LYAPUNOV EXPONENT ESTIMATES OF A CLASS OF HIGHER-ORDER STOCHASTIC ANDERSON MODELS

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ABSTRACT. In this article, we propose a class of high-order stochastic partial differential equations (SPDEs) for spatial dimensions $d \leq 5$ which might be called high-order stochastic Anderson models. This class of the equations is perturbed by a space-time white noise when $d \leq 3$ and by a space-correlated Gaussian noise when $d = 4, 5$. The objectives of this article are to get some estimates on the Lyapunov exponent of the solutions and to study the convergence rates of the chaos expansions of the solutions for the models.

1. INTRODUCTION

In this section, we start at the following well-known Anderson model:

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t} u(t, x) = \Delta u(t, x) + u(t, x) \dot{W}(t, x), & \text{in } [0, \infty) \times \mathbf{R}, \\ u(0, x) = u_0(x), & x \in \mathbf{R}, \end{cases}$$

where $\dot{W}(t, x)$ is a space-time white noise on some probability space $(\Omega, \mathcal{F}, P)$ and satisfies $E[\dot{W}(t, x)\dot{W}(s, y)] = \delta(|t - s|)\delta(|x - y|)$ with $(t, x), (s, y) \in [0, \infty) \times \mathbf{R}$ formally. There have been extensive studies on this model (see e.g. [6, 7, 8, 9, 10, 11]). In those articles, the authors proved the existence and uniqueness of the solution and studied at greater length the asymptotic behaviors of the solution to (1.1). The key role in those approaches is the Itô–Wiener chaos expansion. On the other hand, since the mild solution of (1.1) in Walsh’s [12] sense is function-valued if and only if $d = 1$. For higher dimensions $d \geq 2$, to get a function-valued solution to (1.1), one has to introduce the stochastic integral of Skorohod type, which might be found in more detail in [6, 10, 11].

In this article, we are interested in a class of higher order stochastic partial differential equations (SPDEs), which goes back to the well-known (deterministic) Cahn-Hilliard equations in material sciences. By the deterministic model, Cardon-Weber [3] (see also Bo and Wang [1]) established a class of higher order SPDEs (called Stochastic Cahn-Hilliard equations). Here we focus a concrete linear high-order SPDE, which might be regarded as a higher-order stochastic Anderson model.

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This equation is described by the following:

$$(1.2) \quad \begin{cases} (\frac{\partial}{\partial t} + \Delta^2)u(t, x) = u(t, x)\dot{W}(t, x), & \text{in } [0, \infty) \times \mathbf{D}, \\ u(0, x) = u_0(x), & x \in \mathbf{D}, \\ \frac{\partial u(t, x)}{\partial n} = \frac{\partial \Delta u(t, x)}{\partial n} = 0, & \text{on } [0, \infty) \times \partial \mathbf{D}, \end{cases}$$

where the domain $\mathbf{D} = [0, \pi]^d$ with $d \leq 3$. For Equation (1.2), we say u is a solution of (1.2) in a weak sense as in [12] if u satisfies that

$$(1.3) \quad u(t, x) = \int_{\mathbf{D}} G(t, x, y) u_0(y) dy + \int_0^t \int_{\mathbf{D}} G(t-s, x, y) u(s, y) W(dy, ds),$$

where $G(t, x, y) : (t, x, y) \in [0, \infty) \times \mathbf{D}^2 \rightarrow \mathbf{R}$ is the Green kernel of the operator $\frac{\partial}{\partial t} + \Delta^2$ with a homogeneous Neumann boundary condition as in (1.2). We notice that the existence and uniqueness have been proved for the function-valued mild solution of (1.2) with $d \leq 3$ in Bo et al. [2].

