POLYNOMIAL SPLITTINGS OF METABELIAN VON NEUMANN RHO–INVARIANTS OF KNOTS

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ABSTRACT. We show that if the connected sum of two knots with coprime Alexander polynomials has vanishing von Neumann ρ–invariants associated with certain metabelian representations, then so do both knots. As an application, we give a new example of an infinite family of knots which are linearly independent in the knot concordance group.

1. INTRODUCTION

A knot K in the 3–sphere S^3 is said to be *slice* if there is a locally flat 2–disk D embedded in the 4–ball B^4 with $\partial(B^4, D) = (S^3, K)$. A pair of knots K_1 and K_2 are *concordant* if $K_1 \# (-K_2)$ is slice where $-K$ is the mirror image of K with reversed orientation. The set of concordance classes of knots forms an abelian group under connected sum, called the *knot concordance group* and denoted by $\mathcal{C}$. In $\mathcal{C}$ the identity 0 is represented by slice knots.

By constructing sliceness obstructions using Seifert forms, Levine showed that the knot concordance group surjects to $\mathbb{Z}^\infty \oplus (\mathbb{Z}/2)^\infty \oplus (\mathbb{Z}/4)^\infty$ [13, 14]. A knot with vanishing Levine obstructions is called *algebraically slice*. Using their own invariants, Casson and Gordon showed that there are nonslice knots which are algebraically slice [1]. Using Casson–Gordon invariants Jiang showed that the concordance group of algebraically slice knots is infinitely generated [9]. Gilmer refined Casson–Gordon invariants by combining Casson–Gordon invariants with the Levine obstructions [7].

In [4] Cochran, Orr and Teichner made progress by establishing a geometric filtration of $\mathcal{C}$,

$$0 \subset \cdots \subset \mathcal{F}_{n.5} \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_{1.5} \subset \mathcal{F}_1 \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset \mathcal{C},$$

where the subgroup $\mathcal{F}_h$ is the set of all (h)–*solvable* knots. They showed that this filtration is closely related to the known concordance invariants. For instance, a knot lies in $\mathcal{F}_{0.5}$ if and only if the knot is algebraically slice [4, Remark 1.3.2]. Also they proved that all previously known concordance invariants vanish on (1.5)–solvable knots [4, Section 9]. In particular, Casson–Gordon–Gilmer invariants [1, 7] vanish on (1.5)–solvable knots. Cochran, Orr and Teichner [4, 5] used von Neumann ρ–invariants (L^2–signature defects) to prove that the quotient group $\mathcal{F}_2 / \mathcal{F}_{2.5}$

has infinite rank. The second author [11] proved that $\mathcal{F}_1/\mathcal{F}_{1.5}$ has an infinite rank subgroup of knots for which Casson–Gordon invariants vanish. In their respective papers Cochran–Orr–Teichner [4] and the second author [11] constructed their examples using genetic modification and showed the linear independence using signature functions whose integrals over S^1 are linearly independent.

Linear independence of knots in $\mathcal{C}$ may be checked in a different manner using relative primeness of Alexander polynomials. Such an approach was first given by Levine [13], who showed that if the connected sum of two knots with coprime Alexander polynomials has vanishing Levine obstructions, then so do both knots. The first author [10] showed that the Casson–Gordon–Gilmer invariants split in this way as well.

In this paper, we prove a similar splitting property for the von Neumann ρ–invariants of knots associated with certain metabelian representations. Let K be a knot and M_K the zero surgery on the knot K in S^3. Let $\Lambda := \mathbb{Q}[t^{\pm 1}]$. For every x in the (rational) Alexander module $H_1(M_K; \Lambda)$, one can obtain the real-valued ρ–invariant $\rho(K, \phi_x)$ where $\phi_x : \pi_1(M_K) \to \mathbb{Q}(t)/\Lambda \rtimes \mathbb{Z}$ is the homomorphism associated with x via the Blanchfield linking form (see Definition 2.3). We say that K has *vanishing ρ–invariants* if there exists a self-annihilating Λ–submodule P of $H_1(M_K; \Lambda)$ with respect to the Blanchfield linking form (hence algebraically slice) such that $\rho(K, \phi_x) = 0$ for all $x \in P$. For the definition of self-annihilating submodule, see Section 2. Cochran, Orr and Teichner showed that (1.5)–solvable knots have vanishing ρ–invariants [4, Theorem 4.6]. This yields a sliceness obstruction since a slice knot is (1.5)–solvable.

