REPRESENTATION OF MEASURES WITH POLYNOMIAL DENSENESS IN $L_p(\mathbb{R}, d\mu)$, $0 < p < \infty$, AND ITS APPLICATION TO DETERMINE MOMENT PROBLEMS

ANDREW G. BAKAN

(Communicated by N. Tomczak-Jaegermann)

ABSTRACT. It has been proved that algebraic polynomials $\mathcal{P}$ are dense in the space $L^p(\mathbb{R}, d\mu)$, $p \in (0, \infty)$, iff the measure μ is representable as $d\mu = w^p d\nu$ with a finite non-negative Borel measure ν and an upper semi-continuous function $w : \mathbb{R} \rightarrow \mathbb{R}^+ := [0, \infty)$ such that $\mathcal{P}$ is a dense subset of the space $C_w^0 := \{f \in C(\mathbb{R}) : w(x)f(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty\}$ equipped with the seminorm $\|f\|_w := \sup_{x \in \mathbb{R}} w(x)|f(x)|$. The similar representation $(1+x^2)d\mu = w^2 d\nu$ ($(1+x)d\mu = w^2 d\nu$) with the same ν and w ($w(x) = 0, x < 0$, and $\mathcal{P}$ is also a dense subset of $C_{\sqrt{x} \cdot w}^0$) corresponds to all those measures (supported by $\mathbb{R}^+$) that are uniquely determined by their moments on $\mathbb{R}(\mathbb{R}^+)$. The proof is based on de Branges' theorem (1959) on weighted polynomial approximation. A more general question on polynomial denseness in a separable Fréchet space in the sense of Banach $L^\Phi(\mathbb{R}, d\mu)$ has also been examined.

1. INTRODUCTION AND MAIN RESULTS

Let $\mathcal{M}^+(\mathbb{R})$ denote the set of non-zero finite non-negative Borel measures on $\mathbb{R}$. Each space $L_p(\mathbb{R}, d\mu)$, $0 < p < \infty$, of complex-valued functions contains the family $\mathcal{P}[\mathbb{C}]$ of all algebraic polynomials with complex coefficients if μ belongs to the class $\mathcal{M}^*(\mathbb{R})$ of all measures in $\mathcal{M}^+(\mathbb{R})$ with all moments $s_n(\mu) := \int_{\mathbb{R}} x^n d\mu(x)$, $n \in \mathbb{N}_0 := \{0, 1, 2, \dots\}$, finite. In the sequel, we write $\mathcal{M}^*(\mathbb{R}^+)$ for the set of all measures μ in $\mathcal{M}^*(\mathbb{R})$ which are supported by $\mathbb{R}^+ := [0, +\infty)$, i.e. $\text{supp } \mu := \{x \in \mathbb{R} \mid \mu((x-\varepsilon, x+\varepsilon)) > 0 \ \forall \varepsilon > 0\} \subset \mathbb{R}^+$.

The Hamburger (Stieltjes) moment problem consists in finding for a sequence of real numbers $\{\gamma_n\}_{n \in \mathbb{N}_0}$ a measure $\mu \in \mathcal{M}^*(\mathbb{R})$ ($\mathcal{M}^*(\mathbb{R}^+)$) with moments $s_n(\mu) = \gamma_n$, $n \in \mathbb{N}_0$. If the solution is unique it is said that the corresponding moment problem is *determinate*. Measures μ solving such problems are also called *determinate*. In other words, a measure $\mu \in \mathcal{M}^*(\mathbb{R})$ ($\mathcal{M}^*(\mathbb{R}^+)$) is said to be *determinate in the sense of Hamburger (Stieltjes)* (in short: $\mu \in \det \mathcal{H}(\det \mathcal{S})$) if μ is the only measure in $\mathcal{M}^*(\mathbb{R})$ ($\mathcal{M}^*(\mathbb{R}^+)$) with the same moments as μ (see [11]).

In 1923 M. Riesz [25] established a direct connection between determinate Hamburger moment problems and the problem of polynomial denseness in the space

Received by the editors August 21, 2007.

2000 *Mathematics Subject Classification.* Primary 46E30, 41A10; Secondary 44A60, 41A65.

Key words and phrases. Spaces of measurable functions, approximation by polynomials, moment problems.

$L_2(\mathbb{R}, d\mu)$. He proved that (see [11, Prop. 1.3])

$$(1.1) \quad \mu \in \det \mathcal{H} \Leftrightarrow \mathcal{P}[\mathbb{C}] \text{ is dense in } L_2(\mathbb{R}, (1+x^2) d\mu) .$$

In 1991 Ch. Berg and M. Thill [11, Th. 3.8] supplemented (1.1) with

$$(1.2) \quad \mu \in \det \mathcal{S} \Leftrightarrow \mathcal{P}[\mathbb{C}] \text{ is dense in } L_2(\mathbb{R}, (1+x) d\mu) \text{ and } L_2(\mathbb{R}, x \cdot (1+x) d\mu) .$$

Let $\mathcal{W}^+(\mathbb{R})$ be the set of upper semi-continuous and uniformly bounded functions $w : \mathbb{R} \rightarrow \mathbb{R}^+$. For $w \in \mathcal{W}^+(\mathbb{R})$ denote by C_w^0 the semi-normed space composed of the linear set of all $f \in C(\mathbb{R})$ with $\lim_{|x| \rightarrow \infty} w(x)f(x) = 0$ and the semi-norm $\| \cdot \|_w$, where $\|f\|_w := \sup_{x \in \mathbb{R}} w(x)|f(x)|$ and $C(\mathbb{R})$ denotes the linear space of all continuous complex-valued functions on $\mathbb{R}$. In this context a function $w \in \mathcal{W}^+(\mathbb{R})$ is also called a *weight* and $\mathcal{P}[\mathbb{C}] \subset C_w^0$ if w belongs to the set $\mathcal{W}^*(\mathbb{R})$ of all weights w in $\mathcal{W}^+(\mathbb{R})$ satisfying $\lim_{|x| \rightarrow \infty} |x|^n w(x) = 0$ for all $n \in \mathbb{N}_0$.

In 1924 S. Bernstein [13] asked for conditions on $w \in \mathcal{W}^*(\mathbb{R})$ to have the denseness of $\mathcal{P}[\mathbb{C}]$ in C_w^0 . In 1959 L. de Branges [15] obtained a solution to this problem (see also the survey papers of N. Akhiezer [1], S. Mergelyan [23], B. Ya. Levin [21] and M. Sodin [27]). A slightly improved version (see [28]) of his result is as follows. Let $\mathcal{E}_0(\mathbb{R})$ be the family of entire functions B of minimal exponential type having real and simple zeros only, and let Λ_B denote the set of these zeros and $S_w := \{x \in \mathbb{R} : w(x) > 0\}$.

Theorem A ([8, de Branges, 1959]). *For $w \in \mathcal{W}^*(\mathbb{R})$ assume that S_w is unbounded. Then $\mathcal{P}[\mathbb{C}]$ fails to be dense in C_w^0 if and only if there exists a transcendental function $B \in \mathcal{E}_0(\mathbb{R})$ with $\Lambda_B \subseteq S_w$ such that $\sum_{\lambda \in \Lambda_B} \frac{1}{w(\lambda)|B'(\lambda)|} < \infty$.*

The problem of finding conditions on $\mu \in \mathcal{M}^*(\mathbb{R})$ to have a polynomial denseness in $L_p(\mathbb{R}, d\mu)$ for a given $p \in [1, \infty)$ was treated by various authors (see [4, 18, 24, 10, 11, 21, 12, 22], etc.). The first successful attempt to apply Theorem A for solving this problem was made in 1998 by A. Borichev and M. Sodin [14]. However, a complete analogue of Theorem A for $L_p(\mathbb{R}, d\mu)$ was found only for discrete measures $\mu \in \mathcal{M}^*(\mathbb{R})$ with sufficiently thin support: $\sum_{\lambda \in \text{supp } \mu} (1 + |\lambda|)^{-a} < \infty$ for some $a > 0$. The final result on the description of all $\mu \in \mathcal{M}^*(\mathbb{R})$ such that $\mathcal{P}[\mathbb{C}]$ is dense in $L_p(\mathbb{R}, d\mu)$ for some $p \in [1, \infty)$ was obtained in 1998 [5]; the sketch of its proof and an application to the Hamburger moment problem was published in 2001 [6].

