ON THE ISOLATED POINTS OF THE SURJECTIVE SPECTRUM OF A BOUNDED OPERATOR

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ABSTRACT. For a bounded operator T acting on a complex Banach space, we show that if $T - \lambda$ is not surjective, then λ is an isolated point of the surjective spectrum $\sigma_{su}(T)$ of T if and only if $X = H_0(T - \lambda) + K(T - \lambda)$, where $H_0(T)$ is the quasinilpotent part of T and $K(T)$ is the analytic core for T . Moreover, we study the operators for which $\dim K(T) < \infty$. We show that for each of these operators T , there exists a finite set E consisting of Riesz points for T such that $0 \in \sigma(T) \setminus E$ and $\sigma(T) \setminus E$ is connected, and derive some consequences.

1. INTRODUCTION

Let X be a complex Banach space and let T be a bounded operator acting on X . The *quasinilpotent part* $H_0(T)$ and the *analytic core* $K(T)$ of T are subspaces of X that have been useful in the study of the spectral properties of operators [1, 4, 6, 8, 9, 11, 12, 14]. For example, the following characterization of the isolated points of the spectrum $\sigma(T)$ was obtained in [8].

Theorem 1 ([8, Théorème 1.6]). *Let $T \in B(X)$ and let $\lambda_0 \in \sigma(T)$. Then λ_0 is isolated in $\sigma(T)$ if and only if $H_0(T - \lambda_0)$ and $K(T - \lambda_0)$ are closed and*

$$X = H_0(T - \lambda_0) \oplus K(T - \lambda_0).$$

Ch. Schmoegele [14] and J. J. Koliha [6] improved Theorem 1 by showing that it is not necessary to assume that $H_0(T - \lambda_0)$ is closed. Here we give a characterization for the isolated points of the surjective spectrum $\sigma_{su}(T)$. We show that, assuming $\lambda_0 \in \sigma_{su}(T)$, the point λ_0 is isolated in $\sigma_{su}(T)$ if and only if $X = H_0(T - \lambda_0) + K(T - \lambda_0)$.

In Section 3 we describe the properties of the operators T for which $K(T)$ is finite dimensional. We show that there exists a finite set E (eventually empty), consisting of Riesz points for T , such that $\sigma(T) \setminus E$ is connected and $0 \in \sigma(T) \setminus E$; in the case that 0 is an accumulation point for $\sigma(T)$, we show that Weyl's theorem holds for T . We derive several consequences concerning the isolated points of $\sigma(T)$ and the semi-Fredholm spectrum of T .

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Let X be a complex Banach space and let $B(X)$ denote the algebra of all bounded operators acting on X . Given $T \in B(X)$, we denote by $N(T)$, $R(T)$ and $\sigma(T)$ the kernel, the range and the spectrum of T , respectively.

The *quasinilpotent part* of T is defined by

$$H_0(T) := \{x \in X : \lim_{n \rightarrow +\infty} \|T^n x\|^{1/n} = 0\},$$

and the *analytic core* of T , denoted by $K(T)$, is defined as the set of all $x \in X$ for which there exist $a > 0$ and a sequence (x_n) in X satisfying:

- (1) $x_0 = x$, $Tx_{n+1} = x_n$, and
- (2) $\|x_n\| \leq a^n \|x\|$, for all $n \geq 1$.

We observe that in the definition of $K(T)$, condition (2), can be replaced by

- (2') $\limsup_{n \rightarrow \infty} \|x_n\|^{1/n} < \infty$.

The following two lemmas describe the basic properties of $H_0(T)$ and $K(T)$; see [8, 11]. Recall that T is said to be *quasinilpotent* when $\sigma(T) = \{0\}$.

Lemma 2. *Let $T \in B(X)$. Then:*

- (1) $H_0(T)$ is a (not necessarily closed) subspace of X ;
- (2) for each $j \geq 0$, $N(T^j) \subseteq H_0(T)$;
- (3) $x \in H_0(T)$ if and only if $Tx \in H_0(T)$;
- (4) $H_0(T) = X$ if and only if T is quasinilpotent;
- (5) for $\lambda \neq 0$, $(T - \lambda I)(H_0(T)) = H_0(T)$.

