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EXTREME POINTS OF LATTICE INTERVALS IN THE MINKOWSKI-RÅDSTRÖM–HÖRMANDER LATTICE

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Abstract. In this paper we characterize extreme points of any symmetric interval in the Minkowski–Rådström–Hörmander lattice $\widetilde{X}$ over any Hausdorff topological vector space X (Theorem 1). Then we prove that the unit ball in the Minkowski–Rådström–Hörmander lattice $\widetilde{X}$ over any normed space X, $\dim X \geq 2$, has exactly two extreme points (Theorem 2).

Let $(G, +, \leq)$ be a commutative lattice-ordered group [1], where $x \vee y = \sup(x, y)$, $x \wedge y = \inf(x, y)$, $x_+ = x \vee 0$, $x_- = (-x) \vee 0$, and $|x| = x \vee (-x)$ for $x, y \in G$. In a commutative lattice-ordered group G the following properties hold true: $x \leq y$ implies $z + x \leq z + y, |x| = x_+ + x_-$, $x = x_+ - x_-$, $|x| \geq 0$, $x_+ + x_+ = (x + x)_+$, and $x + (y \vee z) = (x + y) \vee (x + z)$ for any $x, y, z \in G$.

The *interval* $[a, b]$ is the set $\{x \in G \mid a \leq x \leq b\}$ where $a, b \in G$, $a \leq b$. In particular, $[-a, a] = \{x \in G \mid |x| \leq a\}$ for $a \in G$, $a \geq 0$.

We say that $x \in G$ is an *extreme point* of a subset A of G if for any $y, z \in A$ the equality $x + x = y + z$ implies that $x = y = z$. By $\operatorname{ext} A$ we denote the set of all extreme points of A. If G is a vector space, this definition of the extreme point of a subset A coincides with the usual one.

The following proposition characterizes extreme points of symmetric intervals in G.

Proposition 1. *Let G be a commutative lattice-ordered group, $a \in G$, $a \geq 0$. Then $x \in \operatorname{ext}[-a, a]$ if and only if $|x| = a$.*

Proof. Let $x \in \operatorname{ext}[-a, a]$. Since $x \leq a$, $x_+ \leq a$ and $x_+ + x_+ - a \leq a$; and since $x_+ \geq 0$, $x_+ + x_+ - a \geq -a$. Denote $y = x_+ + x_+ - a$. We have $|y| \leq a$. Also $-x \leq a$. Hence $-x_- \geq -a$ and $-x_- \leq a$. Then $a - x_- - x_- \geq -a$. Since $-x_- \leq 0$, $a - x_- - x_- \leq a$. Denote $z = a - x_- - x_-$. We have $|z| \leq a$. Since $y + z = x + x$, $x = y = x_+ + x_+ - a$. Hence $x + |x| = x + (x \vee (-x)) = (x + x) \vee 0 = (x + x)_+ = x_+ + x_+ = x + a$. Then $|x| = a$.

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Assume that $|x| = a$, $y, z \in [-a, a]$, $x + x = y + z$. Denote $w = y - x$. Then $y = x + w$, $z = x - w$. We have

$$\begin{aligned} |x| = a \geq |y| \vee |z| &= (x+w) \vee (-x-w) \vee (x-w) \vee (-x+w) \\ &= [(x+w) \vee (x-w)] \vee [(-x-w) \vee (-x+w)] \\ = [x + (w \vee (-w))] \vee [-x + (w \vee (-w))] &= (x + |w|) \vee (-x + |w|) = |x| + |w|. \end{aligned}$$

Hence $|w| = 0$ and $w = 0$. Therefore, $x = y = z$ and $x \in \operatorname{ext}[-a, a]$. □

Remark. In the case of convex subsets in a real vector space the classical definition of the extreme points coincides with our definition. The intervals in a vector lattice are convex sets. Therefore, the following corollary holds true.

Corollary 1. *Let Y be a vector lattice, $a \in Y$, $a \geq 0$. Then we have* $\operatorname{ext}[-a, a] = \{x \in Y \mid |x| = a\}$.

A subset E of a vector lattice Y is *solid* if for all $x \in E$ the interval $[-|x|, |x|]$ is contained in E. A topological vector lattice Y is *locally solid* if there exists a neighborhood basis of the origin consisting of solid sets [3].