We state the main theorem:

Theorem 1.1. *Let K_1 and K_2 be knots with coprime Alexander polynomials. If $K_1 \# K_2$ has vanishing ρ–invariants, then so do both K_1 and K_2.*

We give a stronger form of this theorem in Theorem 3.1.

To demonstrate the strength of this result, in Section 4 we give a new example of infinitely many knots with vanishing Casson–Gordon invariants which are linearly independent in $\mathcal{F}_1/\mathcal{F}_{1.5}$ and hence in $\mathcal{C}$.

2. Preliminaries

In this section we briefly review the machinery necessary for this paper. In [4], Cochran, Orr and Teichner established a filtration $\{\mathcal{F}_h\}_{h \in \frac{1}{2}\mathbb{N}_0}$ of $\mathcal{C}$ indexed by nonnegative half-integers where $\mathcal{F}_h$ is the subgroup ofdefinition (h)–*solvable* knots defined below. Recall that for a group G and a nonnegative integer n, the n–*th derived group of G, $G^{(n)}$,* is defined inductively by the relations $G^{(0)} := G$ and $G^{(k)} := [G^{(k-1)}, G^{(k-1)}]$ for $k \geq 1$.

For a CW-complex W, we define $W^{(n)}$ to be the regular covering corresponding to the subgroup $(\pi_1(W))^{(n)}$. Suppose W is an oriented 4–manifold. Then there is an intersection form

$$\lambda_n : H_2(W^{(n)}) \times H_2(W^{(n)}) \to \mathbb{Z}\left[\pi_1(W)/\pi_1(W)^{(n)}\right].$$

Also there is a self-intersection form μ_n on $H_2(W^{(n)})$. For more details about these forms refer to [17] and [4, Section 7]. For a nonnegative integer n, an (n)–*Lagrangian* is a submodule $L \subset H_2(W^{(n)})$ on which λ_n and μ_n vanish and which

maps onto a Lagrangian of λ_0 under the homomorphism induced by the covering map.

Definition 2.1 ([4, Section 8])**.** Let $n \in \mathbb{N}_0$. A knot K is called (n)–*solvable* if M_K bounds a spin 4–manifold W such that the inclusion map $M_K \to W$ induces an isomorphism on the first homology and such that W admits an (n)–Lagrangian with (n)–duals. This means that the intersection form λ_n pairs the (n)–Lagrangian and the (n)–duals nonsingularly and that their images together freely generate $H_2(W)$. The 4–manifold W is called an (n)–*solution for* K, and we say K is (n)–*solvable via* W.

Similarly, we define $(n.5)$–*solvable* knots for $n \in \mathbb{N}_0$ in such a way that an $(n.5)$–solution W is required to admit an $(n+1)$–Lagrangian with (n)–duals. For more details, refer to [4, Definitions 8.5 and 8.7].

Cochran, Orr and Teichner showed that every slice knot is (h)–solvable for all nonnegative half-integers h [4, Remark 1.3.1]. They detect $(n.5)$–solvable knots using the von Neumann ρ–invariants [4, Theorem 4.2].

Although the von Neumann ρ–invariants are defined in a more general setting, for our purpose, henceforth we study the von Neumann ρ–invariants associated with the representations to the metabelian group $\Gamma := \mathbb{Q}(t)/\Lambda \rtimes \mathbb{Z}$ where $\mathbb{Z}$ is generated by t acting on $\mathbb{Q}(t)/\Lambda$ by multiplication. In [4] the group Γ is called the *first rationally universal group* and is denoted by Γ_1^U. Since Γ is poly-torsion-free-abelian, the group ring $\mathbb{Z}\Gamma$ is a right Ore domain and $\mathbb{Q}\Gamma$ embeds into its classical right ring of quotients $\mathcal{K}_\Gamma$ [4, Proposition 2.5].