Lemma 3. *Let $T \in B(X)$. Then:*

- (1) $K(T)$ is a (not necessarily closed) subspace of X ;
- (2) $T(K(T)) = K(T)$;
- (3) if X_0 is a closed subspace of X and $T(X_0) = X_0$, then $X_0 \subseteq K(T)$;
- (4) if T is quasinilpotent, then $K(T) = \{0\}$;
- (5) if T is surjective, then $K(T) = X$.

For $T \in B(X)$ and $x \in X$, the *local resolvent* of T at x , denoted $\rho_T(x)$, is defined as the set of all $\lambda \in \mathbb{C}$ for which there exist a neighborhood V_λ of λ and an analytic function $\tilde{x} : V_\lambda \rightarrow X$ satisfying

$$(T - \mu)\tilde{x}(\mu) = x \text{ for all } \mu \in V_\lambda.$$

The set $\sigma_T(x) := \mathbb{C} \setminus \rho_T(x)$ is called the *local spectrum* of T at x (see [3, 7, 9, 15]). The operator T has the *single-valued extension property* (the SVEP for short) if for every open subset V of $\mathbb{C}$, the only analytic function $\phi : V \rightarrow X$ that satisfies $(T - \mu)\phi(\mu) = 0$ for every $\mu \in V$ is $\phi \equiv 0$.

For every subset F of $\mathbb{C}$, the *local subspace* $X_T(F)$ is defined by

$$X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}.$$

For a closed subset F of $\mathbb{C}$, the *glocal subspace* $\mathcal{X}_T(F)$ is defined as the set of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus F \rightarrow X$ satisfying $(T - \lambda)f(\lambda) = x$ on $\mathbb{C} \setminus F$.

Obviously, for F a closed set, $\mathcal{X}_T(F) \subseteq X_T(F)$, and we have equality when T satisfies the SVEP.

The *surjective spectrum* $\sigma_{su}(T)$ of $T \in B(X)$ is defined as the set of all $\lambda \in \mathbb{C}$ such that $R(T - \lambda) \neq X$. It is well-known that $\sigma_{su}(T)$ is a compact subset of $\mathbb{C}$ that contains the boundary of $\sigma(T)$.

The next lemma describes $K(T)$ and $H_0(T)$ in terms of the local and global subspaces.

Lemma 4. (See [7, 9]).

- (1) $K(T) = X_T(\mathbb{C} \setminus \{0\}) = \{x \in X : 0 \notin \sigma_T(x)\}.$
- (2) $H_0(T) = \mathcal{X}_T(\{0\}) \subseteq X_T(\{0\}) = \{x \in X : \sigma_T(x) \subseteq \{0\}\}.$
- (3) $\mathcal{X}_T(\sigma_{su}(T)) = X_T(\sigma_{su}(T)) = X$ ([7, Section 3.2]).

2. ISOLATED POINTS OF THE SURJECTIVE SPECTRUM

Next we give a characterization of the isolated points of $\sigma_{su}(T)$ in terms of $H_0(T)$ and $K(T)$. Recall that $0 \notin \sigma_{su}(T)$ if and if $K(T) = X$.

Theorem 5. *Let $T \in B(X)$ such that $0 \in \sigma_{su}(T)$. Then 0 is isolated in $\sigma_{su}(T)$ if and only if $X = H_0(T) + K(T)$.*

Proof. Suppose that 0 is an isolated point in $\sigma_{su}(T)$. By [7, Proposition 3.3.1] we can write

$$\mathcal{X}_T(\sigma_{su}(T)) = \mathcal{X}_T(\{0\}) + \mathcal{X}_T(\sigma_{su}(T) \setminus \{0\}).$$

By Lemma 4, $\mathcal{X}_T(\sigma_{su}(T)) = X$ and $\mathcal{X}_T(\sigma_{su}(T) \setminus \{0\}) \subseteq X_T(\mathbb{C} \setminus \{0\}) = K(T)$; hence we get $X = H_0(T) + K(T)$.

