of the remaining variables, all of the above relations disappear except for the relations

$$A_{12}^1 A_{12}^2 = A_{12}^2 A_{12}^1 = s \operatorname{Id}_{r-d_1-d_2},$$

and the result follows. $\square$

4. CONCLUSION OF THE MAIN THEOREMS

Zhang [11] has shown, using the theory of Hodge algebras of de Concini, Eisenbud and Procesi [1], that spaces such as those arising in Theorem 3.2 are Cohen-Macaulay. In particular, we have:

Theorem 4.1 (Zhang). *Let k be a field. Then $M_{\ell,k}(2,0)$ is Cohen-Macaulay.*

Proof. This is [11, Thm. 22]. Since this reference is difficult to obtain, we are grateful to the referee for explaining the following alternative argument:

Let $W(k_1, k_2)$ denote the closed subscheme of $M_{\ell,k}(2,0)$ consisting of those pairs of $\ell \times \ell$ matrices M, N such that $MN = NM = 0$, M has rank at most k_1 , and N has rank at most k_2 . Strickland [10, p. 524, §2] has shown that the $W(k_1, k_2)$ are reduced, and when $k_1 + k_2 \leq \ell$, that they are also Cohen-Macaulay. Setting $k_1 = k_2 = \ell$ in the first result, we see in particular that $M_{\ell,k}(2,0)$ is reduced, hence scheme-theoretically the union of $W(j, \ell - j)$ for $j = 0, \dots, \ell$.

Define

$$W^i := \bigcup_{j=0}^i W(j, \ell - j),$$

and observe that

$$W^{i-1} \cap W(i, \ell - i) = W(i-1, \ell - i)$$

(scheme-theoretically). Moreover, $W(i-1, \ell - i)$ has codimension one in both W^{i-1} and $W(i, \ell - i)$. Thus by Goertz [4, Lemma 4.22], if W^{i-1} is Cohen-Macaulay, then so is W^i . By induction, we find that W^ℓ is Cohen-Macaulay. But W^ℓ is equal to $M_{\ell,k}(2,0)$. $\square$

We are thus able to show the following.

Theorem 4.2. *Suppose that S is integral and Cohen-Macaulay, and let $LG := LG(r, \{\mathcal{E}_i\}_i, \{f_i, g_i\}_i)$ be any linked Grassmannian scheme over S . Then LG is flat over S , and reduced and Cohen-Macaulay, with reduced fibers.*

Proof. We first show that LG is Cohen-Macaulay whenever $S = \operatorname{Spec} k$. We prove this by induction on n , starting with the case $n = 2$. In this case, by Theorem 3.2, we can conclude that LG is Cohen-Macaulay if we know that $M_{\ell,S}(2,s)$ is Cohen-Macaulay. If $s \neq 0$, we have $\ell = 0$, and $M_{\ell,S}(2,s)$ is just a point. If $s = 0$, we have the space of pairs of $\ell \times \ell$ matrices M, N with coefficients in k and with $MN = NM = 0$. By Theorem 4.1, this space is Cohen-Macaulay.

We now induct on n . We let LG be a linked Grassmannian of length n , and denote by LG_{n-1} (respectively, LG') the truncated linked Grassmannians of length

$n - 1$ (respectively, 2) obtained from LG by forgetting $\mathcal{E}_n, f_n, g_n$ (respectively, $\mathcal{E}_i, f_i, g_i$ for all $i < n - 1$). Then LG_{n-1} naturally lies in the product of Grassmannians $G_{n-1} := G(r, \mathcal{E}_1) \times \cdots \times G(r, \mathcal{E}_{n-1})$, and LG' in $G' := G(r, \mathcal{E}_{n-1}) \times G(r, \mathcal{E}_n)$. Let

$$\Delta : G := G(r, \mathcal{E}_1) \times \cdots \times G(r, \mathcal{E}_n) \rightarrow G_{n-1} \times G'$$

be the map which doubles the $(n - 1)$ st coordinate; it is then clear by considering the functors of points that we have that $LG \subseteq G$ is the fiber product of Δ with the inclusion map $LG_{n-1} \times LG' \rightarrow G_{n-1} \times G'$.

Now, the image of Δ is a locally complete intersection of codimension $r(d - r)$ in $G_{n-1} \times G'$, so we conclude that LG is cut out by $r(d - r)$ equations inside of $LG_{n-1} \times LG'$, and since this is also the codimension of LG , we find that the induction hypothesis that LG_{n-1} and LG' are Cohen-Macaulay then implies that LG is Cohen-Macaulay as well. This shows that LG is always Cohen-Macaulay over $\text{Spec } k$.

Next, still assuming that $S = \text{Spec } k$, by Theorem 2.2, we have that LG is smooth and hence reduced on an open dense subset, and since LG is Cohen-Macaulay, it cannot have imbedded components, and we conclude that LG is also reduced as long as the base is a point.

Finally, suppose S is an arbitrary integral, Cohen-Macaulay scheme. The base change of LG to any integral, Cohen-Macaulay scheme is another linked Grassmannian, so since we have reducedness when $S = \text{Spec } k$, we have that condition (i) of the following lemma is satisfied for any base. Similarly, by Theorem 2.2 every topological component of LG surjects onto the base after base change to $\text{Spec } A$ with A a DVR, and condition (ii) of the following lemma is satisfied. Thus, the lemma implies that any linked Grassmannian is reduced and flat. Finally, we can conclude Cohen-Macaulayness for arbitrary S by [7, Cor., p. 181]. $\square$

Lemma 4.3. *Let $f : X \rightarrow S$ be a morphism locally of finite presentation with S reduced and Noetherian, and suppose that:*

- (i) *the fibers of f are geometrically reduced;*
- (ii) *after arbitrary base change to $\text{Spec } A$, with A a DVR, no (topological) component of $X \times_S \text{Spec } A$ is supported over the special fiber.*

Then X is reduced, and f is flat.

