ANALOGUES OF THE ARTIN FACTORIZATION FORMULA FOR THE AUTOMORPHIC SCATTERING MATRIX AND SELBERG ZETA-FUNCTION ASSOCIATED TO A KLEINIAN GROUP

JOSHUA S. FRIEDMAN

(Communicated by Wen-Ching Winnie Li)

ABSTRACT. For Kleinian groups acting on a hyperbolic three-space, we prove factorization formulas for both the Selberg zeta-function and the automorphic scattering matrix. We extend results of Venkov and Zograf from Fuchsian groups to Kleinian groups, and we give a proof that is simple and extendable to more general groups.

1. INTRODUCTION

In [VZ83] Venkov and Zograf gave an analogue of Artin’s well-known factorization formula. More specifically, they gave factorization formulas for both the Selberg zeta-function and automorphic scattering matrix that are associated to a Fuchsian group (in the context of the Selberg spectral theory of automorphic functions).

Let $\Gamma < \mathrm{PSL}(2, \mathbb{C})$ be a cofinite Kleinian group, and let Γ_1 be a finite-index normal subgroup of index n . Let $\chi \in \mathrm{Rep}(\Gamma_1, V)$ be a finite-dimensional unitary representation of Γ_1 in V , and let $\psi = U^\chi \in \mathrm{Rep}(\Gamma, V^n)$ be its induced representation to Γ .

Let $\mathfrak{S}(s, \Gamma_1, \chi), \mathfrak{S}(s, \Gamma, \psi)$ be automorphic scattering matrices (defined in §5), and set

$$\begin{aligned}\phi(s, \Gamma_1, \chi) &\equiv \det \mathfrak{S}(s, \Gamma_1, \chi), \\ \phi(s, \Gamma, \psi) &\equiv \det \mathfrak{S}(s, \Gamma, \psi).\end{aligned}$$

Theorem. *Let Γ be a cofinite Kleinian group, and let Γ_1 be a finite-index normal subgroup of Γ . Let $\chi \in \mathrm{Rep}(\Gamma_1, V)$, and $\psi = U^\chi$. Then for all regular $s \in \mathbb{C}$,*

$$\phi(s, \Gamma_1, \chi) = \phi(s, \Gamma, \psi).$$

In [VZ83] a slightly weaker version of this equation was derived. The authors showed that

$$\phi(s, \Gamma_1, \chi) \Omega(\Gamma_1, \chi)^{1-2s} = \phi(s, \Gamma, U^\chi) \Omega(\Gamma, U^\chi)^{1-2s},$$

where $\Omega(\cdot, \cdot)$ is a constant depending on the group and unitary representation.

We also have

Received by the editors March 2, 2007.

2000 *Mathematics Subject Classification.* Primary 11F72.

©2008 American Mathematical Society
 Reverts to public domain 28 years from publication

Theorem. Let Γ be an arbitrary Kleinian group, and let Γ_1 be a finite-index normal subgroup of Γ . Let $\chi \in \text{Rep}(\Gamma_1, V)$, and $\psi = U^\chi$. Then for $\text{Re } s > 1$,

$$Z(s, \Gamma_1, \chi) = Z(s, \Gamma, \psi).$$

Here $Z(s, \cdot, \cdot)$ is the Selberg zeta-function (defined in §2.2).

Theorem. Let Γ be an arbitrary Kleinian group, and let Γ_1 be a finite-index normal subgroup of Γ . Then for $\text{Re } s > 1$,

$$Z(s, \Gamma_1, \mathbf{1}) = Z(s, \Gamma, U^{\mathbf{1}}) = \prod_{\vartheta \in (\Gamma_1 \backslash \Gamma)^*} Z(s, \Gamma, \vartheta)^{n_\vartheta}.$$

Here $(\Gamma_1 \backslash \Gamma)^*$ is the set of all pairwise inequivalent irreducible unitary representations of the group $\Gamma_1 \backslash \Gamma$; for $\vartheta \in (\Gamma_1 \backslash \Gamma)^*$, n_ϑ is the dimension of ϑ ; $Z(s, \Gamma, \vartheta)$ is the Selberg zeta-function; $\mathbf{1}$ is the trivial representation of Γ_1 , and $U^{\mathbf{1}}$ is its induced representation to Γ .

2. PRELIMINARIES

In this section we state the preliminary results that will be needed. Our main references are [Fri05a, Fri05b], [EGM98], and [Ven82], and [Hej76, Hej83]. Unless stated otherwise, throughout this section $\Gamma < \text{PSL}(2, \mathbb{C})$ is a cofinite Kleinian group and $\chi \in \text{Rep}(\Gamma, V)$ is a finite-dimensional unitary representation of Γ in V .