Our main results are the following two theorems. Let $\mathcal{B}(\mathbb{R})$ ($\mathcal{B}(\mathbb{R}^+)$) denote the family of all Borel subsets of $\mathbb{R}$ ($\mathbb{R}^+$) and $\mathcal{W}^*(\mathbb{R}^+)$ the set of all weights $w \in \mathcal{W}^*(\mathbb{R})$ with $w(x) = 0$, $x < 0$.

Theorem 1.1. *Let $0 < p < \infty$ and $\mu \in \mathcal{M}^*(\mathbb{R})$ have unbounded support. Then $\mathcal{P}[\mathbb{C}]$ is dense in $L_p(\mathbb{R}, d\mu)$ if and only if the measure μ can be represented in the following form:*

$$(1.3) \quad \mu(A) = \int_A w(x)^p d\nu(x) , \quad A \in \mathcal{B}(\mathbb{R}) ,$$

(in short: $d\mu(x) = w(x)^p d\nu(x)$) for some $\nu \in \mathcal{M}^+(\mathbb{R})$ and $w \in \mathcal{W}^*(\mathbb{R})$ such that $\mathcal{P}[\mathbb{C}]$ is dense in C_w^0 .

Theorem 1.2. *Let $\mu \in \mathcal{M}^*(\mathbb{R})$ ($\mathcal{M}^*(\mathbb{R}^+)$). There is no other measure in $\mathcal{M}^*(\mathbb{R})$ ($\mathcal{M}^*(\mathbb{R}^+)$) with the same moments as μ , i.e. $\mu \in \det \mathcal{H}(\det \mathcal{S})$, if and only if there exist a finite non-negative Borel measure ν on $\mathbb{R}$ ($\mathbb{R}^+$) and a weight $w \in \mathcal{W}^*(\mathbb{R})$*

$(\mathcal{W}^*(\mathbb{R}^+))$ such that algebraic polynomials $\mathcal{P}[\mathbb{C}]$ are dense in the seminormed space C_w^0 (spaces C_w^0 and $C_{\sqrt{x} \cdot w}^0$) and

$$\mu(A) = \int_A \frac{w(x)^2}{1+x^2} d\nu(x) \left(\int_A \frac{w(x)^2}{1+x} d\nu(x) \right), \quad A \in \mathcal{B}(\mathbb{R}) \ (\mathcal{B}(\mathbb{R}^+)) .$$

For $1 \leq p < \infty$ Theorem 1.1 coincides with [5, Th. 4.1, p. 37] and [6, Th. 2.1, p. 38]. Representation of $\mu \in \det \mathcal{H}$ given in Theorem 1.2 was first obtained in [6, Cor. 2.1, p. 39].

Theorem 1.1 is a special case of the general result proved in Theorem 1.3 below. For arbitrary $\mu \in \mathcal{M}^+(\mathbb{R})$ we generalize the spaces $L_p(\mathbb{R}, d\mu)$, $0 < p < \infty$, in the following way. Let $\Phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a strictly increasing continuous function such that $\lim_{x \rightarrow +\infty} 1/\Phi(x) = \Phi(0) = 0$. Then the inverse function of Φ , denoted by φ , has precisely the same properties. It has been proved in Lemmas 2.1, 2.2 below that the linear space $L^\Phi(\mathbb{R}, d\mu)$ of all complex-valued Borel measurable functions f satisfying

$$\int_{\mathbb{R}} \Phi(n|f(x)|) d\mu(x) < +\infty, \quad n = 1, 2, \dots,$$

with the local base at zero $U_n^{\Phi, \mu} := \{f \in L^\Phi(\mathbb{R}, d\mu) \mid \|\Phi(n|f|)\|_{L^1(\mathbb{R}, d\mu)} < 1/n\}$, $n \in \mathbb{N} := \{1, 2, \dots\}$, is a separable Fréchet space in the sense of Banach, provided that two functions are identified whenever they are equal to each other almost everywhere with respect to the measure μ (in short: a.e. μ). In the case $\Phi(x) = x^p$, $0 < p < \infty$, we obviously have $L^\Phi(\mathbb{R}, d\mu) = L_p(\mathbb{R}, d\mu)$. We are going to characterize all those measures $\mu \in \mathcal{M}^+(\mathbb{R})$ that $\mathcal{P}[\mathbb{C}]$ is a dense subset of $L^\Phi(\mathbb{R}, d\mu)$.

To have all power functions x^n , $n \in \mathbb{N}_0$, in the space $L^\Phi(\mathbb{R}, d\mu)$ we consider the set $\mathcal{M}_\Phi^*(\mathbb{R})$ of all measures in $\mathcal{M}^+(\mathbb{R})$ such that $s_n^\Phi(\mu) := \int_{\mathbb{R}} \Phi(|x|^n) d\mu(x) < \infty$ for all $n \in \mathbb{N}_0$. It follows easily from (2.1) and Lemma 2.2 below that for every measure $\mu \in \mathcal{M}_\Phi^*(\mathbb{R})$ with bounded support algebraic polynomials are dense in the space $L^\Phi(\mathbb{R}, d\mu)$. The present paper contains a complete proof of the following result.

Theorem 1.3. *Let $\mu \in \mathcal{M}_\Phi^*(\mathbb{R})$ have unbounded support. The set of algebraic polynomials $\mathcal{P}[\mathbb{C}]$ is dense in the space $L^\Phi(\mathbb{R}, d\mu)$ iff $d\mu(x) = w(x) d\nu(x)$, where $\nu \in \mathcal{M}^+(\mathbb{R})$ and $w \in \mathcal{W}_\Phi^*(\mathbb{R})$ are such that $\mathcal{P}[\mathbb{C}]$ is dense in all seminormed spaces $C_{w_n}^0$, $n \in \mathbb{N}$, where*

$$(1.4) \quad w_n(x) := 1/\varphi\left(\frac{1}{n \cdot w(x)}\right), \quad x \in \mathbb{R}, \quad n \in \mathbb{N},$$

φ is the inverse function of Φ and it is assumed that $1/0 := +\infty$, $1/+\infty := 0$.

Additional restriction to Φ essentially simplifies the conditions of Theorem 1.3 in the following corollary.

Corollary 1.1. *Let $\mu \in \mathcal{M}_\Phi^*(\mathbb{R})$ have unbounded support, and there exists a constant $\lambda > 1$ satisfying*

$$(1.5) \quad \lim_{x \rightarrow +\infty} \frac{\Phi(\lambda \cdot x)}{\Phi(x)} > 1.$$

Then $\mathcal{P}[\mathbb{C}]$ is dense in $L^\Phi(\mathbb{R}, d\mu)$ if and only if there exist a measure $\nu \in \mathcal{M}^+(\mathbb{R})$ and a weight $w \in \mathcal{W}^(\mathbb{R})$ such that*

$$d\mu(x) = \frac{d\nu(x)}{\Phi\left(\frac{1}{w(x)}\right)}$$

and $\mathcal{P}[\mathbb{C}]$ is dense in C_w^0 .