On the other hand, we shall consider Equation (1.2) for $d = 4, 5$. In order to get a function-valued solution for (1.2), we have to replace the space-time white noise in (1.2) by a space-correlated Gaussian noise. That is, for $d = 4, 5$, we consider such an SPDE as follows:

$$(1.4) \quad \begin{cases} (\frac{\partial}{\partial t} + \Delta^2)u(t, x) = u(t, x)\dot{F}(t, x), & \text{in } [0, \infty) \times \mathbf{D}, \\ u(0, x) = u_0(x), & x \in \mathbf{D}, \\ \frac{\partial u(t, x)}{\partial n} = \frac{\partial \Delta u(t, x)}{\partial n} = 0, & \text{on } [0, \infty) \times \partial \mathbf{D}, \end{cases}$$

where the random field $F := (F(\varphi), \varphi \in \mathcal{D}(\mathbf{R}_+ \times \mathbf{D}))$ is an $L^2(\Omega)$ -valued Gaussian field. For $\varphi, \psi \in \mathcal{D}(\mathbf{R}_+ \times \mathbf{D})$, the covariance of $F(\varphi)$ and $F(\psi)$ is given by

$$(1.5) \quad J(\varphi, \psi) := E[F(\varphi)F(\psi)] = \int_0^\infty \int_{\mathbf{D}^2} \varphi(t, y) f(y-z) \psi(t, z) dy dz dt,$$

where $f : (\mathbf{D} - \mathbf{D})^* \rightarrow [0, +\infty)$ is a continuous kernel function with $(\mathbf{D} - \mathbf{D})^* := \{y - z; y, z \in \mathbf{D}, y \neq z\}$. The bilinear form J defined by (1.5) is non-negative definite if and only if f is the Fourier transform of a non-negative tempered distribution μ on $\mathbf{D}$ (see [5]). Thus F defines a martingale measure on which the stochastic integrals are well defined in Walsh's [12] sense. Throughout the paper, we take $f(r) = |r|^{-\alpha}$ for some $\alpha \in (0, d - \frac{4}{3})$. It follows from the standard argument in [4] that Equation (1.4) admits a unique function-valued solution for $d = 4, 5$.

Our central objective is to estimate the Lyapunov exponents in $L^2(\Omega)$ for the solutions of Equations (1.2) and (1.4). Besides this, the convergence rate of the solutions by their finite chaos expansion terms will also be given in the latter.

2. LYAPUNOV EXPONENT ESTIMATES FOR THE SOLUTION OF (1.2)

At first we have to address the existence and the uniqueness of the solution to Equation (1.2) (the proof refers to Bo et al. [2]).

Proposition 2.1. *Let $\mathbf{D} = [0, \pi]^d$ with $d \leq 3$. Consider the following SPDE:*

$$(2.1) \quad \begin{cases} (\frac{\partial}{\partial t} + \Delta^2)u = b(u) + a(u)\dot{W}, & \text{in } [0, \infty) \times \mathbf{D}, \\ u(0, x) = u_0(x), & x \in \mathbf{D}, \\ \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0, & \text{on } [0, \infty) \times \partial \mathbf{D}. \end{cases}$$

Assume that $b(\cdot), a(\cdot)$ are all Lipschitzian on $\mathbf{R}$ satisfying linear growth conditions and the initial $u_0 \in L^q(\mathbf{D})$ with $2 \leq q \leq \infty$.

(1) If $q = \infty$, then for every $T > 0$, (2.1) admits a unique solution $(u(t, x))_{(t, x) \in [0, T] \times \mathbf{D}}$ such that for $p \in [2, \infty)$,

$$\sup_{(t, x) \in [0, T] \times \mathbf{D}} E[|u(t, x)|^p] < \infty.$$

(2) If $2 \leq q \leq p < \infty$ and $d \in \{1, 2\}$, then for every $T > 0$, (2.1) admits a unique solution $(u(t, x))_{(t, x) \in [0, T] \times \mathbf{D}}$ such that

$$\sup_{t \in [0, T]} E\left[\|u(t, \cdot)\|_q^p\right] < \infty.$$

(3) If $2 \leq q \leq p < \infty$, $\frac{1}{q} - \frac{1}{p} < \frac{1}{6}$ and $d = 3$, then for every $T > 0$, (2.1) admits a unique solution $(u(t, x))_{(t, x) \in [0, T] \times \mathbf{D}}$ such that

$$\sup_{t \in [0, T]} E\left[\|u(t, \cdot)\|_q^p\right] < \infty.$$

In our model (1.2), set $a(u) = u$ and $b(u) \equiv 0$ in (2.1). Then by Proposition 2.1, Equation (1.2) possesses a unique solution $(u(t, x))_{(t, x) \in [0, T] \times \mathbf{D}}$ for every $T > 0$. For this solution, we shall have a Lyapunov exponent uniform estimate, which is explicitly stated as follows:

Theorem 2.1. Let $d \leq 3$ and the initial $u_0 \in L^\infty(\mathbf{D})$. Then for all small $\varepsilon > 0$,

$$(2.2) \quad \limsup_{t \rightarrow \infty} \frac{\log \left\{ \sup_{x \in \mathbf{D}} E \left[|u(t, x)|^2 \right] \right\}}{t^{1+\varepsilon}} \leq h(\varepsilon, d),$$

where the upper bound $h(\varepsilon, d)$ is given by

$$(2.3) \quad h(\varepsilon, d) = \frac{1 - d/4}{1 + \varepsilon} \left[\frac{\varepsilon + d/4}{\varepsilon(1 - d/4)} \right]^{\frac{\varepsilon + d/4}{1 - d/4}}.$$

To prove Theorem 2.1, we need the following estimates on the Green function G corresponding to the operator $\frac{\partial}{\partial t} + \Delta^2$ with a homogeneous Neumann boundary condition as in (1.2).

Lemma 2.1 ([3]). For $(t, x, y) \in (0, \infty) \times \mathbf{D}^2$, there exist positive constants $k_1 := k_1(d)$, $k_2 := k_2(d)$ such that

$$|G(t, x, y)| \leq \frac{k_1}{t^{\frac{d}{4}}} \exp \left(-k_1 \frac{|x - y|^{\frac{4}{3}}}{t^{\frac{1}{3}}} \right)$$

and

$$\int_{\mathbf{R}^d} \exp \left(-\frac{|x|^{\frac{4}{3}}}{t^{\frac{1}{3}}} \right) dx = k_2 t^{\frac{d}{4}}.$$

Now we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. We first introduce the series $\{H_i(t, x), i \geq 0\}$ and $\{I_i(t, x), i \geq 0\}$. For $i = 1, 2, \dots$, and $(t, x) \in [0, \infty) \times \mathbf{D}$, define

$$(2.4) \quad \begin{aligned} H_0(t, x) &= \int_{\mathbf{D}} G(t, x, y) u_0(y) dy, \\ H_i(t, x) &= \int_0^t \int_0^{s_1} \cdots \int_0^{s_{i-1}} \int_{\mathbf{D}^{i+1}} G(t - s_1, x, y_1) \times \cdots \times G(s_{i-1} - s_i, y_{i-1}, y_i) \\ &\quad \times G(s_i, y_i, y_{i+1}) u_0(y_{i+1}) dy_{i+1} W(ds_i, dy_i) \cdots W(ds_1, dy_1) \end{aligned}$$

and

$$\begin{aligned}
 I_0(t, x) &= \left[\int_{\mathbf{D}} G(t, x, y) u_0(y) dy \right]^2, \\
 I_i(t, x) &= \int_0^t \int_0^{s_1} \cdots \int_0^{s_{i-1}} \int_{\mathbf{D}^i} G^2(t - s_1, x, y_1) \times \cdots \times G^2(s_{i-1} - s_i, y_{i-1}, y_i) \\
 (2.5) \quad &\times \left[\int_{\mathbf{D}} G(s_i, y_i, y_{i+1}) u_0(y_{i+1}) dy_{i+1} \right]^2 dy_i ds_i \cdots dy_1 ds_1.
 \end{aligned}$$

Next we are going to estimate $I_i(t, x)$ with $i = 1, 2, \dots$. Note that $u_0 \in L^\infty(\mathbf{D})$. Then by Lemma 2.1,

$$\begin{aligned}
 \int_{\mathbf{D}} G(s_i, y_i, y_{i+1}) u_0(y_{i+1}) dy_{i+1} &\leq \|u_0\|_\infty \left[\int_{\mathbf{D}} |G(s_i, y_i, y_{i+1})| dy_{i+1} \right] \\
 &= \text{const.}
 \end{aligned}$$