2.1. Cofinite Kleinian groups. Let $\Gamma < \text{PSL}(2, \mathbb{C})$ be a cofinite Kleinian group acting on a hyperbolic three-space $\mathbb{H}^3$. Let V be a finite-dimensional complex inner product space with inner-product $\langle \cdot, \cdot \rangle_V$, and let $\text{Rep}(\Gamma, V)$ denote the set of finite-dimensional unitary representations of Γ in V . Let $\mathcal{F} \subset \Gamma$ be a fundamental domain for the action of Γ in $\mathbb{H}^3$.

Let $\chi \in \text{Rep}(\Gamma, V)$. The Hilbert space of χ -*automorphic* functions is the set of measurable functions

$$\begin{aligned} \mathcal{H}(\Gamma, \chi) \equiv \{ f : \mathbb{H}^3 \rightarrow V \mid f(\gamma P) = \chi(\gamma)f(P) \ \forall \gamma \in \Gamma, P \in \mathbb{H}^3, \\ \text{and } \langle f, f \rangle \equiv \int_{\mathcal{F}} \langle f(P), f(P) \rangle_V \, dv(P) < \infty \}. \end{aligned}$$

Finally, let $\Delta = \Delta(\Gamma, \chi)$ be the corresponding positive self-adjoint Laplace-Beltrami operator on $\mathcal{H}(\Gamma, \chi)$.

Next we briefly define the concept of a *singular* unitary representation. Let $\mathbb{P}$ be the boundary of $\mathbb{H}^3$, the Riemann sphere. For every $\zeta \in \mathbb{P}^1$ let Γ_ζ denote the stabilizer subgroup of ζ in Γ ,

$$\Gamma_\zeta \equiv \{ \gamma \in \Gamma \mid \gamma\zeta = \zeta \},$$

and let Γ'_ζ be the maximal torsion-free parabolic subgroup of Γ_ζ (the maximal parabolic subgroup of Γ_ζ that does not contain elliptic elements). A point $\zeta \in \mathbb{P}^1$ is called a *cuspid* of Γ if Γ'_ζ is a free abelian group of rank two. Two cusps ζ_1, ζ_2 are Γ -equivalent if $\zeta_1 \in \Gamma\zeta_2$, that is, if their Γ -orbits coincide.

Every cofinite Kleinian group has finitely many equivalence classes of cusps, so we fix a set $\{\zeta_\alpha\}_{\alpha=1}^{\kappa(\Gamma)}$ of representatives of these equivalence classes. For notational convenience we set $\Gamma_\alpha \equiv \Gamma_{\zeta_\alpha}$ and $\Gamma'_\alpha \equiv \Gamma'_{\zeta_\alpha}$.

Each cusp ζ_α has an associated lattice Λ_α (see [EGM98, Theorem 2.1.8]). For each cusp ζ_α of Γ , define the *singular* space $V_\alpha \equiv \{v \in V \mid \chi(\gamma)v = v, \ \forall \gamma \in \Gamma_\alpha\}$, where $1 \leq \alpha \leq \kappa(\Gamma)$.

A representation $\chi \in \text{Rep}$ is *singular* at the cusp ζ_α of Γ iff the subspace $V_\alpha \neq \{0\}$. If a cusp is not singular, it is called *regular*.

For each cusp ζ_α , set $k_\alpha = \dim_{\mathbb{C}} V_\alpha$, and $k(\Gamma, \chi) \equiv \sum_{\alpha=1}^{\kappa(\Gamma)} k_\alpha$.

2.2. Selberg zeta-function. Let $Z(s, \Gamma, \chi)$ denote the Selberg zeta-function associated to Γ and $\chi \in \text{Rep}(\Gamma, \chi)$. We allow Γ to be an *arbitrary* Kleinian group. See [Fri05a] and [EGM98, Sections 5.2, 5.4] for the details on its construction.

In [Fri05a], we gave the meromorphic continuation of $Z(s, \Gamma, \chi)$ to the left half plane under certain technical assumptions. A practical way of understanding the zeta-function is via its logarithmic derivative. For $\text{Re}(s) > 1$,

$$(2.1) \quad \frac{d}{ds} \log Z(s, \Gamma, \chi) = \sum_{\{T\}_{\text{lox}}} \frac{\text{tr}(\chi(T)) \log N(T_0)}{m(T) |a(T) - a(T)^{-1}|^2} N(T)^{-s}.$$

See [Fri05a] for the definition of the notation used above.

Let $W(s, \Gamma, \chi) = \frac{d}{ds} \log Z(s, \Gamma, \chi)$. Then, for $\text{Re } s > 1$,

$$(2.2) \quad Z(s, \Gamma, \chi) = e^{\int W(s, \Gamma, \chi) ds + C},$$

where C is chosen so

$$\lim_{s \rightarrow \infty} Z(s, \Gamma, \chi) = 1$$

will be satisfied.

2.3. Selberg theory of Δ . In this section we assume that Γ is cofinite and we state some needed results concerning the Selberg trace formula associated to Γ and $\chi \in \text{Rep}(\Gamma, V)$. We will not need the full trace formula found in [Fri05a, Fri05b]. Rather, only parts of its proof will be needed. More details can be found in [Fri05a, Fri05b], [EGM98], and [Ven82].