Let K be a knot and $\phi : \pi_1(M_K) \to \Gamma$ a homomorphism. Then one can define the *von Neumann ρ–invariant* $\rho(M_K, \phi) \in \mathbb{R}$ associated with ϕ which was introduced by Cheeger and Gromov [3]. When M_K bounds an oriented compact 4–manifold W with a homomorphism $\psi : \pi_1(W) \to \Gamma$ extending ϕ (i.e., $(M_K, \phi) = \partial(W, \psi)$), the von Neumann ρ–invariant is computed as

$$\rho(M_K, \phi) = \sigma_\Gamma^{(2)}(W, \psi) - \sigma_0(W),$$

where $\sigma_\Gamma^{(2)}(W, \psi)$ is the L^2–signature of the intersection form on $H_2(W; \mathcal{U}\Gamma)$ and $\sigma_0(W)$ is the ordinary signature of W. Here $\mathcal{U}\Gamma$ is the algebra of (unbounded) operators affiliated to the von Neumann algebra $\mathcal{N}\Gamma$ of the group Γ. Often we simply denote $\rho(M_K, \phi)$ by $\rho(K, \phi)$. We refer the reader to [4, Section 5] for more discussion of L^2–signatures. The following theorem gives an obstruction for a knot being (1.5)–solvable. Note that since $\Gamma^{(2)} = 0$, Γ is (1)–solvable.

Theorem 2.2 ([4, Theorem 4.2])**.** *In the above setting, if W is an (1.5)–solution for K, then $\rho(K, \phi) = 0$.*

In this paper we are interested in the following representations to Γ. We construct a representation $\phi_x : \pi_1(M_K) \to \Gamma$ associated with a given $x \in H_1(M; \Lambda)$ via the Blanchfield linking form in the following manner. Recall that there is a nonsingular form called the *Blanchfield linking form*:

$$B\ell : H_1(M_K; \Lambda) \times H_1(M_K; \Lambda) \to \mathbb{Q}(t)/\Lambda.$$

Let μ be a meridian of K which normally generates $\pi_1(M_K)$ and $\epsilon : \pi_1(M_K) \to \mathbb{Z}$, the abelianization sending μ to 1.

Definition 2.3. The representation $\phi_x : \pi_1(M_K) \to \Gamma$ is defined to be $\phi_x(y) = (B\ell(x, y\mu^{-\epsilon(y)}), \epsilon(y))$ for $y \in \pi_1(M_K)$, where $y\mu^{-\epsilon(y)}$ denotes its image in $H_1(M_K; \Lambda)$ as an abuse of notation.

For a Λ–submodule P of $H_1(M_K; \Lambda)$, define

$$P^\perp := \{y \in H_1(M_K; \Lambda) \mid B\ell(x, y) = 0 \text{ for all } x \in P\}.$$

We say a Λ–submodule P of $H_1(M_K; \Lambda)$ is *self-annihilating* if $P = P^\perp$.

Theorem 2.4 ([4, Theorems 3.6 and 4.4]). *Let W be a (1)–solution for K and $P := \ker\{H_1(M_K; \Lambda) \to H_1(W; \Lambda)\}$. Then*

(1) *P is self-annihilating.*
(2) *ϕ_x extends to $\pi_1(W)$ if and only if $x \in P$.*

3. Polynomial splitting theorem

The following is a stronger form of Theorem 1.1. For two knots K_1 and K_2, note that the Alexander module of $K_1 \# K_2$ is isomorphic with the direct sum of the Alexander modules of K_1 and K_2.