2. PRELIMINARY RESULTS

It follows from the definition of $L^\Phi(\mathbb{R}, d\mu)$ that $f \in L^\Phi(\mathbb{R}, d\mu)$ belongs to the closure $\text{Cl}_\Phi^\mu A$ of $A \subset L^\Phi(\mathbb{R}, d\mu)$ in the space $L^\Phi(\mathbb{R}, d\mu)$ iff

$$(2.1) \quad \inf_{a \in A} \int_{\mathbb{R}} \Phi(n \cdot |f(x) - a(x)|) d\mu(x) = 0, \quad n = 1, 2, \dots$$

Lemma 2.1. *For arbitrary $\mu \in \mathcal{M}^+(\mathbb{R})$ the space $L^\Phi(\mathbb{R}, d\mu)$ is a complete metrizable Hausdorff topological linear space (see [26, Ch. I, 6.1]); i.e. it is a Fréchet space in the sense of Banach (see [16, 6.1.1, Rem. 4]).*

Proof. For arbitrary $f \in L^\Phi(\mathbb{R}, d\mu)$, $R > 0$, $n, M \in \mathbb{N}$, denote $J_R := \mathbb{R} \setminus [-R, R]$, $J_R^f(M) := \{x : |x| \leq R, |f(x)| > M\}$, $I_R^f(M) := \{x : |x| \leq R, |f(x)| \leq M\}$ and $L_n(A; f) := \int_A \Phi(n|f(x)|) d\mu(x)$, $A \in \mathcal{B}(\mathbb{R})$.

It is evident that $U_{n+1}^{\Phi, \mu} \subset U_n^{\Phi, \mu}$ and $\theta \cdot U_n^{\Phi, \mu} \subset U_n^{\Phi, \mu}$ for any $n \in \mathbb{N}$ and $|\theta| \leq 1$, $\theta \in \mathbb{C}$. Obvious inequality $\Phi(|x_1 + x_2|) \leq \Phi(2|x_1|) + \Phi(2|x_2|)$, $x_1, x_2 \in \mathbb{C}$, yields $L_n(\mathbb{R}, f + g) \leq L_{2n}(\mathbb{R}, f) + L_{2n}(\mathbb{R}, g)$, $f, g \in L^\Phi(\mathbb{R}, d\mu)$, and $U_{2n}^{\Phi, \mu} + U_{2n}^{\Phi, \mu} \subset U_n^{\Phi, \mu}$, $n \in \mathbb{N}$. This, combined with $L_n(\mathbb{R}; f) \geq L_1(\mathbb{R}; f) > 0$, $f \neq 0$ a.e. μ , $n \in \mathbb{N}$, allows for arbitrary $g_1, g_2 \in L^\Phi(\mathbb{R}, d\mu)$, $g_1 \neq g_2$ a.e. μ , to choose $p \in \mathbb{N}$ such that $1/p < L_1(\mathbb{R}; g_1 - g_2)$ and to get $g_1 + U_{2p}^{\Phi, \mu} \cap g_2 + U_{2p}^{\Phi, \mu} = \emptyset$.

We claim in addition that $\lim_{\varepsilon \rightarrow 0} L_n(\mathbb{R}, \varepsilon f) = 0$ for every $f \in L^\Phi(\mathbb{R}, d\mu)$ and every positive integer n . Indeed, for every $\delta > 0$ due to the definition of a Lebesgue integral, one can find $R = R(\delta) > 0$ and $M = M(\delta) \in \mathbb{N}$ such that $L_n(J_R; f) + L_n(J_R^f(M); f) \leq \delta/2$. Choosing $\varepsilon > 0$ so that $\Phi(n\varepsilon M) \cdot \mu(\mathbb{R}) < \delta/2$ we get $L_n(\mathbb{R}; \varepsilon f) = L_n(J_R; \varepsilon f) + L_n(J_R^f(M); \varepsilon f) + L_n(I_R^f(M); \varepsilon f) \leq \delta/2 + \Phi(n\varepsilon M) \cdot \mu(\mathbb{R}) < \delta$, which was to be proved. That's why $\bigcup_{\lambda > 0} \lambda \cdot U_n^{\Phi, \mu} = L^\Phi(\mathbb{R}, d\mu)$, $n \in \mathbb{N}$, and in view of [26, Ch. I, 1.2] (see also [3, Ch. III, 1.1, Th. 1]) and [26, Ch. I, 6.1] the properties above imply that $L^\Phi(\mathbb{R}, d\mu)$ is a metrizable Hausdorff topological linear space.

To prove the completeness of $L^\Phi(\mathbb{R}, d\mu)$ observe that if $\{f_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $L^\Phi(\mathbb{R}, d\mu)$, then for any $n \in \mathbb{N}$ it is possible to find $N_n \in \mathbb{N}$ such that $f_k - f_r \in U_{n+1}^{\Phi, \mu}$, $k, r \geq N_n$, and so for these indices and every $\sigma > 0$ we have: $1/n > L_n(\mathbb{R}; f_k - f_r) \geq L_n(\{x \in \mathbb{R} : |f_k(x) - f_r(x)| > \sigma\}; f_k - f_r) \geq \Phi(n\sigma) \cdot \mu(\{x \in \mathbb{R} : |f_k(x) - f_r(x)| > \sigma\})$, whence it follows that $\{f_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence with respect to the convergence in measure μ [19, §22]. Then by theorem 5 in [19, §22] there exists a Borel measurable function f such that the sequence $\{f_k\}_{k \in \mathbb{N}}$ converges to f in measure μ and in view of F. Riesz's theorem [9, II, Th. 4.3] there exists a subsequence $\{f_{r_j}\}_{j \in \mathbb{N}}$ converging to f a.e. μ . For any given $n, m \in \mathbb{N}$ the sequence of non-negative functions $\Phi(n|f_{r_{j+m}}(x) - f_{r_m}(x)|)$, $j \in \mathbb{N}$, converges to $\Phi(n|f(x) - f_{r_m}(x)|)$ a.e. μ as j tends to infinity, and by Fatou's lemma [9, III, Th. 6.2] it follows from $L_{n+1}(\mathbb{R}; f_{r_{j+m}} - f_{r_m}) < 1/(n+1)$, $r_m > N_{n+1}$, that $f - f_{r_m} \in U_n^{\Phi, \mu}$. That's why $f \in L^\Phi(\mathbb{R}, d\mu)$, and furthermore (2.1) and inequalities $L_n(\mathbb{R}; f - f_k) = L_n(\mathbb{R}; f - f_{r_j} + f_{r_j} - f_k) \leq L_{2n}(\mathbb{R}; f - f_{r_j}) + L_{2n}(\mathbb{R}; f_{r_j} - f_k)$, $j, k \in \mathbb{N}$, prove that the initial sequence $\{f_k\}_{k \in \mathbb{N}}$ converges to f in the space $L^\Phi(\mathbb{R}, d\mu)$. The lemma follows. $\square$

Observe that if $M(x)$ is an even convex function on $\mathbb{R}$ strictly increasing on $\mathbb{R}^+$ and satisfying $\lim_{x \rightarrow 0} M(x)/x = 0$ and $\lim_{x \rightarrow +\infty} M(x)/x = +\infty$, then $L^M(\mathbb{R}, d\mu)$

coincides with the space $E_M(\mathbb{R})$ which is the closure of all bounded Borel measurable functions in the Orlicz space $L_M(\mathbb{R})$ (see [3, Ch. IV, 3.6]).