For $P = z + rj$, $P' = z' + r'j \in \mathbb{H}^3$ set

$$\delta(P, P') \equiv \cosh(d(P, P')) = \frac{|z - z'|^2 + r^2 + r'^2}{2rr'},$$

where d is the hyperbolic distance in $\mathbb{H}^3$.

For $k \in \mathcal{S}([1, \infty))$, a Schwartz-class function, set $K(P, P') = k(\delta(P, P'))$. Note that for any $\gamma \in \text{PSL}(2, \mathbb{C})$,

$$(2.3) \quad K(\gamma P, \gamma P') = K(P, P') \quad \text{and} \quad K(P, \gamma P') = K(\gamma^{-1} P, P').$$

For $\Theta \subset \Gamma$, $\chi \in \text{Rep}(\Gamma, V)$, define

$$(2.4) \quad K(P, P', \Theta, \psi) \equiv \sum_{\gamma \in \Theta} \chi(\gamma) K(P, \gamma P').$$

The series above converges absolutely and uniformly on compact subsets of $\mathbb{H}^3 \times \mathbb{H}^3$.

For $\lambda \in \mathbb{C}$, $\lambda = 1 - s^2$, the Selberg–Harish-Chandra transform of k , h is defined by

$$(2.5) \quad h(\lambda) = h(1 - s^2) \equiv \frac{\pi}{s} \int_1^\infty k\left(\frac{1}{2} \left(t + \frac{1}{t}\right)\right) (t^s - t^{-s}) \left(t - \frac{1}{t}\right) \frac{dt}{t}, \quad \lambda = 1 - s^2.$$

In addition, let

$$g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} h(1 + t^2) e^{-itx} dt.$$

We can start with the function h and work backwards to find k [EGM98, Chapter 3]. The pair h, g is said to be *admissible* if h is a holomorphic function on $\{s \in \mathbb{C} \mid |\operatorname{Im}(s)| < 2 + \delta\}$ for some $\delta > 0$, satisfying $h(1 + z^2) = O(1 + |z|^2)^{3/2-\epsilon}$ as $|z| \rightarrow \infty$.

For $v, w \in V$ let $v \otimes \bar{w}$ be the linear operator in V , defined by $v \otimes \bar{w}(x) = \langle x, w \rangle v$, where $x \in V$.

Lemma 2.1 ([Fri05b]). *Let $k \in \mathcal{S}([1, \infty))$ and $h : \mathbb{C} \rightarrow \mathbb{C}$ be the Selberg–Harish-Chandra Transform of k . Then*

$$(2.6) \quad K(P, Q, \Gamma, \chi) = \sum_{m \in \mathcal{D}} h(\lambda_m) e_m(P) \otimes \overline{e_m(Q)} \\ + \frac{1}{4\pi} \sum_{\alpha=1}^{\kappa(\Gamma)} \sum_{l=1}^{k_\alpha} \frac{[\Gamma_\alpha : \Gamma'_\alpha]}{|\Lambda_\alpha|} \int_{\mathbb{R}} h(1+t^2) E_{\alpha l}(P, it) \otimes \overline{E_{\alpha l}(Q, it)} dt.$$

The sum and integrals converge on compact subsets of $\mathbb{H}^3 \times \mathbb{H}^3$. Here $\mathcal{D}$ is an indexing set of the eigenfunctions e_m of Δ with corresponding eigenvalues λ_m , $E_{\alpha l}(P, s)$ are the Eisenstein series associated to the singular cusps of Γ , $k_\alpha = \dim_{\mathbb{C}} V_\alpha$, and $|\Lambda_\alpha|$ is the Euclidean area of a fundamental domain for the lattice Λ_α . If a cusp is regular, it is omitted from the sum in (2.6).

Next, we split up K_Γ as a sum of two kernels. The first kernel,

$$H_\Gamma(P, Q) = \frac{1}{4\pi} \sum_{\alpha=1}^{\kappa} \sum_{l=1}^{k_\alpha} \frac{[\Gamma_\alpha : \Gamma'_\alpha]}{|\Lambda_\alpha|} \int_{\mathbb{R}} h(1+t^2) E_{\alpha l}(P, it) \otimes \overline{E_{\alpha l}(Q, it)} dt,$$

is not of Hilbert–Schmidt class, while the second kernel,

$$L_\Gamma(P, Q) = \sum_{m \in \mathcal{D}} h(\lambda_m) e_m(P) \otimes \overline{e_m(Q)},$$

is of trace class.