Theorem 3.1. *Let K_1 and K_2 be knots and let M_1, M_2, and M be zero surgeries on K_1, K_2, and $K_1 \# K_2$, respectively. Suppose that the Alexander polynomials $\Delta_{K_1}(t)$ and $\Delta_{K_2}(t)$ are coprime. If there exists a self-annihilating submodule P of $H_1(M; \Lambda)$ with respect to $B\ell$ such that $\rho(K_1 \# K_2, \phi_x) = 0$ for all $x \in P$, then there are self-annihilating submodules P_i of $H_1(M_i; \Lambda)$, $i = 1, 2$, such that $P = P_1 \oplus P_2$ and $\rho(K_i, \phi_{x_i}) = 0$ for all $x_i \in P_i$, $i = 1, 2$.*

Proof. First, we show that $P = P_1 \oplus P_2$ for some Λ–submodules $P_i \subset H_1(M_i; \Lambda)$, $i = 1, 2$. To show this, we follow the proof of [10, Lemma 3.1]. Note that $H_1(M; \Lambda) = H_1(M_1; \Lambda) \oplus H_1(M_2; \Lambda)$ and $B\ell = B\ell_1 \oplus B\ell_2$, where $B\ell_1$ and $B\ell_2$ denote the Blanchfield linking forms of M_1 and M_2, respectively. We write an element $z \in H_1(M; \Lambda)$ as $(x, y) \in H_1(M_1; \Lambda) \oplus H_1(M_2; \Lambda)$. Let

$$P_1 := \{x \in H_1(M_1; \Lambda) \mid (x, 0) \in P\} \text{ and } P_2 := \{y \in H_1(M_2; \Lambda) \mid (0, y) \in P\}.$$

Clearly $P_1 \oplus P_2 \subset P$. Conversely, we show $P \subset P_1 \oplus P_2$. For simplicity, denote $\Delta_{K_i}(t)$ by Δ_i for $i = 1, 2$. Since Δ_1 and Δ_2 are coprime, they are also coprime in Λ, and hence there are f and g in Λ such that $f\Delta_1 + g\Delta_2 = 1$ in Λ. Let $z = (x, y) \in P$. Since each Δ_i annihilates $H_1(M_i; \Lambda)$, $\Delta_1 x = 0$ and $\Delta_2 y = 0$. Thus,

$$f\Delta_1 z = (f\Delta_1 x, f\Delta_1 y) = (0, f\Delta_1 y),$$
$$g\Delta_2 z = (g\Delta_2 x, g\Delta_2 y) = (g\Delta_2 x, 0).$$

On the other hand,

$$x = 1x = f\Delta_1 x + g\Delta_2 x = g\Delta_2 x,$$
$$y = 1y = f\Delta_1 y + g\Delta_2 y = f\Delta_1 y.$$

Thus we have

$$(0, y) = (0, f\Delta_1 y) = f\Delta_1 z,$$
$$(x, 0) = (g\Delta_2 x, 0) = g\Delta_2 z.$$

Since P is a Λ–submodule, we conclude that $(x, 0)$ and $(0, y)$ are in P and hence $x \in P_1$ and $y \in P_2$. Then $z = (x, y) \in P_1 \oplus P_2$.
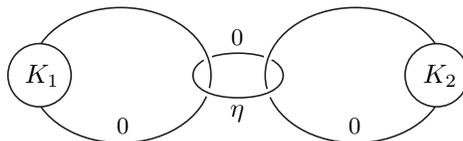
FIGURE 1

Next, we will show that each P_i is self-annihilating with respect to $B\ell_i$, i.e., $P_i = P_i^\perp$. For any $x_1, x_2 \in P_1$, we have $(x_1, 0), (x_2, 0) \in P$; hence

$$B\ell_1(x_1, x_2) = B\ell_1(x_1, x_2) + B\ell_2(0, 0) = B\ell((x_1, 0), (x_2, 0)) = 0.$$

Thus, $P_1 \subset P_1^\perp$. Conversely, let $x \in P_1^\perp$. For any $z \in P$, we can write $z = (x', y)$ for some $x' \in P_1, y \in P_2$. Since $x \in P_1^\perp$, $B\ell_1(x, x') = 0$ and

$$B\ell((x, 0), (x', y)) = B\ell_1(x, x') + B\ell_2(0, y) = 0.$$

Thus, $(x, 0) \in P^\perp$. Since $P = P^\perp$, $(x, 0) \in P$ and hence $x \in P_1$. Thus, we see that $P_1 = P_1^\perp$. Similarly, we can see that $P_2 = P_2^\perp$.

