A 2×2 LATTICE SPACE-TIME CODE OF RANK 5

CHAOPING XING AND WEI LI

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ABSTRACT. For all previous constructions of 2×2 lattice space-time codes with a positive diversity product, the rank was at most 4. In this paper, we give an example of a 2×2 lattice space-time code of rank 5 with a positive diversity product.

1. INTRODUCTION

In the recent years, there has been a lot of research on space-time codes. Many mathematical subjects such as number theory, algebra, combinatorics, etc., have been employed to construct good lattice codes [1, 2, 3, 5, 6, 7].

Denote by $\mathcal{M}_n(\mathbb{C})$ the set of $n \times n$ matrices over $\mathbb{C}$. A *lattice space-time code* over $\mathbb{C}$ is a set $\mathcal{A}$ consisting of matrices in $\mathcal{M}_n(\mathbb{C})$ such that $\mathcal{A}$ is a free abelian group under the matrix addition. The rank of this group is called the *dimension* or *rank* of $\mathcal{A}$.

There are some parameters to measure a lattice space-time code.

Criteria for lattice space-time codes:

- (i) the rank of $\mathcal{A}$ should be as large as possible;
- (ii) the *diversity product* defined by

$$\delta(\mathcal{A}) := \inf\{|\det(A - B)| : A, B \in \mathcal{A}, A \neq B\}$$

should be as large as possible, where $|\cdot|$ stands for the absolute value of a real number or complex number;

- (iii) the discriminant of $\mathcal{A}$ should be as small as possible (see the definition of the discriminant of a complex lattice below).

A natural question is:

what is the maximal rank of a lattice space-time code $\mathcal{A}$ such that $\delta(\mathcal{A}) > 0$? In other words, we want to determine

$$(1.1) \quad r(n) := \max\{\text{rank}(\mathcal{A}) : \mathcal{A} \text{ is a lattice in } \mathcal{M}_n(\mathbb{C}), \delta(\mathcal{A}) > 0\}.$$

Determining the exact value for $r(n)$ seems difficult. One can imagine that it is also not easy to give some reasonable bounds. One obvious lower bound is $r(n) \geq 2n$ (see [8, 1, 2, 6, 7]). In this paper, we give an upper bound, i.e., $r(n) \leq 2n^2$.

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The case $n = 2$ is of particular interest. However, even in this case, we only know $r(n) \geq 4$, and so far we have not constructed any 2×2 lattice space-time codes of rank larger than 4 with a positive diversity product. In this paper, we produce an example of a 2×2 lattice space-time code of rank 5 with a positive diversity product.

2. AN UPPER BOUND ON RANK OF LATTICE SPACE-TIME CODES

The main purpose of this section is to give an upper bound on $r(n)$.

Lemma 2.1. *Let $\mathcal{A}$ be a lattice space-time code in $\mathcal{M}_n(\mathbb{C})$. If the diversity product $\delta(\mathcal{A}) > 0$, then the rank of $\mathcal{A}$ is at most $2n^2$. Therefore, $r(n) \leq 2n^2$.*

Proof. We can identify the space $\mathcal{M}_n(\mathbb{C})$ with $\mathbb{C}^{n^2}$. Thus, we introduce the usual distance in $\mathcal{M}_n(\mathbb{C})$ defined by

$$(2.1) \quad d(A, B) = \sqrt{\sum_{i,j} |a_{ij} - b_{ij}|^2}$$

for two matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathcal{M}_n(\mathbb{C})$. It is clear that we have the obvious inequality.

$$\begin{aligned} |d(A, B)| &= \left(\sum_{i=1}^n \sum_{j=1}^n |a_{ij} - b_{ij}|^2 \right)^{1/2} \\ &\geq \left(n \left(\prod_{i=1}^n \sum_{j=1}^n |a_{ij} - b_{ij}|^2 \right)^{1/n^2} \right)^{1/2} \\ &= \sqrt{n} |\det(A - B)|^{1/n^2}. \end{aligned}$$

Hence, we have

$$(2.2) \quad |\det(A - B)| \leq n^{-n^2/2} |d(A, B)|^{n^2}.$$

Let $\{A_1, \dots, A_t\}$ be a $\mathbb{Z}$ -basis with $t = \text{rank}(\mathcal{A})$. Then $\{A_1, \dots, A_t\}$ is linearly independent over $\mathbb{Q}$.

We conclude that $\{A_1, \dots, A_t\}$ is linearly independent over $\mathbb{R}$ as well. Suppose that $\{A_1, \dots, A_t\}$ is linearly dependent over $\mathbb{R}$. Then one of the matrices is a $\mathbb{R}$ -linear combination of others. We may assume that

$$(2.3) \quad A_t = \sum_{i=1}^{t-1} \alpha_i A_i \quad \text{for some } \alpha_i \in \mathbb{R}.$$

Define the set

$$\mathcal{T} := \left\{ \sum_{i=1}^{t-1} a_i A_i : 0 \leq a_i < 1, 1 \leq i \leq t-1 \right\}.$$

Then $\mathcal{T}$ is a bounded domain. For each $m \geq 1$, by (2.3), we have

$$mA_t = \sum_{i=1}^{t-1} m\alpha_i A_i = \sum_{i=1}^{t-1} c_i A_i + \sum_{i=1}^{t-1} b_i A_i$$

with $c_i = m\alpha_i - \lfloor m\alpha_i \rfloor$ and $b_i = \lfloor m\alpha_i \rfloor$. Put $B_m = \sum_{i=1}^{t-1} b_i A_i$ and $C_m = \sum_{i=1}^{t-1} c_i A_i$. Then we have $B_m \in \mathcal{A}$ and $C_m \in \mathcal{T}$. Consider the sequence $\{mA_t - B_m = C_m\}_{m=1}^\infty \subset \mathcal{A} \cap \mathcal{T}$. It is clear that all matrices in the sequence are distinct, as $\{A_1, \dots, A_t\}$ is linearly independent over $\mathbb{Z}$.