Suppose $Y > 0$ is sufficiently large. Then for all $A > Y$, there exists a compact set $\mathcal{F}_A \subset \mathbb{H}^3$ such that

$$(2.7) \quad \mathcal{F} \equiv \mathcal{F}_A \cup \mathcal{F}_1(A) \cup \cdots \cup \mathcal{F}_\kappa(A)$$

is a fundamental domain for Γ . The sets $\mathcal{F}_\alpha(A)$ are cusp sectors (see [EGM98], Proposition 2.3.9). It follows that

$$(2.8) \quad \lim_{A \rightarrow \infty} \left(\int_{\mathcal{F}_A} \operatorname{tr}_V(K_\Gamma(P, P)) dv(P) - \int_{\mathcal{F}_A} \operatorname{tr}_V(H_\Gamma(P, P)) dv(P) \right) \\ = \int_{\mathcal{F}} \operatorname{tr}_V(L_\Gamma(P, P)) dv(P) = \sum_{m \in \mathcal{D}} h(\lambda_m) < \infty.$$

The infinite sum is absolutely convergent.

Let $\mathfrak{S}(s)$ denote the *automorphic scattering matrix* associated to Γ and χ (see [Fri05b]), and let

$$\phi(s) = \det \mathfrak{S}(s).$$

Upon applying the vector form of the Maaß-Selberg relations (see [Roe66], [Ven82], [Fri05b]), we obtain

$$(2.9) \quad \int_{\mathcal{F}_A} \operatorname{tr}_V(H_\Gamma(P, P)) dv(P) \\ = g(0)k(\Gamma, \chi) \log(A) - \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\phi'}{\phi}(it) h(1+t^2) dt + \frac{h(1) \operatorname{tr} \mathfrak{S}(0)}{4} + o(1)_{A \rightarrow \infty}.$$

The integral on the right-hand side converges absolutely.

3. INDUCED REPRESENTATIONS

Let Γ be an *arbitrary* Kleinian group, and let $\Gamma_1 \triangleleft \Gamma$ be a finite-index normal subgroup of index n . Let $\mathcal{F}, \mathfrak{F}$ be fundamental domains of Γ, Γ_1 respectively, with $[\alpha_i]_{i=1}^n$ a complete set of representatives for the right-cosets of $\Gamma_1 \setminus \Gamma$, satisfying

$$(3.1) \quad \mathfrak{F} = \bigcup_{i=1}^n \alpha_i(\mathcal{F}).$$

Let V be a finite-dimensional hermitian vector space, and let χ be a finite-dimensional unitary representation of Γ_1 in V . Set

$$\overline{\chi}(\gamma) = \begin{cases} \chi(\gamma), & \gamma \in \Gamma_1, \\ 0, & \gamma \notin \Gamma_1. \end{cases}$$

Let $\psi \equiv U^\chi$ be the induced representation of χ from Γ_1 to Γ . More explicitly,

$$\psi : \Gamma \mapsto \operatorname{GL}(V^n),$$

and for $\gamma \in \Gamma$ and $v_i \in V$,

$$(3.2) \quad \psi(\gamma) \left(\sum_{i=1}^n \oplus v_i \right) = \sum_{i=1}^n \oplus \sum_{j=1}^n \overline{\chi}(\alpha_i \gamma \alpha_j^{-1}) v_j.$$

It follows from (3.2) that for $\gamma \in \Gamma$,

$$(3.3) \quad \operatorname{tr}_{V^n} \psi(\gamma) = \sum_{i=1}^n \operatorname{tr}_V \overline{\chi}(\alpha_i \gamma \alpha_i^{-1}).$$

We will need the following result (see [RS71] and [VZ83, Theorem 2.1]):

Lemma 3.1. *There exists an isometry between the Hilbert spaces $\mathcal{H}(\Gamma_1, \chi)$ and $\mathcal{H}(\Gamma, \psi)$, which takes the operator $\Delta(\Gamma_1, \chi)$ to $\Delta(\Gamma, \psi)$.*

We abuse notation and call a set $\Theta \subset \Gamma$ *normal* if for all $\gamma \in \Gamma$, we have $\gamma\Theta\gamma^{-1} = \Theta$.

Lemma 3.2. *Let $\Gamma_1 \triangleleft \Gamma$ be a finite-index, normal subgroup of Γ of index n , $\chi \in \operatorname{Rep}(\Gamma_1, V)$, $\psi \equiv U^\chi \in \operatorname{Rep}(\Gamma, V^n)$, and let $\Theta \subset \Gamma$ be a normal subset. Suppose that*

$$\int_{\mathcal{F}} \operatorname{tr}_{V^n} K(P, P, \Theta, \psi) dv(P)$$

converges absolutely. Then

$$\int_{\mathcal{F}} \operatorname{tr}_{V^n} K(P, P, \Theta, \psi) dv(P) = \int_{\mathfrak{F}} \operatorname{tr}_V K(P, P, \Theta \cap \Gamma_1, \chi) dv(P).$$