Conversely, suppose that $0 \in \sigma_{su}(T)$ and $X = H_0(T) + K(T)$. For every $x \in X$ there exist $x_1 \in H_0(T)$ and $x_2 \in K(T)$ such that $x = x_1 + x_2$. These vectors satisfy $\sigma_T(x_1) \subseteq \{0\}$, $0 \notin \sigma_T(x_2)$ and

$$\sigma_T(x) \subseteq \sigma_T(x_1) \cup \sigma_T(x_2) \subseteq \{0\} \cup \sigma_T(x_2).$$

Since $\sigma_T(x_2)$ is closed, 0 is isolated in $\{0\} \cup \sigma_T(x)$. Thus, since there exists $x_0 \in X$ such that $\sigma_T(x_0) = \sigma_{su}(T)$ (see [7, Proposition 1.3.2]), 0 is isolated in $\sigma_{su}(T)$. $\square$

Let us denote by $\text{acc}(\sigma_T(x))$ the set of accumulation points of $\sigma_T(x)$.

Corollary 6. *Let $T \in B(X)$. Then $X = H_0(T) + K(T)$ if and only if $0 \notin \text{acc}(\sigma_T(x))$ for every $x \in X$.*

Proof. The direct implication is clear in the proof of Theorem 5.

For the converse, note that $K(T) = X$ when $0 \notin \sigma_{su}(T)$ and that $\sigma_T(x) \subseteq \sigma_{su}(T)$ for all $x \in X$. So it is a direct consequence of Theorem 5. $\square$

Corollary 7. *Let $T \in B(X)$. Then $X = H_0(T - \lambda) + K(T - \lambda)$ for all $\lambda \in \mathbb{C}$ if and only if $\sigma(T)$ is finite.*

Proof. Since a compact set consisting of isolated points is finite, by Theorem 5, $\sigma_{su}(T)$ is finite; hence $\sigma(T)$ is finite. $\square$

Recall that the *approximate point spectrum* of $T \in B(X)$, denoted $\sigma_{ap}(T)$, is the set of all $\lambda \in \mathbb{C}$ such that the operator $T - \lambda$ is not bounded below. It is well-known that $\sigma_{ap}(T)$ is a compact subset of $\mathbb{C}$ that contains the boundary of $\sigma(T)$.

Let $T^* \in B(X^*)$ denote the conjugate operator of $T \in B(X)$.

Corollary 8. *Suppose that $0 \in \sigma_{ap}(T)$. Then 0 is an isolated point of $\sigma_{ap}(T)$ if and only if $X^* = H_0(T^*) + K(T^*)$.*

Proof. Since $\sigma_{ap}(T) = \sigma_{su}(T^*)$, it is a direct consequence of Theorem 5. $\square$

Remark. Since $H_0(T)$ and $K(T)$ are closely related with the annihilators of $K(T^*)$ and $H_0(T^*)$, respectively (see [2]), it is tempting to conjecture that 0 is isolated in $\sigma_{ap}(T)$ if and only if $H_0(T) \cap K(T) = \{0\}$ and $H_0(T) + K(T)$ is closed. However, the following example shows that the converse implication fails.

We denote $k(T) := \inf\{\|Tx\| : x \in X, \|x\| = 1\}$ and $i(T) := \lim_{n \rightarrow \infty} k(T^n)^{1/n}$. Moreover, $r(T)$ is the spectral radius of T .

We consider the weighted right shift $V : \ell^2 \rightarrow \ell^2$ defined by $Ve_n = s_n e_{n+1}$, where $0 < s_n \leq 1$.

Observe that we can choose the sequence of weights (s_n) so that $i(V) = 0$ and $r(V) > 0$ (see [7]). Thus, by [7, Proposition 1.6.15],

$$\sigma_{ap}(V) = \{\lambda \in \mathbb{C} : i(V) \leq |\lambda| \leq r(V)\};$$

hence 0 is not isolated in $\sigma_{ap}(V)$.

Moreover, V satisfies Dunford's condition (C) (see [7, Theorem 1.6.16]); in particular, $H_0(V)$ is closed, and $K(V) = \bigcap_{n \geq 0} R(V^n) = \{0\}$.

Problem. Find conditions in terms of $H_0(T)$ and $K(T)$ implying that 0 is isolated in $\sigma_{ap}(T)$.

The following result could be helpful in the solution of this problem. Here, for $c > 0$ we denote $\mathbb{D}(0, c) := \{\lambda \in \mathbb{C} : 0 \leq |\lambda| < c\}$ and $\mathbb{D}^*(0, c) := \mathbb{D}(0, c) \setminus \{0\}$.