Let $\mathcal{K}(\mathbb{R})$ denote the linear space of all $f \in C(\mathbb{R})$ with compact support (see [16, Ch. IV, 4.1]) and $\text{lin } S$ the linear hull of S .

The following lemma implies that the space $L^\Phi(\mathbb{R}, d\mu)$ is *separable*.

Lemma 2.2. *Let $\mu \in \mathcal{M}^+(\mathbb{R})$. Then*

$$(2.2) \quad \text{Cl}_\Phi^\mu \text{lin} \left\{ \frac{1}{(x \pm i)^n} \right\}_{n \in \mathbb{N}} = \text{Cl}_\Phi^\mu \mathcal{K}(\mathbb{R}) = L^\Phi(\mathbb{R}, d\mu) .$$

Proof. To prove the second equality of (2.2) a standard scheme of approximation can be used. We first prove that $\text{Cl}_\Phi^\mu \text{lin} \{\chi_A\}_{A \in \mathcal{B}(\mathbb{R})} = L^\Phi(\mathbb{R}, d\mu)$ (see [9, Ch. VI, Th. 8.4]), where χ_A denotes the characteristic function of $A \subset \mathbb{R}$. In addition to the notation introduced in the proof of Lemma 2.1, we use for a given $f \in L^\Phi(\mathbb{R}, d\mu)$ and arbitrary $R > 0$, $n, N, M \in \mathbb{N}$, the following: $I_0 := [-R, R] \cap f^{-1}(\{0\})$, $I_{\pm k} := [-R, R] \cap f^{-1}((\pm 1) \cdot (M - \frac{k}{N}, M - \frac{k-1}{N}])$, $\omega_{\pm k}(x) := (\pm M \mp \frac{k}{N}) \chi_{I_{\pm k}}(x)$, $1 \leq k \leq NM$, $\omega_0(x) := 0$, $\psi_N(x) := \sum_{k=-NM}^{NM} \omega_k(x)$, $x \in \mathbb{R}$.

The definition of the Lebesgue integral implies for a given $n \in \mathbb{N}$ an existence of $R = R(n) > 0$ and $M = M(n) \in \mathbb{N}$ such that $L_n(J_R; f) + L_n(J_R^f(M); f) \leq 1/(2n)$. Since the sets $J_R, J_R^f(M), I_0$ and $I_{\pm k}$, $1 \leq k \leq NM$, are disjoint, $|f(x) - \psi_N(x)| \leq \frac{1}{N}$ for every $x \in I_R^f(M)$ and there exist $N = N(n) \in \mathbb{N}$ such that $\Phi(n/N) \cdot \mu(\mathbb{R}) < 1/(2n)$; then $L_n(\mathbb{R}; f - \psi_N) = L_n(J_R; f) + L_n(J_R^f(M); f) + \sum_{k=-NM}^{NM} L_n(I_k, f - \omega_k) \leq 1/(2n) + \Phi(n/N) \cdot \mu(\mathbb{R}) < 1/n$, i.e. $f - \psi_N \in U_n^{\Phi, \mu}$. Equality $\text{Cl}_\Phi^\mu \text{lin} \{\chi_A\}_{A \in \mathcal{B}(\mathbb{R})} = L^\Phi(\mathbb{R}, d\mu)$ is proved.

It remains to prove that for arbitrary bounded $A \in \mathcal{B}(\mathbb{R})$ it is possible to approximate χ_A by functions in $\mathcal{K}(\mathbb{R})$. Due the regularity of any $\mu \in \mathcal{M}^+(\mathbb{R})$ (see [19, §52, Th. 7]) for every $\varepsilon > 0$ one can find a compact set F_ε and an open set G_ε such that $F_\varepsilon \subset A \subset G_\varepsilon$ and $\mu(G_\varepsilon \setminus F_\varepsilon) < \varepsilon$. Then the known function (see [9, Ch. VI, Th. 8.6]) $\rho_\varepsilon(x) := \rho(x, \mathbb{R} \setminus G_\varepsilon) / [\rho(x, \mathbb{R} \setminus G_\varepsilon) + \rho(x, F_\varepsilon)] \in \mathcal{K}(\mathbb{R})$, where $\rho(x, B) := \inf_{y \in B} |x - y|$, $B \subset \mathbb{R}$, equals χ_A when $x \in F_\varepsilon \cup (\mathbb{R} \setminus G_\varepsilon)$ and has its values in $[0, 1]$ when $x \in G_\varepsilon \setminus F_\varepsilon$. Therefore for every $n \in \mathbb{N}$: $L_n(\mathbb{R}, \chi_A - \rho_\varepsilon) = L_n(G_\varepsilon \setminus F_\varepsilon, \chi_A - \rho_\varepsilon) \leq \Phi(n) \cdot \mu(G_\varepsilon \setminus F_\varepsilon) < \Phi(n)\varepsilon$, and so $\chi_A - \rho_{1/(n\Phi(n))} \in U_n^{\Phi, \mu}$, which finishes the proof of the right-hand side equality of (2.2).

The left-hand side equality of (2.2) follows from the denseness of $(x \pm i)^{-n}$, $n \in \mathbb{N}$, in the normed linear space $C^0(\mathbb{R})$ defined as C_w^0 with $w \equiv 1$ and the inclusion $\mathcal{K}(\mathbb{R}) \subset C^0(\mathbb{R})$. The proof of this denseness can be achieved by the method given in the proof of Theorem 2.3.2 in [2]. Arguing by contradiction we get the existence of a linear continuous functional on $C^0(\mathbb{R})$ (see [16, Ex. 4.45], [7, Th. 1]) vanishing on all $(x \pm i)^{-n}$, $n \in \mathbb{N}$, i.e. (see [26, Ch. 1, 7.1]) there exist two real-valued finite Borel measures σ_1 and σ_2 such that $F_\pm^{(n-1)}(i) = 0$, $n \in \mathbb{N}$, where $F_\pm(z) := \int_{\mathbb{R}} (x - z)^{-1} d\sigma_\pm(x)$ and $\sigma_\pm := \sigma_1 \pm i\sigma_2$. Since functions $F_\pm$ are analytical in the upper half-plane, $F_\pm(z) = 0$ for any $\text{Im } z > 0$, and by the Stieltjes-Perron inversion formula [2, Ch. III, 1α] we get a contradiction. The lemma is proved. $\square$

A suitable criterion for polynomial denseness in the space $L^\Phi(\mathbb{R}, d\mu)$ is established in the following lemma.