Proof. By (3.3),

$$\begin{aligned}
 & \int_{\mathcal{F}} \mathrm{tr}_{V^n} K(P, P, \Theta, \psi) dv(P) \\
 &= \sum_{i=1}^n \int_{\mathcal{F}} \sum_{\gamma \in \Theta} \mathrm{tr}_V \bar{\chi}(\alpha_i \gamma \alpha_i^{-1}) K(P, \gamma P) dv(P) \\
 &= \sum_{i=1}^n \int_{\mathcal{F}} \sum_{\substack{\gamma \in \Theta \\ \alpha_i \gamma \alpha_i^{-1} \in \Gamma_1}} \mathrm{tr}_V \chi(\alpha_i \gamma \alpha_i^{-1}) K(P, \gamma P) dv(P) \\
 &= \sum_{i=1}^n \int_{\mathcal{F}} \sum_{\gamma \in \Theta \cap \Gamma_1} \mathrm{tr}_V \chi(\alpha_i \gamma \alpha_i^{-1}) K(P, \gamma P) dv(P) \quad (\text{by normality}).
 \end{aligned}$$

Now, since Θ and Γ_1 are normal, $\gamma \in \Theta \cap \Gamma_1$ implies that $\alpha_i \gamma \alpha_i^{-1} \in \Theta \cap \Gamma_1$. Also, as γ goes through each element of $\Theta \cap \Gamma_1$, so does $\alpha_i \gamma \alpha_i^{-1}$. So, for each i , setting $\beta = \alpha_i \gamma \alpha_i^{-1}$ we obtain

$$\begin{aligned}
 & \sum_{i=1}^n \int_{\mathcal{F}} \sum_{\gamma \in \Theta \cap \Gamma_1} \mathrm{tr}_V \chi(\alpha_i \gamma \alpha_i^{-1}) K(P, \gamma P) dv(P) \\
 &= \sum_{i=1}^n \int_{\mathcal{F}} \sum_{\beta \in \Theta \cap \Gamma_1} \mathrm{tr}_V \chi(\beta) K(P, \alpha_i^{-1} \beta \alpha_i P) dv(P) \\
 (3.4) \quad &= \sum_{i=1}^n \int_{\mathcal{F}} \sum_{\beta \in \Theta \cap \Gamma_1} \mathrm{tr}_V \chi(\beta) K(\alpha_i P, \beta \alpha_i P) dv(P)
 \end{aligned}$$

$$(3.5) \quad = \sum_{i=1}^n \int_{\alpha_i(\mathcal{F})} \sum_{\beta \in \Theta \cap \Gamma_1} \mathrm{tr}_V \chi(\beta) K(Q, \beta Q) dv(Q)$$

$$\begin{aligned}
 (3.6) \quad &= \int_{\mathfrak{F}} \sum_{\beta \in \Theta \cap \Gamma_1} \mathrm{tr}_V \chi(\beta) K(Q, \beta Q) dv(Q) \\
 &= \int_{\mathfrak{F}} \mathrm{tr}_V K(P, P, \Theta \cap \Gamma_1, \chi) dv(P),
 \end{aligned}$$

where in (3.5) we set $Q = \alpha_i P$. In (3.4) we used Equation (2.3), and we used the fact that $dv(\alpha_i Q) = dv(Q)$; and in (3.6) we *tiled* the fundamental domain $\mathcal{F}$ according to (3.1). $\square$

Lemma 3.2 works well when the point-pair-invariant k gives rise to an integral kernel of trace-class. However, a careful look at the proof shows that we could replace $\mathcal{F}$ by a *truncated* fundamental domain $\mathcal{F}_A$.

For the rest of this section, we assume that Γ is cofinite. Let $Y > 0$ be sufficiently large so that for $A > Y$, $\mathcal{F}$ decomposes into $\mathcal{F} = \mathcal{F}_A \cup \mathcal{F}^A$, where $\mathcal{F}_A$ is compact and $\mathcal{F}^A$ is a union of cusp sectors (2.7). Since

$$\mathfrak{F} = \bigcup_{i=1}^n \alpha_i(\mathcal{F})$$

it follows that $\mathfrak{F} = \mathfrak{F}_A \cup \mathfrak{F}^A$ with $\mathfrak{F}_A = \bigcup_{i=1}^n \alpha_i(\mathcal{F}_A)$ and $\mathfrak{F}^A = \bigcup_{i=1}^n \alpha_i(\mathcal{F}^A)$; $\mathfrak{F}^A$ is also a union of cusp sectors.

Lemma 3.3. *Let $\Gamma_1 \triangleleft \Gamma$ be a finite-index, normal subgroup of Γ of index n , $\chi \in \text{Rep}(\Gamma_1, V)$, $\psi \equiv U^\chi \in \text{Rep}(\Gamma, V^n)$, and let $\Theta \subset \Gamma$ be a normal subset. Suppose that*

$$\int_{\mathcal{F}_A} \text{tr}_{V^n} K(P, P, \Theta, \psi) dv(P)$$

converges absolutely. Then

$$\int_{\mathcal{F}_A} \text{tr}_{V^n} K(P, P, \Theta, \psi) dv(P) = \int_{\mathfrak{F}_A} \text{tr}_V K(P, P, \Theta \cap \Gamma_1, \chi) dv(P).$$

4. FACTORIZATION FORMULA OF THE SELBERG ZETA-FUNCTION

In this section we prove the analogue of the Venkov-Zograf factorization formula ([VZ83]) for *arbitrary* Kleinian groups.