From Lemma 2.1, it follows that for all $(t, x) \in [0, \infty) \times \mathbf{D}$,

$$\begin{aligned}
 I_i(t, x) &\leq \text{const.} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{i-1}} \frac{1}{|t - s_1|^{\frac{d}{4}}} \times \cdots \times \frac{1}{|s_{i-2} - s_{i-1}|^{\frac{d}{4}}} \times \frac{1}{|s_{i-1} - s_i|^{\frac{d}{4}}} ds_1 \cdots ds_i \\
 &= \text{const.} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{i-2}} \frac{1}{|t - s_1|^{\frac{d}{4}}} \times \cdots \times \frac{1}{|s_{i-2} - s_{i-1}|^{\frac{d}{4}}} \times s_{i-1}^{1-\frac{d}{4}} ds_1 \cdots ds_{i-1}. \\
 (2.6)
 \end{aligned}$$

A routine calculation shows that, for $\alpha > 0$ and $j = 1, 2, \dots$,

$$\begin{aligned}
 \int_0^{s_j} \frac{s_{j+1}^\alpha}{|s_j - s_{j+1}|^{\frac{d}{4}}} ds_{j+1} &= \int_0^{s_j} \frac{s_{j+1}^\alpha s_j^{-\frac{d}{4}}}{|1 - \frac{s_{j+1}}{s_j}|^{\frac{d}{4}}} ds_{j+1} \\
 &= s_j^{\alpha + (1-\frac{d}{4})} \int_0^1 a^\alpha (1-a)^{-\frac{d}{4}} da.
 \end{aligned}$$

Thus by (2.6), for all $(t, x) \in [0, \infty) \times \mathbf{D}$,

$$(2.7) \quad I_i(t, x) \leq \text{const.} \quad t^{i(1-\frac{d}{4})} \prod_{k=1}^{i-1} \int_0^1 a^{k(1-\frac{d}{4})} (1-a)^{-\frac{d}{4}} da,$$

where we set $\prod_{k=1}^0 \equiv 1$ by the usual convention. Note that for $k = 1, 2, \dots$, and $1 < p < \frac{4}{d}$ with $\frac{1}{q} + \frac{1}{p} = 1$,

$$\begin{aligned}
 \int_0^1 a^{k(1-\frac{d}{4})} (1-a)^{-\frac{d}{4}} da &\leq \left[\int_0^1 (1-a)^{-\frac{dp}{4}} da \right]^{\frac{1}{p}} \times \left[\int_0^1 (1-a)^{(1-\frac{d}{4})kq} da \right]^{\frac{1}{q}} \\
 (2.8) \quad &= \left[\frac{1}{1 - \frac{dp}{4}} \right]^{\frac{1}{p}} \left[\frac{1}{1 + (1 - \frac{d}{4})kq} \right]^{\frac{1}{q}}.
 \end{aligned}$$

Therefore by (2.7),

$$\begin{aligned}
 I_i(t, x) &\leq \text{const. } t^{i(1-\frac{d}{4})} \prod_{k=1}^{i-1} \int_0^1 a^{k(1-\frac{d}{4})} (1-a)^{-\frac{d}{4}} da \\
 (2.9) \quad &\leq \text{const. } \left[\frac{1}{1-\frac{dp}{4}} \right]^{\frac{i-1}{p}} \frac{t^{i(1-\frac{d}{4})}}{\left[(1-\frac{d}{4})q\right]^{\frac{i-1}{q}} [(i-1)!]^{\frac{1}{q}}}.
 \end{aligned}$$

Let

$$(2.10) \quad u(t, x) = \sum_{i=0}^{\infty} H_i(t, x),$$

and so

$$(2.11) \quad E[|u(t, x)|^2] = \sum_{i=0}^{\infty} I_i(t, x).$$

From (2.9) and the Hölder inequality, it follows that there exist constants $c_1, c_2 > 0$ such that