Proposition 9. *Let $T \in B(X)$ such that 0 is an isolated point of $\sigma_{ap}(T)$. Then*

- (1) $H_0(T)$ and $K(T)$ are closed subspaces;
- (2) $H_0(T) \cap K(T) = \{0\}$;
- (3) $H_0(T) \oplus K(T)$ is closed and there exists $\lambda_0 \neq 0$ such that

$$H_0(T) \oplus K(T) = K(T - \lambda_0) = \bigcap_{n=0}^{\infty} R((T - \lambda_0)^n).$$

Proof. Since 0 is isolated in $\sigma_{ap}(T)$, there exists $\delta > 0$ so that $T - \lambda$ is bounded below for all $\lambda \in \mathbb{D}^*(0, \delta)$. By [11, Proposition 2.11], the map $\lambda \rightarrow K(T - \lambda)$ is constant on $\mathbb{D}^*(0, \delta)$. Moreover, fixing $\lambda_0 \in \mathbb{D}^*(0, \delta)$, by [11, Lemme 2.4],

$$K(T - \lambda_0) = \bigcap_{n=0}^{\infty} R((T - \lambda_0)^n).$$

We denote $X_0 := \bigcap_{n=0}^{\infty} R((T - \lambda_0)^n)$ and $T_0 : X_0 \rightarrow X_0$ the operator induced by T on X_0 . Observe that X_0 is a Banach space.

Clearly $T_0 - \lambda$ is bijective for $\lambda \in \mathbb{D}^*(0, \delta)$. Moreover, since T is not surjective, $K(T) \neq X$; hence $H(T) \neq \{0\}$ by Theorem 5. From (5) in Lemma 2 we obtain $H_0(T) \subset X_0$; hence $0 \in \sigma(T_0)$ and, by Theorem 1,

$$X_0 = H_0(T_0) \oplus K(T_0),$$

with $H_0(T_0)$ and $K(T_0)$ closed.

Since it is clear that $H_0(T) = H_0(T_0)$, to finish the proof it is enough to show that $K(T) = K(T_0)$.

Let $x_0 \in K(T)$. Then there exists a sequence (x_n) in X and $c > 0$ so that $T(x_1) = x_0$, $T(x_{n+1}) = x_n$ and $\|x_n\| \leq c^n \|x_0\|$ for $n \geq 1$. So

$$\phi(\lambda) := \sum_{n=0}^{\infty} x_{n+1} \lambda^n$$

defines an analytic function on $\mathbb{D}(0, c^{-1})$ that satisfies

$$(T - \lambda)\phi(\lambda) = \sum_{n=0}^{\infty} x_n \lambda^n - \sum_{n=0}^{\infty} x_{n+1} \lambda^{n+1} = x_0, \quad \text{for all } \lambda \in \mathbb{D}(0, c^{-1}).$$

In particular, $x_0 \in R(T - \lambda)$ for all $\lambda \in \mathbb{D}(0, c^{-1})$. Therefore, by [10, Proposition 1.12], $x_0 \in K(T - \lambda_0)$. Thus $K(T) \subset X_0$. Note also that

$$\phi(\lambda) = (T_0 - \lambda)^{-1} x_0 \in X_0, \quad \text{for } \lambda \in \mathbb{D}^*(0, c^{-1}).$$

Hence, by continuity, $x_1 = \phi(0) \in X_0$. A similar argument shows that $x_n \in X_0$ for all $n \geq 1$; thus $x_0 \in K(T_0)$, and we conclude that $K(T) = K(T_0)$. $\square$

3. OPERATORS WITH FINITE DIMENSIONAL ANALYTIC CORE

For an operator $T \in B(X)$, the point spectrum $\sigma_p(T)$ is the set of all $\lambda \in \mathbb{C}$ such that $N(T - \lambda) \neq \{0\}$. A point $\lambda \in \mathbb{C}$ which is isolated in the spectrum of T is said to be a *Riesz point for T* if the spectral projection associated with the spectral set $\{\lambda\}$ has finite dimensional range.