We say that a point x of a topological vector space Y is a *strongly extreme point* of $E \subset Y$ if for all generalized sequences $(y_\alpha)_{\alpha\in\Lambda}$, $(z_\alpha)_{\alpha\in\Lambda} \subset E$ such that $\lim_{\alpha\in\Lambda} \frac{y_\alpha + z_\alpha}{2} = x$, we have $\lim_{\alpha\in\Lambda} y_\alpha = \lim_{\alpha\in\Lambda} z_\alpha = x$.

The following proposition identifies extreme points and strongly extreme points of symmetric intervals in topological vector lattices.

Proposition 2. *Let Y be a locally solid topological vector lattice. Then for any $a \in Y$, $a \geq 0$, every extreme point of $[-a, a]$ is strongly extreme.*

Proof. Let $x \in \operatorname{ext}[-a, a]$. By Proposition 1, $|x| = a$. Let $y_\alpha, z_\alpha \in [-a, a]$, $\alpha \in \Lambda$, $\lim_{\alpha\in\Lambda} \frac{y_\alpha + z_\alpha}{2} = x$. We have

$$|x| = a \geq |y_\alpha| \vee |z_\alpha| = |\frac{y_\alpha + z_\alpha}{2}| + |\frac{y_\alpha - z_\alpha}{2}|.$$

Since Y is locally solid, $|\cdot|$ is continuous and $\lim_{\alpha\in\Lambda} |\frac{y_\alpha + z_\alpha}{2}| = a$. Since $|\frac{y_\alpha - z_\alpha}{2}| \leq a - |\frac{y_\alpha + z_\alpha}{2}|$ and $\lim_{\alpha\in\Lambda}(a - |\frac{y_\alpha + z_\alpha}{2}|) = 0$, by the corollary after Proposition 1 in Section 7.1.3 of [3], $\lim_{\alpha\in\Lambda} \frac{y_\alpha - z_\alpha}{2} = 0$. Therefore,

$$\lim_{\alpha\in\Lambda} y_\alpha = \lim_{\alpha\in\Lambda} \frac{y_\alpha + z_\alpha}{2} + \lim_{\alpha\in\Lambda} \frac{y_\alpha - z_\alpha}{2} = x.$$

In a similar way $\lim_{\alpha\in\Lambda} z_\alpha = x$. Hence the point x is a strongly extreme point of the interval $[-a, a]$. □

Let $X = (X, \tau)$ be a Hausdorff topological vector space over the field $\mathbb{R}$ and $\mathcal{B}(X)$ be the family of all nonempty closed bounded convex subsets of X. For $A, B, C, D \in \mathcal{B}(X)$ we have $A + B = \{a + b \mid a \in A, b \in B\}$, $A \dot{+} B = \overline{A + B}$, $A \vee B = \overline{\operatorname{conv}}(A \cup B)$, $A \dot{-} B = \{x \in X \mid x + B \subset A\}$, $(A, B) \sim (C, D)$ if and only if $A \dot{+} D = B \dot{+} C$. The relation "$\sim$" is a relation of equivalence and $[A, B] = [(A, B)]_\sim$ denotes a class of equivalence. Also $[A, B] \leq [C, D]$ if and only if $A \dot{+} D \subset B \dot{+} C$, $[A, B] + [C, D] = [A \dot{+} C, B \dot{+} D]$, $[A, B] \vee [C, D] = \sup\{[A, B], [C, D]\} = [(A \dot{+} D) \vee (B \dot{+} C), B \dot{+} D]$.

The quotient space $\widetilde{X} = \mathcal{B}^2(X)/_\sim$ is called the *Minkowski–Rådström–Hörmander* (M–R–H) *space* over X. Moreover, $(\widetilde{X}, +, \leq)$ is a topological vector lattice. For the definition of an M–R–H lattice we refer to [14] and to paragraph 3.4 in [10].

The Minkowski–Rådström–Hörmander lattice [9], [12] is very useful in studying bounded-valued correspondences [4], [6]; in quasidifferential calculus [5], [10], [13]; and in calculating an Aumann integral [2]. The M–R–H lattices were also studied in a number of papers, for example [7], [11] and [14].