Since this sequence is in the bounded domain $\mathcal{T}$, we can find a matrix $D \in \mathcal{M}_n(\mathbb{C})$ such that there is a subsequence $\{C_{m_i}\}_{i=1}^\infty$ with a limit

$$\lim_{i \rightarrow \infty} C_{m_i} = D.$$

Thus, for any $\varepsilon > 0$, there exists N such that $|d(C_{m_N}, C_{m_{N+1}})| \leq \sqrt{n}\varepsilon^{1/n^2}$. Hence, by (2.2) we have $|\det(C_{m_N} - C_{m_{N+1}})| \leq \varepsilon$. This implies that $\delta(\mathcal{A}) = 0$. This contradiction means that $\{A_1, \dots, A_t\}$ is linearly independent over $\mathbb{R}$. The desired result follows from the fact that

$$t = \text{rank}(\mathcal{A}) = \dim_{\mathbb{R}}(\mathcal{A}_{\mathbb{R}}) \leq \dim_{\mathbb{R}}(\mathcal{M}_n(\mathbb{C})) = 2n^2,$$

where $\mathcal{A}_{\mathbb{R}} := \{\sum_{i=1}^t \alpha_i A_i : \alpha_i \in \mathbb{R}\}$ is the linear space over $\mathbb{R}$ generated by $\{A_1, \dots, A_t\}$. $\square$

3. CONSTRUCTION

We need the following result from elementary number theory for this section (see [4, Th. 3 of p. 391]).

Lemma 3.1. *A positive integer can be expressed as a sum of three squares of integers only if it is not of the form $4^r(8t+7)$ for some integers $r, t \geq 0$.*

Consider the five 5×5 matrices over $\mathbb{C}$:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad A_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \quad A_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

and

$$A_4 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}; \quad A_5 = \begin{pmatrix} \sqrt{7} & 0 \\ 0 & -\sqrt{7} \end{pmatrix}.$$

Theorem 3.2. *Let $\mathcal{A}$ be the lattice generated by the above five matrices. Then the rank of $\mathcal{A}$ is 5 and the diversity product of $\mathcal{A}$ is 1.*

Proof. First of all, we have $\det(A_1) = 1$. Hence, the diversity product of $\mathcal{A}$ is at most 1.

To show that the rank of $\mathcal{A}$ is 5 and the diversity product of $\mathcal{A}$ is at least 1, it is sufficient to prove that for any five integers a, b, c, d, e with $(a, b, c, d, e) \neq (0, 0, 0, 0, 0)$ we have $|\det(A)| \geq 1$ with $A = aA_1 + bA_2 + cA_3 + dA_4 + eA_5$.

It is easy to verify that

$$\begin{aligned} |\det(A)| &= |a^2 + b^2 + c^2 + d^2 - 7e^2 - 2\sqrt{7}bei| \\ &= \sqrt{(a^2 + b^2 + c^2 + d^2 - 7e^2)^2 + 28b^2e^2}. \end{aligned}$$

We can show $|\det(A)| \geq 1$ by distinguishing several cases:

- (i) If $be \neq 0$, then we have $|\det(A)| \geq \sqrt{28}$.
- (ii) If e is equal to 0, then $|\det(A)| = \sqrt{a^2 + b^2 + c^2 + d^2} \geq 1$.

- (iii) Finally we assume that $b = 0$; then $|\det(A)|^2 = |a^2 + c^2 + d^2 - 7e^2|$. Write e into the form $2^r f$ with f being an odd integer. Then $7e^2 = 4^r(7f^2) = 4^r(8t+7)$ for some $t \geq 0$. By Lemma 3.1, $7e^2$ cannot be equal to $a^2 + c^2 + d^2$. Thus, $|\det(A)|^2 = |a^2 + c^2 + d^2 - 7e^2| > 0$. Since $|\det(A)|^2$ is an integer, we have $|\det(A)|^2 \geq 1$.

This completes the proof. $\square$

Remark 3.3. For our lattice $\mathcal{A}$ in Theorem 3.2, the dimension of the tensor product $\mathcal{A} \otimes_{\mathbb{Q}} \mathbb{R}$ has dimension 5 over $\mathbb{R}$. Although there exists a 2×2 lattice $\mathcal{B}$ of rank 8 such that the determinant of difference of every two distinct matrices in $\mathcal{B}$ is not vanishing, its inferior limit is still equal to 0. As far as we know, the lattice in this paper is the first 2×2 one of rank larger than 4 with a positive inferior limit.

By Lemma 2.1, we know that a 2×2 lattice space-time code with positive diversity is at most 8, and our Theorem 3.2 shows the existence of a 2×2 lattice space-time code of rank 5 with a positive diversity product. This raises the following question.

Open Problem. Is there a 2×2 lattice space-time code of rank bigger than 5 with a positive diversity product?

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DIVISION OF MATHEMATICAL SCIENCE, SCHOOL OF PHYSICAL AND MATHEMATICAL SCIENCES,
NANYANG TECHNOLOGICAL UNIVERSITY, SINGAPORE 637616, REPUBLIC OF SINGAPORE
E-mail address: xingcp@ntu.edu.sg

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING, PEOPLE'S REPUBLIC OF CHINA