Lemma 10. *Let $T \in B(X)$ and let $\lambda \in \mathbb{C}$ be an isolated point in $\sigma(T)$. Then λ is a Riesz point for T if and only if $\dim H_0(T - \lambda) < \infty$.*

Proof. Suppose that λ is a Riesz point for T . Then there exists an integer d such that $X = N(T - \lambda)^d \oplus R(T - \lambda)^d$ and $\dim N(T - \lambda) < \infty$.

Since $N(T - \lambda)^d \subseteq H_0(T - \lambda)$, we get

$$H_0(T - \lambda) = N(T - \lambda)^d \oplus H_0(T - \lambda) \cap R(T - \lambda)^d.$$

Moreover, the restriction of $T - \lambda$ to $R(T - \lambda)^d$ is invertible. Therefore

$$H_0(T - \lambda) \cap R(T - \lambda)^d = \{0\}.$$

So we have $H_0(T - \lambda) = N(T - \lambda)^d$; hence $\dim H_0(T - \lambda) < \infty$.

Conversely, suppose that $\dim H_0(T - \lambda) < \infty$. Since

$$\bigcup_{n \geq 0} N(T - \lambda)^n \subseteq H_0(T - \lambda),$$

we have $\dim \bigcup_{n \geq 0} N(T - \lambda)^n < \infty$. Thus there exists an integer d such that

$$H_0(T - \lambda) = \bigcup_{n \geq 0} N(T - \lambda)^n = N(T - \lambda)^d.$$

Since λ is isolated in $\sigma(T)$, by Theorem 1 we have

$$X = K(T - \lambda) \oplus N(T - \lambda)^d.$$

Moreover,

$$R(T - \lambda)^d = (T - \lambda)^d(X) = (T - \lambda)^d(K(T - \lambda)) = K(T - \lambda).$$

Thus $X = N(T - \lambda)^d \oplus R(T - \lambda)^d$ and $\dim N(T - \lambda) < \infty$; hence λ is a Riesz point for T . $\square$

The set of all $\lambda \in \mathbb{C}$ which are Riesz points for T is denoted by $\Pi_0(T)$. It is well-known that $\lambda \in \Pi_0(T)$ if and only if λ is isolated in $\sigma(T)$ and $T - \lambda$ is a Fredholm operator, equivalently, if there exists an integer d such that

$$X = N(T - \lambda)^d \oplus R(T - \lambda)^d$$

and $\dim N(T - \lambda) < \infty$.

Proposition 11. *Let $T \in B(X)$ such that $\dim K(T) < \infty$. Then*

- (1) *If M is a closed subspace of X , invariant under T and such that $0 \notin \sigma_{su}(T|_M)$, then $\dim M < \infty$.*
- (2) *$\sigma_p(T) \subseteq \{0, \lambda_1, \dots, \lambda_n\}$ with $\dim N(T - \lambda_j) < \infty$ for $j = 1, \dots, n$.*
- (3) *T has the SVEP.*

Proof. (1) Note that M is closed and, since $0 \notin \sigma_{su}(T|_M)$, we have $T(M) = M$. Thus $M \subseteq K(T)$; hence $\dim M < \infty$.

(2) For each $\lambda \neq 0$, we have $N(T - \lambda) \subseteq K(T)$. Moreover, a set of eigenvectors, each of them corresponding to a different eigenvalue of T , is linearly independent. Therefore, $\dim K(T) < \infty$ implies $\sigma_p(T) \subseteq \{0, \lambda_1, \dots, \lambda_n\}$ and $\dim N(T - \lambda_j) < \infty$ for $j = 1, \dots, n$.

(3) This is a consequence of (2). $\square$

Let us denote by $iso\sigma(T)$ the set of all isolated points of T and by $\Pi_{00}(T)$ the set of all $\lambda \in iso\sigma(T)$ for which $0 < \dim N(T - \lambda) < \infty$.

Recall that the *Weyl spectrum* of an operator $T \in B(X)$ is the set $\sigma_w(T)$ of all $\lambda \in \mathbb{C}$ such that $T - \lambda$ is not Fredholm of index zero. An operator $T \in B(X)$ is said to *satisfy Weyl's Theorem* if $\sigma_w(T) = \sigma(T) \setminus \Pi_{00}(T)$.