Let $\mathcal{U}$ be a neighborhood basis of the orgin in X. For $U \in \mathcal{U}$ denote $\widetilde{U} = \bigcup\{[-\widetilde{a}, \widetilde{a}\,] \mid \widetilde{a} = [K, \{0\}],\, 0 \in K \in \mathcal{B}(X),\, K \subset U\}$. Then $\{\widetilde{U} \mid U \in \mathcal{U}\}$ is a neighborhood basis of $\widetilde{0}$ in the topological vector space $\widetilde{X}$ [14]. Notice that $\widetilde{X}$ is a locally solid topological vector lattice.

The following theorem is central for this paper. It characterizes extreme points of symmetric intervals in the M–R–H lattice.

Theorem 1. *Let $\widetilde{X}$ be the Minkowski–Rådström–Hörmander lattice over a Hausdorff topological vector space X and $K, L \in \mathcal{B}(X)$, $L \subset K$, $\widetilde{a} = [K, L]$. Then*

$$\operatorname{ext}[-\widetilde{a}, \widetilde{a}\,] = \{\widetilde{x} \in \widetilde{X} \mid \widetilde{x} = [A, B],\, A \cup B = K,\, A \cap B = L\}.$$

Moreover, all extreme points of the interval $[-\widetilde{a}, \widetilde{a}]$ are strongly extreme.

Proof. Let $A, B \in \mathcal{B}(X)$, $A \cup B = K$, $A \cap B = L$, $\widetilde{x} = [A, B]$. Then

$$|\widetilde{x}| = \widetilde{x} \vee (-\widetilde{x}) = [A, B] \vee [B, A] = [2(A \cup B), A \dot{+} B].$$

By Corollary 4.5.5 in [10] $A \cup B \dot{+} A \cap B = A \dot{+} B$. Hence $|\widetilde{x}| = [A \cup B, A \cap B] = [K, L] = \widetilde{a}$. By Corollary 1, $\widetilde{x} \in \operatorname{ext}[-\widetilde{a}, \widetilde{a}\,]$.

Assume that $\widetilde{x} = [A_1, B_1] \in \operatorname{ext}[-\widetilde{a}, \widetilde{a}\,]$. Let $v \in A_1 \dot{+} B_1$. Denote $A_2 = A_1 \dot{+} A_1 \vee B_1 \dot{+} K - v$, $B_2 = B_1 \dot{+} A \vee B_1 \dot{+} K - v$. Then $\widetilde{x} = [A_2, B_2]$ and $K \subset A_2$, $L \subset K \subset B_2$. By Corollary 1 we have $|\widetilde{x}| = \widetilde{a}$. Hence $|[A_2, B_2]| = [A_2 \cup B_2, A_2 \cap B_2] = [K, L]$. By Theorem 6.3.3, p. 136 in [10], there exist $A, B \in \mathcal{B}(X)$ such that $\widetilde{x} = [A, B]$, $A \cup B = K$, $A \cap B = L$. The last assertion of the theorem follows from Proposition 2 and local solidity of the M–R–H lattice $\widetilde{X}$. □

Remark. Let $\widetilde{a} = [K, L]$, $K, L \in \mathcal{B}(X)$, $L \subset K$. Every extreme point $\widetilde{x} = [A, B]$, $A, B \in \mathcal{B}(X)$, $A \cup B = K$, $A \cap B = L$ of the interval $[-\widetilde{a}, \widetilde{a}]$ is determined by the family of all connected components of $K \setminus L$ that are contained in A and disjoint with B. Therefore, the number of all extreme points of the interval $[-\widetilde{a}, \widetilde{a}]$ is equal to the number of all subsets of the family of all connected components of $K \setminus L$.

Example. If K is the closed disc circumscribed around the square L in the plane, then the set $K \setminus L$ has four connected components: C_1, C_2, C_3 and C_4. The interval $[-\widetilde{a}, \widetilde{a}], \widetilde{a} = [K, L]$ has $2^4 = 16$ extreme points. Four extreme points, $\widetilde{a}, \widetilde{x}_1 = [L \cup C_3 \cup C_4, L \cup C_1 \cup C_2], \widetilde{x}_2 = [C_2 \vee C_4, C_1 \vee C_3]$ and $\widetilde{x}_3 = [C_1 \vee C_3 \vee C_4, L \cup C_2]$, are presented in Figure 1.