Throughout this section, Γ is an arbitrary Kleinian group and $\Gamma_1 \triangleleft \Gamma$ is a finite-index normal subgroup of index n .

Theorem 4.1. *Suppose that $\Gamma_1 \triangleleft \Gamma$, $\chi \in \text{Rep}(\Gamma_1, V)$, and that $\psi = U^\chi \in \text{Rep}(\Gamma, V^n)$. Then for $\text{Re}(s) > 1$,*

$$Z(s, \Gamma_1, \chi) = Z(s, \Gamma, \psi).$$

Proof. By (2.2), it suffices to show that

$$(4.1) \quad W(s, \Gamma_1, \chi) = W(s, \Gamma, \psi).$$

Let $\Theta = \Gamma^{\text{lox}} \subset \Gamma$ be the set of *all* loxodromic elements of Γ . Note that Θ is *normal* since the conjugate of a loxodromic element is loxodromic. For $\text{Re } s > 1$, $\delta > 1$, set

$$k_s(\delta) \equiv \frac{1}{4\pi} \frac{(\delta + \sqrt{\delta^2 - 1})^{-s}}{\sqrt{\delta^2 - 1}}.$$

For $P, P' \in \mathbb{H}^3$, set $K_s(P, P') = k_s(\delta(P, P'))$, and, using (2.4), define $K_s(P, P', \Theta, \psi)$ and $K_s(P, P', \Theta \cap \Gamma_1, \chi)$. By [EGM98, pages 185-198] and [Fri05a, Lemma 6.3], it follows that

$$W(s, \Gamma_1, \chi) = \int_{\mathfrak{F}} \text{tr}_V K_s(P, P, \Theta \cap \Gamma_1, \chi) dv(P)$$

and

$$W(s, \Gamma, \psi) = \int_{\mathcal{F}} \text{tr}_{V^n} K_s(P, P, \Theta, \psi) dv(P).$$

The result now follows from Lemma 3.2. $\square$

Compare our proof with the proof given in [VZ83, Theorem 3.1].

If we let $\chi = \mathbf{1}$, the trivial one-dimensional representation, it follows that $\psi = U^{\mathbf{1}}$ can be decomposed into irreducible sub-representations, explicitly:

$$\psi = \bigoplus_{\vartheta \in (\Gamma_1 \setminus \Gamma)^*} n_{\vartheta} \vartheta,$$

where $(\Gamma_1 \setminus \Gamma)^*$ denotes the set of all pairwise inequivalent irreducible unitary representations of the group¹ $\Gamma_1 \setminus \Gamma$, and n_{ϑ} is the dimension of ϑ . For more details see [Kir76].

Next, by (2.1) and (2.2) it follows that for $\vartheta_1, \vartheta_2 \in \text{Rep}(\Gamma, V_i)$ ($i = 1, 2$),

$$Z(s, \Gamma, \vartheta_1 \oplus \vartheta_2) = Z(s, \Gamma, \vartheta_1) Z(s, \Gamma, \vartheta_2).$$

¹Of course, each χ must be extended from $\Gamma_1 \setminus \Gamma$ to Γ .

We have

Theorem 4.2. *Let $\Gamma_1 \triangleleft \Gamma$. Then for $\operatorname{Re} s > 1$,*

$$Z(s, \Gamma_1, \mathbf{1}) = Z(s, \Gamma, U^1) = \prod_{\vartheta \in (\Gamma_1 \backslash \Gamma)^*} Z(s, \Gamma, \vartheta)^{n_\vartheta}.$$

5. FACTORIZATION FORMULA FOR THE DETERMINANT OF THE SCATTERING MATRIX

In this section we prove the analogous result to Theorem 4.1 for the automorphic scattering matrix. Throughout this section, Γ is a cofinite Kleinian group.

Let $\Gamma_1 \triangleleft \Gamma$, $\chi \in \operatorname{Rep}(\Gamma_1, V)$, and $\psi = U^\chi$. Let $k \in \mathcal{S}([1, \infty))$, and let h be given by (2.5). It follows from Lemma 3.1 that $\Delta(\Gamma_1, \chi)$ and $\Delta(\Gamma, \psi)$ have the same eigenvalues. Hence (from §2.3)

$$\int_{\mathcal{F}} \operatorname{tr}_{V^n} L(P, P, \Gamma, \psi) dv(P) = \int_{\mathfrak{F}} \operatorname{tr}_V L(P, P, \Gamma_1, \chi) dv(P) = \sum_{m \in \mathcal{D}} h(\lambda_m).$$