Proof. By [6, Thm. 11.8.1], we can prove that f is flat by checking that it is flat after arbitrary base change under morphisms $\text{Spec } A \rightarrow S$, where A is a DVR. Since f has geometrically reduced fibers, the fibers remain reduced under base change, and assumption (ii) allows us to conclude that $(X \times_S \text{Spec } A)_{\text{red}}$ is flat over $\text{Spec } A$ by [6, Prop. 14.3.8].

It then follows by [9, Lem. 6.13] that $X \times_S \text{Spec } A$ is reduced, hence flat, giving us that f is flat. The flatness of f then also implies by [5, Cor. 3.3.5] that X is reduced, as desired. $\square$

Finally, we use the previous theorem to conclude a similar flatness result for the limit linear series schemes introduced in [9]; see that paper for the appropriate definitions.

Theorem 4.4. *Fix integers r, d, n , and ramification sequences $\alpha^1, \dots, \alpha^n$, and let X_0 be a curve of compact type over $\text{Spec } k$ having no more than two components, with marked points $P_1, \dots, P_n$. Suppose that the space $G_d^r(X_0) := G_d^r(X_0/k)$;*

$\{(P_i, \alpha^i)\}_i$ of limit linear series on X_0 of degree d and dimension r , with prescribed ramification at the P_i , has exactly the expected dimension $\rho := (r+1)(d-r) - rg - \sum_{i,j} \alpha_j^i$.

Then $G_d^r(X_0)$ is Cohen-Macaulay, and if we have a smoothing family of curves $X/B, \{\tilde{P}_i\}_i$ (see [9, Def. 3.1]) where $X_0, \{P_i\}_i$ occurs as a fiber over $b_0 \in B$, then $G_d^r(X) := G_d^r(X/B; \{(\tilde{P}_i, \alpha^i)\}_i)$, the relative space of limit linear series on X/B , is Cohen-Macaulay and flat over B at every point over b_0 .

We remark that the expected dimension hypothesis is satisfied for a general curve with marked points in characteristic 0; see [8].

Proof. Indeed, in the proof of [9, Thm. 5.3] the space $G_d^r(X)$ of limit linear series on a smoothing family X/B (where we may allow $B = \text{Spec } k, X = X_0$) is constructed inside a product of Grassmannians by intersecting a linked Grassmannian with a finite collection of Schubert cycles, and this has dimension ρ in the fiber over b_0 if and only if this intersection has maximal codimension over b_0 . Since we now know that linked Grassmannians are Cohen-Macaulay, at any point over b_0 we have a proper-dimensional intersection of Cohen-Macaulay schemes inside a regular one, and this must be Cohen-Macaulay by the lemma below. Since B is regular as part of the hypothesis of a smoothing family, we also obtain that $G_d^r(X)$ is flat over B (at least, at any point over b_0) by [7, Thm. 23.1]. $\square$

We recall the following well-known lemma and include the proof for lack of a suitable reference:

Lemma 4.5. *Let X be regular, and $Y_1, \dots, Y_n$ closed subschemes of X , each Cohen-Macaulay, of codimensions $c_1, \dots, c_n$. Let $Z = Y_1 \cap \dots \cap Y_n$, and suppose that Z has codimension $\sum_i c_i$ in X . Then Z is also Cohen-Macaulay.*

Proof. By induction, it is clearly enough to consider the case that $n = 2$. Also, the question is purely local, so we are reduced to the following: let R be a regular local ring, and I, J ideals of codimension d, e such that R/I and R/J are Cohen-Macaulay, and $I + J$ has codimension $d + e$. Then we want to see that $R/(I + J)$ is Cohen-Macaulay. By [2, Cor. 19.15], the Cohen-Macaulayness of the rings $R/I, R/J, R/(I + J)$ is equivalent to having free resolutions over R of lengths $d, e, d + e$ respectively. Thus, we may assume we have free resolutions $\mathcal{F}^\bullet, \mathcal{G}^\bullet$ of R/I and R/J having lengths d, e respectively. We claim that the tensor complex $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet$, which has length $d + e$, is still exact, thus giving a free resolution of $R/(I + J)$ of length $d + e$ and showing that $R/(I + J)$ is Cohen-Macaulay.

By [2, Thm. 20.9], it is enough to check that $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet$ is exact after localizing at an arbitrary prime $\mathfrak{p}$ of codimension strictly less than $d + e$: indeed, condition (1) of *loc. cit.* is automatically satisfied for the tensor product of two free resolutions, and to verify (2), if we had (in the notation of *loc. cit.*) $\text{depth } I(\varphi_k) < k$ for some $k \leq d + e$, we could choose $\mathfrak{p}$ to be a minimal prime of $I(\varphi_k)$ having codimension strictly less than $d + e$, and (2) would still be violated after localizing at $\mathfrak{p}$.

Now, let $\mathfrak{p}$ be arbitrary of codimension less than $d + e$. It cannot contain $I + J$ and thus does not contain both I and J ; without loss of generality, say that I is not contained in $\mathfrak{p}$. But if we localize $\mathcal{F}^\bullet$ at $\mathfrak{p}$, R/I itself becomes 0, so the sequence becomes split exact by [2, Lem. 20.1], and in this case, tensoring with $(\mathcal{G}^\bullet)_{\mathfrak{p}}$ necessarily preserves exactness. $\square$

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