Lemma 2.3. *Let $\mu \in \mathcal{M}_{\Phi}^*(\mathbb{R})$ have unbounded support. Then $\text{Cl}_{\Phi}^{\mu} \mathcal{P}[\mathbb{C}] = L^{\Phi}(\mathbb{R}, d\mu)$ iff there exists a sequence $\{P_n\}_{n \in \mathbb{N}} \subset \mathcal{P}[\mathbb{C}]$ such that*

$$(2.3) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \Phi(n \cdot |\frac{1}{x+i} - P_n(x)|) d\mu(x) = 0.$$

Proof. It follows from (2.3) and (2.2) that to prove the polynomial denseness in $L^{\Phi}(\mathbb{R}, d\mu)$ it is sufficient to verify that $(x+i)^{-n}$ belongs to $\text{Cl}_{\Phi}^{\mu} \text{lin} \{(x+i)^{-1} \cup \mathcal{P}[\mathbb{C}]\}$ for every integer $n \geq 2$. But this is a corollary of the evident inequalities (see [2, Th. 2.3.2])

$$|\frac{1}{(x+i)^{n+1}} - \frac{A}{x+i} - P(x)| \leq |\frac{1}{(x+i)^n} - A - (x+i)P(x)|, \quad n \in \mathbb{N}, \quad x \in \mathbb{R},$$

where $A \in \mathbb{C}$ and $P \in \mathcal{P}[\mathbb{C}]$. Triviality of the converse implication finishes the proof. $\square$

Let $\mathcal{W}_{\Phi}^*(\mathbb{R})$ be the set of $w \in \mathcal{W}^+(\mathbb{R})$ satisfying $\lim_{|x| \rightarrow +\infty} w(x)\Phi(|x|^n) = 0$ for all $n \in \mathbb{N}_0$. It is assumed that $|x|^0 = 1$ for every $x \in \mathbb{R}$. For $w \in \mathcal{W}^+(\mathbb{R})$ we introduce a pseudometrizable topological linear space $C_{w,\Phi}^0(\mathbb{R})$ composed of the linear space $\{f \in C(\mathbb{R}) \mid \Phi(n|f|) \in C_w^0, n \in \mathbb{N}\}$ and the local base at zero $W_n^{\Phi,w} := \{f \in C_{w,\Phi}^0(\mathbb{R}) \mid \|\Phi(n|f|)\|_w \leq 1/n\}$, $n \in \mathbb{N}$ (see [17, 4.1], [26, Ch. I, §6]). In the case where $\Phi(x) = x$, we have $C_{w,\Phi}^0(\mathbb{R}) = C_w^0$.

Lemma 2.4. *For $w \in \mathcal{W}^+(\mathbb{R})$ and $n \in \mathbb{N}$ define $w_n(x) = 1/\varphi(\frac{1}{n \cdot w(x)})$, $x \in \mathbb{R}$. Then $C_{w,\Phi}^0(\mathbb{R})$ coincides with the linear space $\bigcap_{n \in \mathbb{N}} C_{w_n}^0$ equipped with the locally convex topology determined by the semi-norms $\|\cdot\|_{w_n}$, $n \in \mathbb{N}$ (see [26, Ch. II, §4]).*

Proof. For arbitrary $\omega \in \mathcal{W}^+(\mathbb{R})$ observe that $f \in C_{\omega}^0$ iff $f \in C(\mathbb{R})$ and $\lim_{M \rightarrow \infty} \|f\|_{\omega}^M = 0$, where $\|f\|_{\omega}^M := \sup_{|x| \geq M} \omega(x)|f(x)|$. It is easy to see that for every $n, m \in \mathbb{N}$, $f \in C(\mathbb{R})$ and $x \in \mathbb{R}$ the inequality $w(x)\Phi(n|f(x)|) \leq 1/m$ holds iff $w_m(x)|f(x)| \leq 1/n$. Therefore

$$(2.4) \quad \|\Phi(n|f|)\|_w^M \leq \frac{1}{m} \Leftrightarrow \|f\|_{w_m}^M \leq \frac{1}{n}, \quad M \geq 0, \quad n, m \in \mathbb{N}, \quad f \in C(\mathbb{R}).$$

Properties (2.4) and $w_n \leq w_{n+1}$, $n \in \mathbb{N}$, imply that

$$(2.5a) \quad W_n^{\Phi,w} = \frac{1}{n} \cdot U_{w_n}, \quad U_{w_n} := \{f \in C(\mathbb{R}) \mid \|f\|_{w_n} \leq 1\}, \quad n \in \mathbb{N},$$

$$(2.5b) \quad W_{n+m}^{\Phi,w} = \frac{1}{n+m} \cdot U_{w_{n+m}} \subset \frac{1}{n+m} \cdot U_{w_n}, \quad n \in \mathbb{N}, \quad m \in \mathbb{N}_0,$$

and for $f \in C(\mathbb{R})$ the following assertions are equivalent: 1) $\Phi(n|f|) \in C_w^0 \forall n \in \mathbb{N}$; 2) $\forall n \in \mathbb{N} : \lim_{M \rightarrow \infty} \|\Phi(n|f|)\|_{\omega}^M = 0$; 3) $\forall n \in \mathbb{N} \forall m \in \mathbb{N} \exists M_{n,m} > 0 \forall M \geq M_{n,m} : \|\Phi(n|f|)\|_{\omega}^M \leq 1/m$; 4) $\forall m \in \mathbb{N} \forall n \in \mathbb{N} \exists M_{n,m} > 0 \forall M \geq M_{n,m} : \|f\|_{w_m}^M \leq 1/n$; 5) $\forall m \in \mathbb{N} : \lim_{M \rightarrow \infty} \|f\|_{w_m}^M = 0$; 6) $f \in C_{w_m}^0 \forall m \in \mathbb{N}$. The proof of Lemma 2.4 is complete. $\square$

The equality $C_{w,\Phi}^0(\mathbb{R}) = \bigcap_{n \in \mathbb{N}} C_{w_n}^0$, proved in Lemma 2.4, implies, in particular, that w belongs to $\mathcal{W}_{\Phi}^*(\mathbb{R})$ iff $w_n \in \mathcal{W}^*(\mathbb{R})$ for every $n \in \mathbb{N}$. Furthermore, if $\text{Cl}_{\Phi}^w A$ denotes the closure of $A \subset C_{w,\Phi}^0(\mathbb{R})$ in the space $C_{w,\Phi}^0(\mathbb{R})$, then by (2.5b):

$$(2.6) \quad f \in \text{Cl}_{\Phi}^w A \Leftrightarrow f \in C(\mathbb{R}) \text{ and } \inf_{a \in A} \|f - a\|_{w_n} = 0, \quad n = 1, 2, \dots;$$

whence it follows that $\mathcal{P}[\mathbb{C}]$ is dense in $C_{w,\Phi}^0(\mathbb{R})$ iff $\mathcal{P}[\mathbb{C}]$ is dense in all $C_{w_n}^0$, $n \in \mathbb{N}$. Since $1/(x+i) \in C_{w,\Phi}^0(\mathbb{R})$ for any $w \in \mathcal{W}^+(\mathbb{R})$, the known equivalence of polynomial denseness in C_{ω}^0 , $\omega \in \mathcal{W}^*(\mathbb{R})$, and the possibility of approximating the

single function $1/(x+i)$ by polynomials in C_ω^0 (see [8, p. 238]) gives for arbitrary $w \in \mathcal{W}_\Phi^*(\mathbb{R})$:

$$(2.7) \quad \text{Cl}_\Phi^w \mathcal{P}[\mathbb{C}] = C_{w,\Phi}^0(\mathbb{R}) \Leftrightarrow \mathcal{P}[\mathbb{C}] \text{ is dense in all } C_{w_n}^0, n \in \mathbb{N} \stackrel{(2.6)}{\Leftrightarrow} \frac{1}{x+i} \in \text{Cl}_\Phi^w \mathcal{P}[\mathbb{C}].$$

We recall the definition of the so-called *upper Baire function* M_F of $F : \mathbb{R} \rightarrow \mathbb{R}$ as $M_F(x) := \lim_{\delta \downarrow 0} \sup_{y \in (x-\delta, x+\delta)} F(y)$. If F is locally bounded from above, then M_F is an upper semi-continuous function.

Denote by $\overline{\mathcal{P}}$ the set of topological linear spaces of complex-valued functions on $\mathbb{R}$ which have $\mathcal{P}[\mathbb{C}]$ as a dense subset. The crucial tool in the proof of Theorem 1.3 will be the following theorem contained in [8] as Lemma 6 with a proof based on the result of Theorem A.