The following proposition characterizes the subset $\widetilde{U}$ of $\widetilde{X}$, where U is a closed convex neighborhood of the origin in a topological vector space X that is not normable.

Proposition 3. *If a Hausdorff topological vector space X is not normable and U is a closed convex neighborhood of the origin, then the set $\widetilde{U}$ in $\widetilde{X}$ has no extreme points.*

Proof. Let $\widetilde{x} \in \widetilde{U}$. Then $|\widetilde{x}| \leq \widetilde{a}$, for some $\widetilde{a} = [K, \{0\}]$, $0 \in K \in \mathcal{B}(X)$, $K \subset U$. By the theorem of Kolmogorov the set U is not bounded. Then there exists $u \in U \setminus K$. The convex hull of the union of two bounded convex sets in X is a bounded set.

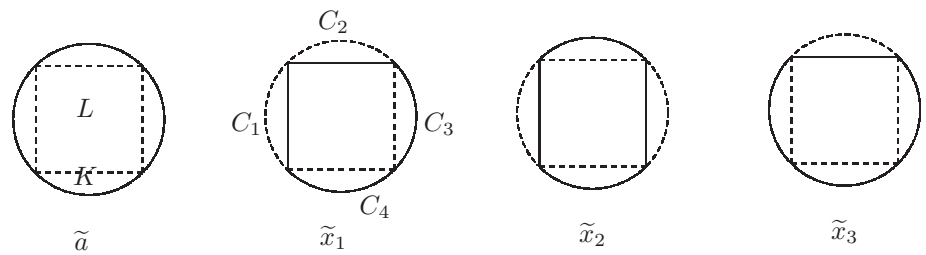

FIGURE 1. Four extreme points from the example

Therefore, the set $K \vee u$ is bounded and $K_1 = \overline{K \vee u} \in \mathcal{B}(X)$. Also $K_1 \subset U$ and K is a proper subset of K_1. Then $|\widetilde{x}| \leq \widetilde{b} = [K_1, \{0\}], |\widetilde{x}| \neq \widetilde{b}$ and $\widetilde{x} \notin \operatorname{ext}[-\widetilde{b}, \widetilde{b}]$, where $[-\widetilde{b}, \widetilde{b}] \subset \widetilde{U}$. $\square$

Let $(X, \|\cdot\|)$ be a normed space and $\mathbb{B}$ be the closed unit ball in X. If $x^* \in X^*$, then $\|x^*\| = \sup\{x^*(x) \mid x \in X, \|x\| \leq 1\}$. The pair $(X^*, \|\cdot\|)$ is a Banach space. Let $\widetilde{x} = [A, B]$. The function $\|\cdot\| : \widetilde{X} \longrightarrow \mathbb{R}_+$ defined by $\|\widetilde{x}\|_H = d_H(A, B) = \inf\{\alpha > 0 \mid A \subset B + \alpha\mathbb{B},\ B \subset A + \alpha\mathbb{B}\}$ is a norm in $\widetilde{X}$. Let us notice that d_H is a Hausdorff metric in $\mathcal{B}(X)$ and the topology in $\widetilde{X}$ generated by $\|\cdot\|_H$ is identical with the topology introduced before Theorem 1. The normed M–R–H lattice $(\widetilde{X}, \|\cdot\|_H)$ is not a Banach space with the exception of $\dim X = 1$. The unit ball $\widetilde{\mathbb{B}}$ in the M–R–H lattice $\widetilde{X}$ is equal to the interval $[-\widetilde{a}, \widetilde{a}]$ where $\widetilde{a} = [\mathbb{B}, \{0\}]$ ([12], [14]). The following theorem is the major consequence of Theorem 1.