Finally, we will show that $\rho(K_i, \phi_{x_i}) = 0$ for all $x_i \in P_i$, $i = 1, 2$. We construct a cobordism C between the disjoint union $M_1 \cup M_2$ and M. Though this construction is well-known, it is an essential step here and is briefly described below. (For details, refer to [5, Section 4].) Attach a 1–handle at the top level between $M_1 \times [0, 1]$ and $M_2 \times [0, 1]$ so that the upper boundary is the zero surgery on the split link $K_1 \cup K_2$. Then attach a 2–handle along η with zero framing as indicated in Figure 1. The resulting 4–manifold is the desired C. Its boundary at the bottom, $\partial_- C$, is the disjoint union $-(M_1 \cup M_2)$. To see that $\partial_+ C = M$, slide the 2–handle attached along K_2 over the 2–handle attached along K_1 to get a 2–handle attached along $K_1 \# K_2$. After the sliding one obtains a surgery diagram where η is a meridian of K_1 and is unlinked from $K_1 \# K_2$. Using η one can unlink and unknot K_1, and therefore one can see that the zero surgery on $\eta \cup K_1$ is homeomorphic with the zero surgery on the Hopf link which is homeomorphic with S^3. Thus K_1 and η can be discarded. (Or one can use the *slam-dunk* move in [8, p. 163].) The result is a surgery diagram for M, and this shows that $\partial_+ C = M$.

Let $x \in P_1$. Then $(x, 0) \in P$. We obtain the representations $\phi_x : \pi_1(M_1) \to \Gamma$ and $\phi_{(x,0)} : \pi_1(M) \to \Gamma$ as defined in Definition 2.3. From the above construction, it is easy to see that $\pi_1(C) = \pi_1(M_1) * \pi_1(M_2)/\langle \mu_1 \mu_2^{-1} \rangle$, where $*$, $\langle\ \rangle$, and μ_i stand for the free product, the subgroup normally generated by the given elements, and the meridian of K_i at the point that η goes over, respectively. Also, we see that $\pi_1(M) = \pi_1(E_1) * \pi_1(E_2)/\langle \mu_1 \mu_2^{-1}, l_1 l_2 \rangle$, where each E_i is the exterior of K_i in S^3 and each l_i is the longitude of K_i. Since $\pi_1(C) = \pi_1(M)/\langle l_1 \rangle = \pi_1(M)/\langle l_2 \rangle$ and $\phi_{(x,0)}(l_1) = \phi_{(x,0)}(l_2) = 0$, the representation $\phi_{(x,0)}$ extends over $\pi_1(C)$; i.e., there is a homomorphism $\tilde{\phi} : \pi_1(C) \to \Gamma$ such that $\phi_{(x,0)} = \tilde{\phi} \circ i_*$, where $i_* : \pi_1(M) \to \pi_1(C)$ is the homomorphism induced by the inclusion $i : M \to C$. From the construction of C, it immediately follows that $\tilde{\phi}|_{\pi_1(M_1)} = \phi_x$ and $\tilde{\phi}|_{\pi_1(M_2)}$ is the zero map.

By [5, Lemma 4.2] $H_*(C; \mathcal{K}_\Gamma) = 0$, and one can easily show that the inclusion induced map $i_* : H_2(\partial C; \mathbb{Z}) \to H_2(C; \mathbb{Z})$ is surjective. Therefore $\sigma_\Gamma^{(2)}(C, \tilde{\phi}) = 0$ and $\sigma_0(C) = 0$. Hence

$$\begin{aligned} \rho(K_1 \# K_2, \phi_{(x,0)}) &= \rho(K_1, \tilde{\phi}|_{\pi_1(M_1)}) + \rho(K_2, \tilde{\phi}|_{\pi_1(M_2)}) \\ &= \rho(K_1, \phi_x) + \rho(K_2, 0) \\ &= \rho(K_1, \phi_x). \end{aligned}$$