Theorem 12. *Let X be an infinite dimensional Banach space and let $T \in B(X)$ such that $\dim K(T) < \infty$. Then*

- (1) *$\Pi_0(T)$ is finite;*
- (2) *$\sigma_w(T) = \sigma(T) \setminus \Pi_0(T)$ contains 0 and is connected;*
- (3) *if $0 \in acc(\sigma(T))$, Weyl's theorem holds for T .*

Proof. (1) follows from part (2) in Proposition 11.

(2) Since X is infinite dimensional and $\dim K(T) < \infty$, $0 \notin \Pi_0(T)$. Suppose that $\sigma(T) = \sigma_1 \cup \sigma_2$ with $\sigma_1 \cap \sigma_2 = \emptyset$. Since $K(T) \neq X$, $0 \in \sigma(T)$.

Suppose $0 \in \sigma_1$. Using the Riesz-Dunford functional calculus, we get two closed subspaces X_1, X_2 of X , which are invariant under T , $\sigma(T|_{X_1}) = \sigma_1$, $\sigma(T|_{X_2}) = \sigma_2$ and $X = X_1 \oplus X_2$.

Since $0 \notin \sigma_{su}(T|_{X_2})$, by Proposition 11, $\dim X_2 < \infty$. Thus $\sigma_2 = \{\lambda_1, \dots, \lambda_n\} \subseteq \sigma_p(T)$ and $n \leq \dim K(T)$. Therefore $\sigma(T)$ consists of the connected component containing 0 and a finite subset of isolated eigenvalues with finite multiplicity.

Moreover, since each $\lambda_j \in \sigma_2$ is isolated, by Theorem 1, $H_0(T - \lambda_j)$ is closed. Since $T(H_0(T - \lambda_j)) = H_0(T - \lambda_j)$ (see Lemma 2), from Lemma 3 we get $H_0(T - \lambda_j) \subseteq K(T)$. Thus $\dim H_0(T - \lambda_j) < \infty$. Now Lemma 10 allows us to conclude that λ_j is a Riesz point for T . Thus $\Pi_0(T)$ is finite and $\sigma(T) \setminus \Pi_0(T)$ is a connected subset containing 0. Moreover, since T satisfies the SVEP, [13, Theorem 2.9] implies that $\sigma_w(T) = \sigma(T) \setminus \Pi_0(T)$, which proves (1) and (2).

(3) Since $0 \in acc(\sigma(T))$, $iso\sigma(T) \cap \sigma_w(T) = \emptyset$. Hence, $iso\sigma(T) \subseteq \Pi_0(T)$, and so $iso\sigma(T) = \Pi_{00}(T) = \Pi_0(T)$. This completes the proof. $\square$

Let us see some consequences of Theorem 12.

Given an operator $T \in B(X)$, we denote by $\rho_{SF}(T)$ the set of all $\lambda \in \mathbb{C}$ for which $R(T - \lambda)$ is closed and $N(T - \lambda)$ or $X/R(T - \lambda)$ is finite dimensional. For each $\lambda \in \rho_{SF}(T)$ we can define the index

$$\text{ind}(T - \lambda) := \dim N(T - \lambda) - \dim X/R(T - \lambda) \in \mathbb{Z} \cup \{\pm\infty\}.$$

Corollary 13. *Let $T \in B(X)$ such that $\dim K(T) < \infty$.*

- (1) *0 is isolated in $\sigma(T)$ if and only if $\sigma(T)$ is a finite set. In this case T is a Riesz operator; i.e., every $\lambda \neq 0$ is a Riesz point for T .*
- (2) *$\text{ind}(T - \lambda) \leq 0$ for all $\lambda \in \rho_{SF}(T)$.*

Proof. (1) This is a direct consequence of part (2) in Theorem 12.

(2) Suppose there were $\lambda_0 \in \rho_{SF}(T)$ for which $\text{ind}(T - \lambda_0) > 0$. By the continuity of the index, we would get that there exists a nonempty open subset of $\mathbb{C}$ consisting of eigenvalues of T , which is not possible by Proposition 11. $\square$

Observation 14. When X is a separable Hilbert space, condition (2) in Corollary 13 is equivalent to the adjoint operator T^* being quasitriangular [5, Theorem 6.4].

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