Now, since $\int_{\mathcal{F}_A} \operatorname{tr}_{V^n} K(P, P, \Gamma, \psi) dv(P) = \int_{\mathfrak{F}_A} \operatorname{tr}_V K(P, P, \Gamma_1, \chi) dv(P)$, it follows from Equation (2.8) that

$$\int_{\mathcal{F}_A} \operatorname{tr}_{V^n} H(P, P, \Gamma, \psi) dv(P) = \int_{\mathfrak{F}_A} \operatorname{tr}_V H(P, P, \Gamma_1, \chi) dv(P) + o(1)_{A \rightarrow \infty}.$$

However, from Equation (2.9), we obtain

$$\begin{aligned} (5.1) \quad & \int_{\mathcal{F}_A} \operatorname{tr}_{V^n} H(P, P, \Gamma, \psi) dv(P) \\ &= g(0)k(\Gamma, \psi) \log(A) - \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\phi'_\Gamma}{\phi_\Gamma}(it) h(1+t^2) dt + \frac{h(1) \operatorname{tr} \mathfrak{S}_\Gamma(0)}{4} + o(1)_{A \rightarrow \infty} \end{aligned}$$

and

$$\begin{aligned} (5.2) \quad & \int_{\mathcal{F}_A} \operatorname{tr}_V H(P, P, \Gamma_1, \chi) dv(P) \\ &= g(0)k(\Gamma_1, \chi) \log(A) - \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\phi'_{\Gamma_1}}{\phi_{\Gamma_1}}(it) h(1+t^2) dt + \frac{h(1) \operatorname{tr} \mathfrak{S}_{\Gamma_1}(0)}{4} + o(1)_{A \rightarrow \infty}. \end{aligned}$$

Hence, we have

$$\begin{aligned} (5.3) \quad & g(0)k(\Gamma, \psi) \log(A) - \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\phi'_\Gamma}{\phi_\Gamma}(it) h(1+t^2) dt + \frac{h(1) \operatorname{tr} \mathfrak{S}_\Gamma(0)}{4} \\ &= g(0)k(\Gamma_1, \chi) \log(A) - \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\phi'_{\Gamma_1}}{\phi_{\Gamma_1}}(it) h(1+t^2) dt + \frac{h(1) \operatorname{tr} \mathfrak{S}_{\Gamma_1}(0)}{4} + o(1)_{A \rightarrow \infty}. \end{aligned}$$

Equation (5.3) is true for *all* admissible pairs h, g . Hence we must have

$$\begin{aligned} k(\Gamma, \psi) &= k(\Gamma_1, \chi), \\ \operatorname{tr} \mathfrak{S}_\Gamma(0) &= \operatorname{tr} \mathfrak{S}_{\Gamma_1}(0), \end{aligned}$$

and

$$(5.4) \quad \frac{\phi'_\Gamma}{\phi_\Gamma}(z) = \frac{\phi'_{\Gamma_1}}{\phi_{\Gamma_1}}(z).$$

We will prove (5.4) shortly.

Theorem 5.1. *Let $\Gamma_1 \triangleleft \Gamma$, $\chi \in \text{Rep}(\Gamma_1, V)$, and $\psi = U^\chi \in \text{Rep}(\Gamma, V^n)$. Let*

$$\begin{aligned}\phi_{\Gamma_1}(s) &\equiv \det \mathfrak{S}_{\Gamma_1}(s) \equiv \det \mathfrak{S}(s, \Gamma_1, \chi), \\ \phi_\Gamma(s) &\equiv \det \mathfrak{S}_\Gamma(s) \equiv \det \mathfrak{S}(s, \Gamma, \psi).\end{aligned}$$

Then for all regular s ,

$$\phi_{\Gamma_1}(s) = \phi_\Gamma(s).$$

Proof. For $r > 0, z \in \mathbb{C}$, let $h(z) = e^{-rz^2}$ (and it follows that)

$$g(x) = \frac{e^{-r}}{\sqrt{4\pi r}} e^{-x^2/(4r)}.$$

By considering the asymptotics of (5.3), it follows that

$$k(\Gamma, \psi) = k(\Gamma_1, \chi)$$

and

$$\text{tr } \mathfrak{S}_\Gamma(0) = \text{tr } \mathfrak{S}_{\Gamma_1}(0).$$

Hence

$$\int_{\mathbb{R}} \frac{\phi'_\Gamma}{\phi_\Gamma}(it) h(1+t^2) dt = \int_{\mathbb{R}} \frac{\phi'_{\Gamma_1}}{\phi_{\Gamma_1}}(it) h(1+t^2) dt,$$

which implies that

$$\int_{\mathbb{R}} \frac{\phi'_\Gamma}{\phi_\Gamma}(it) e^{-rt^2} dt = \int_{\mathbb{R}} \frac{\phi'_{\Gamma_1}}{\phi_{\Gamma_1}}(it) e^{-rt^2} dt \quad (r > 0).$$