Theorem 2. *Let X be a normed space. Unless $\dim X = 1$ the two endpoints $\widetilde{a} = [\mathbb{B}, \{0\}]$ and $-\widetilde{a}$ are the only extreme points of the unit ball $\widetilde{\mathbb{B}}$ in the Minkowski–Rådström–Hörmander lattice $\widetilde{X}$. In $\widetilde{\mathbb{R}}$ we have $\operatorname{ext}\widetilde{\mathbb{B}} = \{\widetilde{a}, -\widetilde{a}, \widetilde{b}, -\widetilde{b}\}$, where $\widetilde{b} = [\{1\}, \{0\}]$. Moreover, all extreme points of the ball $\widetilde{\mathbb{B}}$ are strongly extreme.*

Proof. Let $\widetilde{x} = [A, B]$ be an extreme point of the ball $\widetilde{\mathbb{B}}$. We may assume that $A \cup B$ is convex. By Theorem 1 we have $|\widetilde{x}| = \widetilde{a}$. Then $[A \cup B, A \cap B] = [\widetilde{\mathbb{B}}, \{0\}]$. Hence $A \cup B = A \cap B \dot{+} \mathbb{B}$. By Theorem 3 of [8], $A \subset B$ or $B \subset A$ or $\mathbb{B}$ is an interval. If $A \subset B$, then $\widetilde{x} = -\widetilde{a}$. If $B \subset A$, then $\widetilde{x} = \widetilde{a}$. If $\mathbb{B}$ is an interval, then $\dim X = 1$. For $X = \mathbb{R}$ the M–R–H lattice $\widetilde{X}$ is two-dimensional and the ball $\widetilde{\mathbb{B}}$ is a square. $\square$

Remarks. 1. The M–R–H lattice $\widetilde{X}$ is not strictly convex nor uniformly convex. Moreover, the ball $\widetilde{\mathbb{B}}$ is not the convex hull of the set $\operatorname{ext}\widetilde{\mathbb{B}}$, with the exception of $\dim X = 1$.

2. If $\widetilde{a} = [K, L]$, $L \subset \operatorname{int} K$, then the interval $[-\widetilde{a}, \widetilde{a}]$ in the completion of $\widetilde{X}$ has exactly the same extreme points. If $L \not\subset \operatorname{int} K$ we do not know whether it is true.

3. The unit ball in the completion of $\widetilde{X}$ has exactly the same extreme points. Hence for $\dim X \geq 2$ the completion of $\widetilde{X}$ is not a reflexive space due to the Krein–Milman theorem.

4. Let $\mathcal{K}(X)$ be the family of all nonempty compact convex subsets of the Hausdorff topological vector space X. The quotient space $\mathcal{K}^2(X)/_\sim$ found an application in the quasidifferential calculus [5]. The space $\mathcal{K}^2(X)/_\sim$ is a sublattice of the M–R–H lattice $\widetilde{X}$. If the normed space X is infinitely dimensional, then the unit ball in $\mathcal{K}^2(X)/_\sim$ has no extreme points by the reasoning analogous to Proposition 3.

Finally, we show that in the case of the dual space X^* being separable, all the extreme points of any interval in $\widetilde{X}$ are exposed.

Theorem 3. *Let $(X, \|\cdot\|)$ be a normed space such that the dual space X^* is separable. If $\widetilde{a} \in \widetilde{X}$, $\widetilde{a} \geq \widetilde{0}$, then* $\operatorname{ext}[-\widetilde{a}, \widetilde{a}] = \exp[-\widetilde{a}, \widetilde{a}]$.

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a dense subset of the unit sphere in the dual space $\{f \in X^* \mid \|f\| = 1\}$. For any mapping $\epsilon : \mathbb{N} \longrightarrow [-1, 1]$ define the function $b_\epsilon : \mathcal{B}(X) \longrightarrow \mathbb{R}$ by

$$b_\epsilon(A) = \sum_{n=1}^{\infty} 2^{-n}\epsilon(n) \sup f_n(A).$$

The function b_ϵ is additive, positively homogeneous and continuous, because $|b_\epsilon(A) - b_\epsilon(B)| \leq d_H(A, B)$.