Here, $\rho(K_2, 0) = 0$ follows from [5, Property (2.5)]. Since $(x, 0) \in P$, the left-hand side is zero by the assumption. Therefore $\rho(K_1, \phi_x) = 0$ for all $x \in P_1$. Similarly, $\rho(K_2, \phi_y) = 0$ for all $y \in P_2$. This completes the proof. $\qquad \square$

4. Examples

As an application of Theorem 3.1, we present a new example of an infinite family of knots with vanishing Casson–Gordon invariants which are linearly independent in $\mathcal{C}$. In fact, we will show that they are linearly independent in $\mathcal{F}_1 / \mathcal{F}_{1.5}$. The authors do not know how to show that these knots are linearly independent in $\mathcal{C}$ without using Theorem 3.1.

Let T be the infinite set of positive integers each of which is divisible by three distinct primes. For $k \in T$, let $\Phi_k(t)$ be the k–th cyclotomic polynomial. As is well-known (for instance see [11, Section 2] and [2, Chapter 5]), there is a slice knot K'_k that has the cyclic rational Alexander module $H_1(M_{K'_k}; \Lambda) = \Lambda / (\Phi_k(t)^2)$. Let J be a knot with Arf invariant zero such that the (averaged) integral of the Levine–Tristram signature function of J is nonzero. For example, one can take J to be the connected sum of two copies of the trefoil. Denote this integral by $\rho(J)$. Let η_k be the unknot in the complement of a Seifert surface for K'_k in S^3 which represents the homology class generating the rational Alexander module for K'_k over Λ, i.e., $H_1(M_{K'_k}; \Lambda) = ([\eta_k])$. Such η_k exists since any element in the Alexander module for a knot can be represented by a simple closed curve in the complement of the knot which represents a commutator in the knot group, and we may assume that the simple closed curve is unknotted by crossing changes (cf. [16]).

We construct a knot $K_k := K'_k(\eta_k, J)$ using the *satellite construction* (or *genetic modification* following the terminology in [5]) as follows. Take the union of the exterior of η_k in S^3 and the exterior of J in S^3 along their boundary (which is a torus) such that the meridian (respectively the longitude) of η_k is identified with the longitude (respectively the meridian) of J. The resulting ambient manifold is S^3 and K_k is defined to be the image of K'_k under this identification. In this case, we say that K_k is the *satellite* of the *companion* J with the *axis* η_k and the *pattern* K'_k. See [5, Section 3] for more details on this construction.

Since η_k lies in the complement of a Seifert surface for K'_k, the knots K'_k and K_k have the isomorphic rational Alexander module. In particular each K_k has the unique nontrivial proper submodule $(\Phi_k(t))$. Using this property, in [11, Section 6] the second author showed the following: each K_k, $k \in T$, is (1)–solvable but not (1.5)–solvable.

Furthermore, we prove the linear independence of K_k, $k \in T$.

Proposition 4.1. *The knots K_k, $k \in T$, are linearly independent in $\mathcal{F}_1/\mathcal{F}_{1.5}$ and hence in the knot concordance group. Moreover, the concordance invariants by Casson–Gordon* [1]*, Letsche* [12] *and Friedl* [6] *vanish on connected sums of copies of K_k, $k \in T$.*

Proof. As mentioned above, all K_k are (1)–solvable. Let $k_1, \ldots, k_l$ be distinct elements in T. Suppose that $K := a_1 K_{k_1} \# \cdots \# a_l K_{k_l}$ (all a_i are integers) is (1.5)–solvable. We only need to show that $a_i = 0$ for all i. Suppose to the contrary that $a_i \neq 0$ for some i. We may assume that $a_1 > 0$ by replacing K_{k_1} by $-K_{k_1}$ if necessary. For convenience let us denote K_{k_1} by K_1.

Then there is a self-annihilating submodule P of $H_1(M_K; \Lambda)$ such that $\rho(M_K, \phi_z) = 0$ for all $z \in P$ by Theorems 2.4 and 2.2. Since $\Phi_k(t)$, $k \in T$, are all irreducible and are pairwise coprime, by Theorem 3.1 there exists a self-annihilating submodule P_1 of $H_1(M_{a_1 K_1}; \Lambda)$ such that $\rho(M_{a_1 K_1}, \phi_x) = 0$ for all $x \in P_1$.