$$\begin{aligned}
 E[|u(t, x)|^2] &\leq c_1 + c_2 \sum_{i=1}^{\infty} t^{i(1-\frac{d}{4})} \left[\frac{1}{1-\frac{dp}{4}} \right]^{\frac{i-1}{p}} \times \frac{1}{\left[(1-\frac{d}{4})q\right]^{\frac{i-1}{q}} [(i-1)!]^{\frac{1}{q}}} \\
 &\leq c_1 + c_2 t^{1-\frac{d}{4}} \sum_{i=1}^{\infty} \left[\frac{t^{1-\frac{d}{4}}}{(1-\frac{dp}{4})^{\frac{1}{p}}} \frac{1}{\left[q(1-\frac{d}{4})\right]^{\frac{1}{q}}} \right]^{i-1} \times \left[\frac{1}{(i-1)!} \right]^{\frac{1}{q}} \\
 &\leq c_1 + c_2 t^{1-\frac{d}{4}} \left\{ \sum_{i=1}^{\infty} \left[\frac{1}{q(1-\frac{d}{4})} \right]^{\frac{(i-1)p}{q}} \right\}^{\frac{1}{p}} \\
 (2.12) \quad &\times \left\{ \sum_{i=1}^{\infty} \left[t^{(1-\frac{d}{4})q} \left[1-\frac{dp}{4} \right]^{-q/p} \right]^{i-1} \left[\frac{1}{(i-1)!} \right] \right\}^{\frac{1}{q}}.
 \end{aligned}$$

On the other hand, since $d \leq 3$, $1 < p < 4/d$, $1/p + 1/q = 1$, and so $q[1-d/4] > 1$. Then

$$c_3(d) =: \left\{ \sum_{i=1}^{\infty} \left[\frac{1}{q(1-\frac{d}{4})} \right]^{\frac{(i-1)p}{q}} \right\}^{\frac{1}{p}} < \infty.$$

Hence by (2.12), we obtain for all $(t, x) \in [0, \infty) \times \mathbf{D}$,

$$(2.13) \quad E[|u(t, x)|^2] \leq c_1 + c_2 c_3(d) t^{1-\frac{d}{4}} \exp \left(\frac{t^{(1-\frac{d}{4})q}}{q(1-\frac{d}{4})^{\frac{q}{p}}} \right).$$

Recall the definitions of $H_i(t, x)$ with $i = 1, 2, \dots$. By (2.10)–(2.13), we get that $u(t, x)$ defined by (2.10) is a series solution of Equation (1.2). Moreover the uniqueness argument shows that it is the unique solution. In the following, we are going

to get an estimate of the Lyapunov exponent for the solution. For any fixed $\epsilon > 0$, set

$$p = \frac{1+\epsilon}{\frac{d}{4}+\epsilon} < \frac{4}{d}, \quad q = \frac{1+\epsilon}{1-\frac{d}{4}}.$$

Then we apply the estimate (2.13) to conclude that

$$\limsup_{t \rightarrow \infty} \frac{\log \left\{ \sup_{x \in \mathbf{D}} E [|u(t, x)|^2] \right\}}{t^{1+\epsilon}} \leq \frac{1-\frac{d}{4}}{1+\epsilon} \left[\frac{\epsilon + \frac{d}{4}}{\epsilon(1-\frac{d}{4})} \right]^{\frac{\epsilon+\frac{d}{4}}{1-\frac{d}{4}}}.$$

Then the desired result follows from the definition of $h(\epsilon, d)$. $\square$

The next assertion gives the convergence estimate for the solution to (1.2) by its first N -finite chaos terms of (2.10).

Proposition 2.2. *Let $d \leq 3$ and $N \in \mathbf{N}$. Define the first N -finite chaos terms of (2.8) by*

$$u_N(t, x) = \sum_{i=0}^N H_i(t, x).$$

For $q > \frac{4}{4-d}$ and $\alpha := T^{(1-\frac{d}{4})q}(1-\frac{d}{4}p)^{-q/p}$ with $T > 0$, define

$$b(d, N) := \left[\frac{\alpha}{q(1-\frac{d}{4})} \right]^{\frac{N}{q}} \times \frac{1}{(N!)^{\frac{1}{q}}} \times \left[\frac{N}{N-\alpha} \right]^{\frac{1}{q}}$$

and $\lim_{N \rightarrow \infty} b(d, N) = 0$. Then there exists a constant $c := c(d) > 0$ such that

$$(2.14) \quad E [|u(t, x) - u_N(t, x)|^2] \leq cb(d, N),$$

for all $(t, x) \in [0, T] \times \mathbf{D}$ and $N > \alpha$.