Theorem B ([8, Lemma 6, p. 238]). *For $w \in \mathcal{W}^*(\mathbb{R})$ with unbounded S_w let $S \subset S_w$ be such that $w(x) = M_w \cdot \chi_S(x)$, $x \in \mathbb{R}$. If for any countable set $G \subset S$ without finite accumulation points we have $C_{w \cdot \chi_G}^0 \in \overline{\mathcal{P}}$, then also $C_w^0 \in \overline{\mathcal{P}}$.*

To prove Theorem 1.3 we need the following auxiliary result.

Lemma 2.5. *Let $\mu \in \mathcal{M}^+(\mathbb{R})$ have unbounded support and $a \in L_1(\mathbb{R}, d\mu)$. Then there exists an upper semi-continuous function $\theta : \mathbb{R} \rightarrow (0, 1]$ such that $\lim_{|t| \rightarrow \infty} \theta(t) = 0$ and $a/\theta \in L_1(\mathbb{R}, d\mu)$.*

Proof. Since $\alpha(x) := 1 + |a(x)| \in L_1(\mathbb{R}, d\mu)$, the sequence of positive numbers

$$t_n := \int_{|x| > n} \alpha(x) d\mu(x), \quad n \in \mathbb{N}_0,$$

tends to zero as $n \rightarrow \infty$ and it is possible to find a subsequence $\{n_k\}_{k \in \mathbb{N}_0}$ such that $n_0 := 0$, $\sum_{k \in \mathbb{N}_0} \sqrt{t_{n_k}} < \infty$ and $t_{n_{k+1}} < t_{n_k}$, $k \in \mathbb{N}_0$. Introduce the function θ defined by

$$1/\theta(x) := \chi_{\{0\}}(x) + \sqrt{t_0} \cdot \sum_{k \in \mathbb{N}_0} \chi_{(n_k, n_{k+1}]}(|x|) / \sqrt{t_{n_k}}, \quad x \in \mathbb{R}.$$

It is evident that θ is an even and upper semi-continuous function on $\mathbb{R}$ which does not increase on $\mathbb{R}^+$, $\theta(x) \in (0, 1]$ for every $x \in \mathbb{R}$, and $\theta(x) \rightarrow 0$ as $|x| \rightarrow +\infty$. Inequalities

$$\int_{\mathbb{R}} \frac{\alpha(x)}{\theta(x)} d\mu(x) = a(0) \cdot \mu(\{0\}) + \sqrt{t_0} \cdot \sum_{k \in \mathbb{N}_0} (t_{n_k} - t_{n_{k+1}}) / \sqrt{t_{n_k}} < \infty$$

and $|a(x)| < \alpha(x)$, $x \in \mathbb{R}$, complete the proof of Lemma 2.5. $\square$

3. PROOF OF THEOREM 1.3

3.1. Necessity. For every $n \in \mathbb{N}$ it is possible to approximate the function $c(x) := 1/(x+i) \in C_{w_n}^0$ by a sequence $\{P_{n,k}\}_{k \in \mathbb{N}} \subset \mathcal{P}[\mathbb{C}]$ such that $c - P_{n,k} \in \frac{1}{k} U_{w_n}$. Therefore by (2.5a) for $Q_n := P_{n,n}$, $c - Q_n \in \frac{1}{n} U_{w_n} = W_n^{\Phi} w$, and so $\{Q_n\}_{n \in \mathbb{N}}$ converges to c in the space $C_{w,\Phi}^0(\mathbb{R})$. That's why $d\mu(x) = w(x) d\nu(x)$ yields

$$\int_{\mathbb{R}} \Phi(n |c(x) - Q_n(x)|) d\mu(x) \leq \|\Phi(n |c - Q_n|)\|_w \cdot \nu(\mathbb{R}) \leq \frac{\nu(\mathbb{R})}{n}.$$

This proves $\text{Cl}_\Phi^\mu \mathcal{P}[\mathbb{C}] = L^\Phi(\mathbb{R}, d\mu)$ by virtue of (2.3).

3.2. Sufficiency. Due to (2.3) for every $n \in \mathbb{N}$ there exists $P_n \in \mathcal{P}[\mathbb{C}]$ such that $(c(x) := 1/(x+i))$:

$$\Phi(n | c(x) - P_n(x) |) \in \frac{1}{2^n} U_{L_1(\mu)}, \quad U_{L_1(\mu)} := \{f \in L_1(\mathbb{R}, d\mu) \mid \|f\|_{L_1(\mathbb{R}, d\mu)} \leq 1\}.$$

Then the non-decreasing sequence of non-negative continuous functions

$$\alpha_N(x) := \sum_{n=1}^N \Phi(n \cdot |c(x) - P_n(x)|) \in U_{L_1(\mu)}, \quad N \in \mathbb{N},$$

has a limit $\alpha_0 \in U_{L_1(\mu)}$ by the Beppo Levi theorem [9, Ch. III, Th. 6.3]. It follows from Theorem 1.4 in [20, Ch. I] that the function α_0 is lower semi-continuous on $\mathbb{R}$. The unboundedness of $\text{supp } \mu$ implies that the numbers $s_n^\Phi(\mu)$ grow to infinity faster than any function $\Phi(Q^n)$, $Q > 1$, as $n \rightarrow \infty$, and therefore ($0^0 := 1$)

(3.1)

$$h(x) := 2\mu(\mathbb{R}) \cdot \sum_{n \in \mathbb{N}_0} \frac{\Phi(|x|^n)}{2^{n+1} s_n^\Phi(\mu)} \in C(\mathbb{R}) \cap U_{L_1(\mu)}, \quad h(x) \geq 1, \quad x \in \mathbb{R}.$$

In addition, by F. Riesz's theorem (see [19, §25, Th. 1]) there exists a subsequence $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ such that the sequence $\{\Phi(n_k | c(x) - P_{n_k}(x) |)\}_{k \in \mathbb{N}}$ converges to zero a.e. μ , i.e.

$$(3.2) \quad \lim_{k \rightarrow \infty} \Phi(n_k | c(x) - P_{n_k}(x) |) = 0, \quad x \in \mathbb{R} \setminus D, \quad \mu(D) = 0, \quad D \in \mathcal{B}(\mathbb{R}).$$

When applied to the function $\alpha_0 + h \in 2U_{L_1(\mu)}$, Lemma 2.5 gives a function θ with values in $(0, 1]$ such that: $1/w_* := (\alpha_0 + h)/\theta \in L_1(\mathbb{R}, d\mu)$; w_* is an upper semi-continuous function on $\mathbb{R}$; $0 \leq w_* = \theta/(\alpha_0 + h) \leq 1$ in view of (3.1); and $w_* \in \mathcal{W}_\Phi^*(\mathbb{R})$ due to $w_*(x)\Phi(|x|^n) \leq \theta(x)\Phi(|x|^n)/h(x) \leq 2^n s_n^\Phi(\mu) \theta(x)/\mu(\mathbb{R}) \rightarrow 0$, $|x| \rightarrow \infty$, $n \in \mathbb{N}$. We introduce the function

$$(3.3) \quad w(x) := M_{w_*} \cdot \chi_{\mathbb{R} \setminus D}(x), \quad x \in \mathbb{R}.$$

Then $0 \leq w \leq w_* \leq 1$ and $w = w_*$ a.e. μ . Therefore $1/w \in L_1(\mathbb{R}, d\mu)$ and $w \in \mathcal{W}_\Phi^*(\mathbb{R})$. Defining the measure ν by $\nu(A) := \int_A [1/w(x)] d\mu(x)$, $A \in \mathcal{B}(\mathbb{R})$, we get the representation of μ in the form $d\mu(x) = w(x) d\nu(x)$, and due to (2.7) it only remains for us to prove that $\mathcal{P}[\mathbb{C}]$ is dense in the space $C_{w, \Phi}^0(\mathbb{R})$.