Let $\overline{b}_\epsilon(\widetilde{x}) = b_\epsilon(A) - b_\epsilon(B)$, where $\widetilde{x} = [A, B]$. The function $\overline{b}_\epsilon$ is well-defined on $\widetilde{X}$, and linear and continuous with $|\overline{b}_\epsilon(\widetilde{x})| \leq \|\widetilde{x}\|$ for $\widetilde{x} \in \widetilde{X}$. Let $\widetilde{a} = [K, L]$, $L \subset K$ and $\widetilde{x} \in \operatorname{ext}[-\widetilde{a}, \widetilde{a}]$. By Theorem 1, $\widetilde{x} = [A, B]$, $A \cup B = K$, $A \cap B = L$. Let

$$\epsilon(n) = \begin{cases} 1 & \text{if} \quad \sup f_n(B) \leq \sup f_n(A), \\ -1 & \text{if} \quad \sup f_n(B) > \sup f_n(A). \end{cases}$$

Let $\epsilon^+(n) = \max(\epsilon(n), 0)$, $\epsilon^-(n) = \max(-\epsilon(n), 0)$. For any $\widetilde{y} \in [-\widetilde{a}, \widetilde{a}]$, $\widetilde{y} = [C, D]$ we have $L \dot{+} D \subset K \dot{+} C$ and $C \dot{+} L \subset D \dot{+} K$. Notice that $b_{\epsilon^+}(A) = b_{\epsilon^+}(A \cup B)$ and $b_{\epsilon^+}(B) = b_{\epsilon^+}(A \cap B)$. Then

$$(*) \qquad \begin{array}{rcccl} b_{\epsilon^+}(C) + b_{\epsilon^+}(B) & = & b_{\epsilon^+}(C) + b_{\epsilon^+}(L) & \leq & b_{\epsilon^+}(D \dot{+} K) \\ & = & b_{\epsilon^+}(D) + b_{\epsilon^+}(K) & = & b_{\epsilon^+}(D) + b_{\epsilon^+}(A). \end{array}$$

Hence $\overline{b}_{\epsilon^+}(\widetilde{y}) \leq \overline{b}_{\epsilon^+}(\widetilde{x})$. In an analogous way, $b_{\epsilon^-}(D) + b_{\epsilon^-}(A) \leq b_{\epsilon^-}(C) + b_{\epsilon^-}(B)$. Hence $\overline{b}_{\epsilon^-}(\widetilde{y}) \geq \overline{b}_{\epsilon^-}(\widetilde{x})$. Therefore,

$$\overline{b}_\epsilon(\widetilde{y}) = \overline{b}_{\epsilon^+}(\widetilde{y}) - \overline{b}_{\epsilon^-}(\widetilde{y}) \leq \overline{b}_{\epsilon^+}(\widetilde{x}) - \overline{b}_{\epsilon^-}(\widetilde{x}) = \overline{b}_\epsilon(\widetilde{x}).$$

Now, assume that $\overline{b}_\epsilon(\widetilde{y}) = \overline{b}_\epsilon(\widetilde{x})$. Then $\overline{b}_{\epsilon^+}(\widetilde{y}) = \overline{b}_{\epsilon^+}(\widetilde{x})$ and $\overline{b}_{\epsilon^-}(\widetilde{y}) = \overline{b}_{\epsilon^-}(\widetilde{x})$. By $(*)$ for any $n \in \mathbb{N}$, such that $\epsilon(n) = 1$,

$$\sup f_n(C \dot{+} B) = \sup f_n(C \dot{+} L) = \sup f_n(D \dot{+} K) = \sup f_n(A \dot{+} D).$$

In a similar way for $\epsilon(n) = -1$,

$$\sup f_n(D \dot{+} A) = \sup f_n(L \dot{+} D) = \sup f_n(K \dot{+} C) = \sup f_n(C \dot{+} B).$$

Then by the fact that $\{f_n \mid n \in \mathbb{N}\}$ is dense in the unit sphere in X^* we have for all $f \in X^*$,

$$\sup f(D \dot{+} A) = \sup f(B \dot{+} C).$$

Hence $A \dot{+} D = B \dot{+} C$ and $\widetilde{y} = \widetilde{x}$. Therefore, $\widetilde{x}$ is an exposed point of the interval $[-\widetilde{a}, \widetilde{a}]$. □

Corollary 2. *Let $(X, \|\cdot\|)$ be a normed space such that the dual space X^* is separable. Then every extreme point of the unit ball $\widetilde{\mathbb{B}}$ in the Minkowski–Rådström–Hörmander lattice $\widetilde{X}$ is exposed.*

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