Proof. By (2.10)–(2.11), we obtain for all $(t, x) \in [0, T] \times \mathbf{D}$,

$$\begin{aligned} & E [|u(t, x) - u_N(t, x)|^2] \\ &= \sum_{i=N+1}^{\infty} I_i(t, x) \\ (2.15) \quad & \leq \text{const.} \sum_{i=N+1}^{\infty} t^{i(1-\frac{d}{4})} \left[\frac{1}{1-\frac{dp}{4}} \right]^{\frac{i-1}{p}} \times \frac{1}{[(1-\frac{d}{4})q]^{\frac{i-1}{q}} [(i-1)!]^{\frac{1}{q}}} \\ & \leq \text{const.} T^{1-\frac{d}{4}} \sum_{i=N+1}^{\infty} \left[T^{1-\frac{d}{4}} \frac{1}{(1-\frac{dp}{4})^{\frac{1}{p}}} \frac{1}{[q(1-\frac{d}{4})]^{\frac{1}{q}}} \right]^{i-1} \times \left[\frac{1}{(i-1)!} \right]^{\frac{1}{q}} \\ & \leq \text{const.} A_1(N) \times A_2(N), \end{aligned}$$

with

$$\begin{aligned} A_1(N) &:= \left\{ \sum_{i=N+1}^{\infty} \left[\frac{1}{q(1-\frac{d}{4})} \right]^{\frac{(i-1)p}{q}} \right\}^{\frac{1}{p}} \\ &= \left[\frac{1}{q(1-\frac{d}{4})} \right]^{\frac{N}{q}} \left\{ \frac{1}{1 - \left[\frac{1}{q(1-\frac{d}{4})} \right]^{\frac{p}{q}}} \right\}^{\frac{1}{p}} \end{aligned}$$

and

$$A_2(N) := \left\{ \sum_{i=N+1}^{\infty} \left[T^{(1-\frac{d}{4})q} \left(1 - \frac{d}{4}p \right)^{-q/p} \right]^{i-1} \times \left[\frac{1}{(i-1)!} \right] \right\}^{\frac{1}{q}}.$$

We take N such that $0 < \alpha := T^{(1-\frac{d}{4})q} (1 - \frac{d}{4}p)^{-q/p} < N$. Therefore,

$$\begin{aligned} \sum_{i=N+1}^{\infty} \frac{\alpha^{i-1}}{(i-1)!} &= \sum_{i=0}^{\infty} \frac{\alpha^{N+i}}{(N+i)!} \\ &\leq \frac{\alpha^N}{N!} \sum_{i=0}^{\infty} \frac{\alpha^i}{N^i} \\ &= \frac{\alpha^N}{N!} \times \frac{N}{N-\alpha}. \end{aligned}$$

Then for $N > \alpha$,

$$A_2(N) \leq \frac{\alpha^{\frac{N}{q}}}{(N!)^{\frac{1}{q}}} \left[\frac{N}{N-\alpha} \right]^{\frac{1}{q}}.$$

Then the conclusion of the proposition follows from (2.15). $\square$

3. LYAPUNOV EXPONENT ESTIMATES FOR $d = 4, 5$

In this section, we consider Equation (1.4) for $d = 4, 5$. As in the weak sense of [12], (1.4) is equivalent to the following mild form:

$$(3.1) \quad u(t, x) = \int_{\mathbf{D}} G(t, x, y) u_0(y) dy + \int_0^t \int_{\mathbf{D}} G(t-s, x, y) u(s, y) F(dy, ds).$$

The existence and uniqueness of (3.1) have been established in [4] in the case of $f(r) = r^{-\alpha}$ (Riesz potential) for $\alpha \in (0, d - \frac{4}{3})$, where f is the spatial correlative kernel of F as in (1.5). For $i = 1, 2, \dots$ and $(t, x) \in [0, \infty) \times \mathbf{D}$, define

$$\begin{aligned} \tilde{H}_0(t, x) &= \int_{\mathbf{D}} G(t, x, y) u_0(y) dy, \\ \tilde{H}_i(t, x) &= \int_0^t \int_0^{s_1} \cdots \int_0^{s_{i-1}} \int_{\mathbf{D}^{i+1}} G(t-s_1, x, y_1) \times \cdots \times G(s_{i-1}-s_i, y_{i-1}, y_i) \\ (3.2) \quad &\times G(s_i, y_i, y_{i+1}) u_0(y_{i+1}) dy_{i+1} F(dy_i, ds_i) \cdots F(dy_1, ds_1) \end{aligned}$$