Let $S := \mathbb{R} \setminus D$, $q \in \mathbb{N}$, $a \vee b := \max\{a, b\}$, $a, b \in \mathbb{R}$, and G be any countable set $G \subset S$ without finite accumulation points. Definition (3.3) implies $w(x) = M_w \cdot \chi_S(x)$, $x \in \mathbb{R}$, and taking into account (1.4) we get $w_n = M_{w_n} \cdot \chi_S$ for every $n \in \mathbb{N}$. It is possible to find $K \in \mathbb{N}$ such that $n_k > q \forall k \geq K$ and $\theta(x) < 1/q \forall |x| \geq K$. Since the set $G \cap [-K, K]$ is finite, (3.2) implies an existence of $k_* > K$ such that

$$\Phi(q | c(x) - P_{n_k}(x) |) < \frac{1}{q}, \quad k \geq k_*, \quad x \in G \cap [-K, K];$$

whence for every integer $k \geq k_*$,

$$\begin{aligned} \sup_{x \in G} w(x) \Phi(q | c(x) - P_{n_k}(x) |) &\leq \sup_{x \in G \cap [-K, K]} w(x) \Phi(q | c(x) - P_{n_k}(x) |) \\ &\vee \sup_{|x| > K} \frac{\theta(x)}{\alpha_0(x) + h(x)} \Phi(q | c(x) - P_{n_k}(x) |) \leq \frac{1}{q} \vee \sup_{|x| > K} \theta(x) \leq \frac{1}{q}, \end{aligned}$$

i.e., $P_{n_k} \in c + W_q^{\Phi, w \cdot \chi_G}$, $k \geq k_*$. By this it is meant that $c \in \text{Cl}_{\Phi}^{w \cdot \chi_G} \mathcal{P}[\mathbb{C}]$, and due to (1.4), (2.7), $\mathcal{P}[\mathbb{C}]$ is dense in all $C_{w_n \cdot \chi_G}^0$, $n \in \mathbb{N}$. The application of Theorem B gives the denseness of $\mathcal{P}[\mathbb{C}]$ in all $C_{w_n}^0$, $n \in \mathbb{N}$. The theorem is proved.

4. PROOF OF COROLLARY 1.1 AND THEOREM 1.1

Condition (1.5) implies an existence of two finite positive numbers q and L such that $\Phi(\lambda \cdot x) \geq (1+q)\Phi(x)$, $x \geq L$, and after the change of variables $x = \varphi(y)$ we get $\varphi((1+q)y) \leq \lambda \cdot \varphi(y)$, $y \geq \Phi(L)$. Successive application of this inequality gives $\varphi((1+q)^k y) \leq \lambda^k \cdot \varphi(y)$, $y \geq \Phi(L)$, $k \in \mathbb{N}$, and hence, for every $n \in \mathbb{N}$ one can find finite positive constants C_n, d_n such that $\varphi(x) \leq C_n \cdot \varphi(x/n)$, $x \geq d_n \Phi(L)$. Since for arbitrary $d > 0$ the function $\varphi(x)/\varphi(x/n)$ is continuous on $[1/d, d_n \Phi(L)]$, it follows that $\varphi(x) \leq C_n(d) \cdot \varphi(x/n)$, $x \geq 1/d$, $n \in \mathbb{N}$, with finite positive numbers $C_n(d)$. Therefore for $d := \|1\|_w$ the formula (1.4) yields $w_n \leq C_n(d) \cdot w_1$, and also $S_{w_n} = S_w$, $w_n \leq w_{n+1}$, for any $n \in \mathbb{N}$, so that the denseness of $\mathcal{P}[\mathbb{C}]$ in all $C_{w_n}^0$, $n \in \mathbb{N}$, is equivalent to the denseness of $\mathcal{P}[\mathbb{C}]$ in $C_{w_1}^0$, and the formula $w = 1/\Phi(1/w_1)$ finishes the proof of Corollary 1.1.

Theorem 1.1 follows readily from Corollary 1.1 when $\Phi(x) = x^p$, $p > 0$.

5. PROOF OF THEOREM 1.2

Sufficiency of both assertions of Theorem 1.2 follows easily from Theorem 1.1 with $p = 2$. The necessity of Theorem 1.2 for $\det \mathcal{H}$ is an obvious consequence of (1.1). It remains to prove the necessity of Theorem 1.2 for $\det \mathcal{S}$.

If $\mu \in \det \mathcal{S}$, then $\mu \in \mathcal{M}^*(\mathbb{R}^+)$, and in any of its representation of type (1.3) we may assume (see [8, Rem. 1, p. 223]) that $S_w, \text{supp } \nu \subset \mathbb{R}^+$. Let $\mathcal{M}_2^*(\mathbb{R}^+)$ denote the set of all $\mu \in \mathcal{M}^*(\mathbb{R}^+)$ such that $L_2(\mathbb{R}, d\mu) \in \overline{\mathcal{P}}$ and $\mathcal{W}_1^*(\mathbb{R}^+)$ the subset of weights $w \in \mathcal{W}^*(\mathbb{R}^+)$ with $C_w^0 \in \overline{\mathcal{P}}$. It follows from (1.2) that $\mu^* \in \mathcal{M}_2^*(\mathbb{R}^+)$, where $d\mu^*(x) := (1+x)d\mu(x)$, and so by Theorem 1.1,

$$(5.1) \quad (1+x)d\mu(x) = w(x)^2 d\nu(x), \quad \nu \in \mathcal{M}^+(\mathbb{R}), \quad w \in \mathcal{W}_1^*(\mathbb{R}^+), \quad \text{supp } \nu \subset \mathbb{R}^+.$$

If the weight w is *regular* (see [14, p. 249]), i.e. $(1+x)^n w \in \mathcal{W}_1^*(\mathbb{R}^+)$ for every $n \in \mathbb{N}_0$, then $(1+x) \geq \sqrt{x}$, $x \in \mathbb{R}^+$, implies $\sqrt{x} \cdot w(x) \in \mathcal{W}_1^*(\mathbb{R}^+)$, as requested. Otherwise, w is called *singular* (see [14, p. 249]), and by Proposition A1.1 in [14, p. 249] there exists $F \in \mathcal{E}_0(\mathbb{R})$ such that $S_w = \Lambda_F$, and in (5.1) we can assume that $\text{supp } \mu = \text{supp } \nu = S_w$. Denote $\lambda_0 := 0$, $\{\lambda_k\}_{k \in \mathbb{N}} := \Lambda_F \setminus \{0\}$, and $\omega_k := \omega(\lambda_k)$, $\sigma_k := \sigma(\{\lambda_k\})$, $k \in \mathbb{N}_0$, for any given $\omega \in \mathcal{W}^*(\mathbb{R})$ and $\sigma \in \mathcal{M}^+(\mathbb{R})$. According to (1.2) and Theorem 1.1 we also have

$$(5.2) \quad x(1+x)d\mu(x) = v(x)^2 d\eta(x), \quad \eta \in \mathcal{M}^+(\mathbb{R}), \quad v \in \mathcal{W}_1^*(\mathbb{R}^+),$$

where $\text{supp } \eta = S_v = \Lambda_F \setminus \{0\}$. Comparison of (5.1) and (5.2) gives

$$(5.3) \quad \lambda_k w_k^2 \nu_k = v_k^2 \eta_k, \quad k \in \mathbb{N}.$$

We introduce the discrete weight $\Omega \in \mathcal{W}^*(\mathbb{R})$ and the discrete measure $\rho \in \mathcal{M}^+(\mathbb{R})$ defined by

$$S_\Omega = \text{supp } \rho \subset \Lambda_F, \quad \Omega_0 := w_0, \quad \Omega_k := w_k \cdot \sqrt{\frac{\nu_k}{\nu_k + \eta_k}}, \quad \rho_0 := \nu_0, \quad \rho_k := \nu_k + \eta_k, \quad k \in \mathbb{N}.$$