Now, by the functional equation for $\mathfrak{S}_\Gamma(s)$ (see [Fri05a])

$$\mathfrak{S}_\Gamma(s) \mathfrak{S}_\Gamma(-s) = I.$$

So

$$\phi_\Gamma(s) \phi_\Gamma(-s) = 1.$$

Hence $\frac{\phi'_\Gamma}{\phi_\Gamma}(it)$ is an even function of t ; so is $\frac{\phi'_{\Gamma_1}}{\phi_{\Gamma_1}}(it)$. Thus

$$\int_0^\infty \frac{\phi'_\Gamma}{\phi_\Gamma}(it) e^{-rt^2} dt = \int_0^\infty \frac{\phi'_{\Gamma_1}}{\phi_{\Gamma_1}}(it) e^{-rt^2} dt \quad (r > 0).$$

Next, the substitution $u = t^2$ allows us to rewrite the above integral as a Laplace transform, and by uniqueness (the Laplace transform is invertible), it follows that $\frac{\phi'_\Gamma}{\phi_\Gamma}(it) = \frac{\phi'_{\Gamma_1}}{\phi_{\Gamma_1}}(it)$, and (by analytic continuation)

$$\frac{\phi'_\Gamma}{\phi_\Gamma}(s) = \frac{\phi'_{\Gamma_1}}{\phi_{\Gamma_1}}(s).$$

Integrating and exponentiating give us a constant C_1 so that

$$\phi_\Gamma = C_1 \cdot \phi_{\Gamma_1}.$$

However, the functional equation (for $\mathfrak{S}_\Gamma(s)$) implies that

$$\phi_\Gamma(0) = (-1)^{(k_\Gamma - \text{tr } \mathfrak{S}_\Gamma(0))/2} = (-1)^{(k_{\Gamma_1} - \text{tr } \mathfrak{S}_{\Gamma_1}(0))/2} = \phi_{\Gamma_1}(0) \neq 0.$$

Hence $C_1 = 1$. □

ACKNOWLEDGEMENTS

I would like to thank Professor Leon Takhtajan for originally suggesting this problem to me and for reading this paper. I would also like to thank Peter Zograf for reading this paper and for useful discussions, and the anonymous referee for correcting some errors.

REFERENCES

- [EGM98] J. Elstrodt, F. Grunewald, and J. Mennicke, *Groups acting on hyperbolic space*, Harmonic analysis and number theory, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998. MR1483315 (98g:11058)
- [Fri05a] Joshua S. Friedman, *The Selberg trace formula and Selberg zeta-function for cofinite Kleinian groups with finite-dimensional unitary representations*, Math. Z. **250** (2005), no. 4, 939–965. MR2180383 (2006g:11099)
- [Fri05b] ———, *The Selberg trace formula and Selberg zeta-function for cofinite Kleinian groups with finite-dimensional unitary representations*. Ph.D. Thesis, Stony Brook University, 2005, <http://arxiv.org/abs/math.NT/0612807>.
- [Hej76] Dennis A. Hejhal, *The Selberg trace formula for $\mathrm{PSL}(2, R)$. Vol. I*, Lecture Notes in Mathematics, vol. 548, Springer-Verlag, Berlin, 1976. MR0439755 (55:12641)
- [Hej83] ———, *The Selberg trace formula for $\mathrm{PSL}(2, \mathbf{R})$. Vol. 2*, Lecture Notes in Mathematics, vol. 1001, Springer-Verlag, Berlin, 1983. MR0711197 (86e:11040)
- [Kir76] A. A. Kirillov, *Elements of the theory of representations*, Springer-Verlag, Berlin, 1976, Translated from the Russian by Edwin Hewitt, Grundlehren der Mathematischen Wissenschaften, Band 220. MR0412321 (54:447)
- [Roe66] Walter Roelcke, *Das Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene. I, II*, Math. Ann. **167** (1966), 292–337; *ibid.* **168** (1966), 261–324. MR1513277
- [RS71] D. B. Ray and I. M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Advances in Math. **7** (1971), 145–210. MR0295381 (45:4447)
- [Ven82] A. B. Venkov, *Spectral theory of automorphic functions*, Proc. Steklov Inst. Math. **1982**, no. 4(153) (1983); a translation of Trudy Mat. Inst. Steklov. **153** (1981). MR665585 (85j:11060a), MR0692019 (85j:11060b)
- [VZ83] A. B. Venkov and P. G. Zograf, *Analogues of Artin's factorization formulas in the spectral theory of automorphic functions associated with induced representations of Fuchsian groups*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), 1150–1158, 1343; translated in Math. USSR-Izvestiya **21** (1983), no. 3, 435–443. MR0682487 (85f:11041)

DEPARTMENT OF MATHEMATICS AND SCIENCES, UNITED STATES MERCHANT MARINE ACADEMY,
300 STEAMBOAT ROAD, KINGS POINT, NEW YORK 11024

E-mail address: CrownEagle@gmail.com, friedmanj@usmma.edu, joshua@math.sunysb.edu