Recall that $H_1(M_{a_1 K_1}; \Lambda) = \overset{a_1}{\bigoplus} H_1(M_{K_1}; \Lambda)$. Pick a nonzero element $x = (x_1, \ldots, x_{a_1}) \in P_1$. Suppose that $x_i \in (\Phi_{k_1}(t))$ for all $i = 1, \ldots, a_1$. We may assume that $x_1 \neq 0$. For the Blanchfield linking form $B\ell$ of K_1, $B\ell(\eta_{k_1}, x_i) \neq 0$ if and only if $x_i \neq 0$, since η_{k_1} generates $H_1(M_{K_1}; \Lambda)$ and $B\ell$ is nonsingular. Define ϵ_i to be 1 if $x_i \neq 0$ and 0 otherwise, for $1 \leq i \leq a_1$. Then

$$
\begin{aligned}
\rho(M_{a_1 K_1}, \phi_x) &= \sum_{i=1}^{a_1} \rho(M_{K_1}, \phi_{x_i}) \\
&= \sum_{i=1}^{a_1} \epsilon_i \rho(J) \\
&= \rho(J) + \sum_{i=2}^{a_1} \epsilon_i \rho(J) \\
&\neq 0,
\end{aligned}
$$

which contradicts Theorem 3.1. The first equality can be shown by constructing a cobordism between the disjoint union of a_1 copies of M_{K_1} and $M_{a_1 K_1}$, as we did in the proof of Theorem 3.1. Here we need to add $(a_1 - 1)$ 1–handles and the same number of 2–handles to construct the cobordism. The argument is almost the same as that in the proof of Theorem 3.1, and hence we do not repeat it here. The second equality of the above equation follows from the proof of [11, Proposition 6.4] and Properties (2.3), (2.4) and (2.5) in [5]. The third equality follows since $x_1 \neq 0$ and $\epsilon_1 = 1$.

Next, if $x_{i_0} \notin (\Phi_{k_1}(t))$ for some i_0, $1 \leq i_0 \leq a_1$, then since P_1 is a Λ–submodule and $\Phi_{k_1}(t)x_{i_0} \neq 0$ in $H_1(M_{K_1}; \Lambda)$,

$$
\Phi_{k_1}(t)x = (\Phi_{k_1}(t)x_1, \ldots, \Phi_{k_1}(t)x_{a_1})
$$

is a nonzero element in P_1 satisfying $\Phi_{k_1}(t)x_i \in (\Phi_{k_1}(t))$ for all i. Now, we are back to the previous case, leading to a contradiction.

The second statement follows since all prime power cyclic branched covers of each K_k are homology spheres by [15, Theorem 1.2]. $\qquad\square$

There are infinitely many knots with vanishing Arf invariant, say J_i, $i \in \mathbb{N}$, such that $\rho(J_i)$, $i \in \mathbb{N}$, are linearly independent over $\mathbb{Z}$ [5, Proposition 2.6]. Using this the second author [11] showed that for a fixed $k \in T$ the knots $K_k^i := K_k'(\eta_k, J_i)$, $i \in \mathbb{N}$, are linearly independent in $\mathcal{F}_1/\mathcal{F}_{1.5}$. Note that the knots in Proposition 4.1 cannot be shown to be linearly independent in this way since they are the satellites of the same companion J. This demonstrates that Theorem 3.1 is definitely required to prove Proposition 4.1.

One can easily prove the following proposition using the ideas in the proof of Proposition 4.1; hence we omit the proof. Note that the family of knots in the following proposition includes our examples of knots in Proposition 4.1.

Proposition 4.2. *The knots K_k^i, $i \in \mathbb{N}$, $k \in T$, are linearly independent in $\mathcal{F}_1/\mathcal{F}_{1.5}$ and hence in $\mathcal{C}$.*

Acknowledgements

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