Then, obviously, $(1+x)d\mu(x) = \Omega^2(x)d\rho(x)$ and $\Omega \in \mathcal{W}_1^*(\mathbb{R}^+)$ in view of $\Omega(x) \leq w(x)$, $x \in \mathbb{R}$. In addition, (5.3) yields

$$\lambda_k \Omega_k^2 = \frac{\lambda_k w_k^2 \nu_k}{\nu_k + \eta_k} = \frac{v_k^2 \cdot \eta_k}{\nu_k + \eta_k} \leq v_k^2, \quad k \in \mathbb{N},$$

i.e. $\sqrt{x} \cdot \Omega(x) \leq v(x)$, whence $\sqrt{x} \cdot \Omega(x) \in \mathcal{W}_1^*(\mathbb{R}^+)$, which was to be proved.

ACKNOWLEDGMENTS

The author thanks Professors Christian Berg, Stephan Ruscheweyh and Mikhail Sodin for useful discussions. He is greatly indebted to the referee for important remarks.

REFERENCES

- [1] N. I. Akhiezer, *On the weighted approximation of continuous functions on the real axis*, Uspekhi Mat. Nauk **11**(1956), 3-43; AMS Transl. Ser. 2, **22**(1962), 95-137.
- [2] N. I. Akhiezer, *The classical moment problem*, Oliver and Boyd, Edinburgh, 1965.
- [3] G. P. Akilov and L. V. Kantorovich, *Functional analysis in normed spaces*, 2nd ed., Pergamon Press, New York, 1982. MR664597 (83h:46002)
- [4] E. J. Akutowicz, *Weighted approximation on the real axis*, Jahresber. Deutsch. Math.-Verein. **68**(1966), 113-139. MR0200645 (34:535)
- [5] A. G. Bakan, *Polynomial Approximation in $L_p(\mathbb{R}^1, d\mu)$* I., Preprint, Nat. Acad. Sci. of Ukraine, Inst. of Math., Kiev, 1998, No. 7, 45 pp. MR1734341 (2000m:41002)
- [6] A. Bakan, *Polynomial density in $L_p(\mathbb{R}^1, d\mu)$ and representation of all measures which generate a determinate Hamburger moment problem*, in: Approximation, Optimization and Mathematical Economics (Pointe-a-Pitre, 1999), 37-46, Physica, Heidelberg, 2001. MR1842874 (2002i:41004)
- [7] A. G. Bakan, *Criterion of polynomial density and the general form of a continuous linear functional on the space C_w^0* , Ukraïn. Mat. Zh. **54**(2002), No. 5, 610-622 (Russian); English transl.: Ukrainian Math. J. **54**(2002), No. 5, 750-762. MR1956458 (2004a:46028)
- [8] A. Bakan and St. Ruscheweyh, *Representation of measures with simultaneous polynomial denseness in $L_p(\mathbb{R}, d\mu)$, $1 \leq p < \infty$* , Arkiv für matematik **43**(2005), No. 2, 221-249. MR2172989 (2006k:28003)
- [9] J. Berezanskii, Z. Sheftel, and G. Us, *Functional Analysis*, vol. 1, Birkhäuser, Basel, 1996. MR1397267 (97i:46001a)
- [10] Ch. Berg and J. P. R. Christensen, *Density questions in the classical theory of moments*, Ann. Inst. Fourier **31**(1981), No. 3, 99-114. MR638619 (84i:44006)
- [11] Ch. Berg and M. Thill, *Rotation invariant moment problems*, Acta Math. **167**(1991), 207-227. MR1120603 (92j:44004)
- [12] Ch. Berg, *Moment problems and polynomial approximation*, Ann. Fac. Sci. Toulouse, Stieltjes special issue (1996), 9-32. MR1462705 (98h:44002)
- [13] S. Bernstein, *Le probleme de l'approximation des fonctions continues sur tout l'axe reel at l'une de ses applications*, Bull. Math. de France **52**(1924), 399-410. MR1504852
- [14] A. Borichev and M. Sodin, *The Hamburger moment problem and weighted polynomial approximation on discrete subsets of the real line*, J. d'Anal. Math. **76**(1998), 219-264. MR1676987 (2000g:44017)
- [15] L. de Branges, *The Bernstein problem*, Proc. Amer. Math. Soc. **10**(1959), 825-832. MR0114080 (22:4907)
- [16] R. E. Edwards, *Functional Analysis*, Holt, Rinehart & Winston, 1965. MR0221256 (36:4308)
- [17] R. Engelking, *General Topology*, Polish Scientific Publishers, 1985. MR0500780 (58:18316b)
- [18] P. K. Geetha, *On Bernstein approximation problem*, J. Math. Anal. Appl. **25**(1969), 450-469. MR0255827 (41:487)
- [19] P. R. Halmos, *Measure Theory*, Holt, Rinehart & Winston, 1950. MR0033869 (11:504d)
- [20] W. K. Hayman and P. B. Kennedy, *Subharmonic Functions*, Academic Press, 1976. MR0460672 (57:665)
- [21] B. Ja. Levin, *Density of functions, quasianalyticity and subharmonic majorants*, Zapiski nauchn. seminarov LOMI, **170**(1989), 102-156 (Russian); English transl.: J. Soviet Math. **63**(1993), 171-201. MR1039577 (91g:46045)
- [22] D. S. Lubinsky, *Bernstein's weighted approximation on $\mathbb{R}$ still has problems*, Electron. Trans. Numer. Anal. **25**(2006), 166-177. MR2280371 (2007k:41024)
- [23] S. N. Mergelyan, *Weighted approximation by polynomials*, Uspekhi Mat. Nauk **11**(1956), 107-152 (Russian); English transl.: AMS Transl. Ser. 2, **10**(1958), 59-106. MR0094633 (20:1146)
- [24] L. D. Pitt, *Weighted L^p closure theorems for spaces of entire functions*, Isr. J. Math. **24**(1976), 94-118. MR0477726 (57:17239)

- [25] M. Riesz, *Sur le problème des moments et le théorème de Parseval correspondant*, Acta Litt. Ac. Sci. Szeged **1**(1923), 209-225.
- [26] H. Schaefer, *Topological Vector Spaces*, Macmillan, N.Y., 1966. MR0193469 (33:1689)
- [27] M. Sodin, *Which perturbations of quasianalytic weights preserve quasianality? How to use de Branges' theorem*, J. d'Anal. Math. **69**(1996), 293-309. MR1428104 (97k:41015)
- [28] M. Sodin and P. Yuditskii, *Another approach to de Branges' theorem on weighted polynomial approximation*, Israel Math. Conf. Proc., Amer. Math. Soc., Providence, RI, **11**(1997), 221-227. MR1476719 (99c:41014)

INSTITUTE OF MATHEMATICS, NATIONAL ACADEMY OF SCIENCES OF UKRAINE, TERESCHENKIVSKA STREET 3, KYIV 01601, UKRAINE

E-mail address: **andrew@bakan.kiev.ua**