and

$$\begin{aligned} \tilde{I}_0(t, x) &= \left[\int_{\mathbf{D}} G(t, x, y) u_0(y) dy \right]^2, \\ \tilde{I}_i(t, x) &= \int_0^t \int_0^{s_1} \cdots \int_0^{s_{i-1}} \int_{\mathbf{D}^{2i}} G(t-s_1, x, y_1) f(y_1-z_1) G(t-s_1, x, z_1) \\ (3.3) \quad &\times \cdots \times G(s_{i-1}-s_i, y_{i-1}, y_i) f(y_i-z_i) \\ &\times G(s_{i-1}-s_i, y_{i-1}, z_i) dy_i dz_i ds_i \cdots dy_1 dz_1 ds_1 \\ &\times \left[\int_{\mathbf{D}^2} G(s_i, y_i, y) u_0(y) f(y-z) G(s_i, y_i, z) u_0(z) dy dz \right]. \end{aligned}$$

Similarly as in Section 2, we shall have

Theorem 3.1. *Let $d = 4, 5$. Assume that the initial $u_0 \in L^\infty(\mathbf{D})$. Then $\tilde{I}_i(t, x)$ is well defined, for all $i \in \mathbf{N}$ and $(t, x) \in [0, \infty) \times \mathbf{D}$. Further, it holds that*

$$(3.4) \quad \limsup_{t \rightarrow \infty} \frac{\log \left\{ \sup_{x \in \mathbf{D}} E \left[|u(t, x)|^2 \right] \right\}}{t^{1 + (\frac{4}{3} - \frac{d}{4})}} \leq \frac{1}{\frac{7}{3} - \frac{d}{4}}.$$

For proving Theorem 3.1, we need the following estimate.

Lemma 3.1. *Let $d = 4, 5$ and $\alpha \in (0, d - \frac{4}{3})$. Define*

$$I(t, x) = \int_0^t \int_{\mathbf{D}^2} |G(t-s, x, y)| |y-z|^{-\alpha} |G(t-s, x, z)| dy dz ds.$$

Then there exists a constant $c > 0$ such that

$$I(t, x) \leq ct^{\frac{4}{3} - \frac{d}{4}}, \quad \forall (t, x) \in [0, \infty) \times \mathbf{D}.$$

Proof. Note that for $\beta \geq 1$ and $x, y, z \in \mathbf{R}$,

$$|y-z|^\beta \leq 2^{\beta-1} [|x-y|^\beta + |x-z|^\beta].$$

Then by Lemma 2.1, for $\lambda \in (0, 1)$,

$$(3.5) \quad \begin{aligned} & I(t, x) \\ & \leq k_1^2 \int_0^t \int_{\mathbf{D}^2} (t-s)^{-\frac{d}{2}} \exp \left(-k_1 \frac{|x-y|^{\frac{4}{3}}}{(t-s)^{\frac{1}{3}}} \right) |y-z|^{-\alpha} \exp \left(-k_1 \frac{|x-z|^{\frac{4}{3}}}{(t-s)^{\frac{1}{3}}} \right) dy dz ds \\ & = k_1^2 \int_0^t \int_{\mathbf{D}^2} s^{-\frac{d}{2}} \exp \left(-k_1 (1-\lambda) \frac{|x-y|^{\frac{4}{3}}}{s^{\frac{1}{3}}} \right) \exp \left(-k_1 \lambda \frac{|x-y|^{\frac{4}{3}}}{s^{\frac{1}{3}}} \right) \\ & \quad \times |y-z|^{-\alpha} \exp \left(-k_1 \lambda \frac{|x-z|^{\frac{4}{3}}}{s^{\frac{1}{3}}} \right) dy dz ds \\ & \leq k_1^2 \int_0^t \int_{\mathbf{D}^2} s^{-\frac{d}{2}} \exp \left(-k_1 (1-\lambda) \frac{|x-y|^{\frac{4}{3}}}{s^{\frac{1}{3}}} \right) \\ & \quad \times |y-z|^{-\alpha} \exp \left(-k_1 \lambda 2^{-\frac{1}{3}} \frac{|y-z|^{\frac{4}{3}}}{s^{\frac{1}{3}}} \right) dy dz ds \\ & \leq k_1^2 \int_0^t s^{-\frac{d}{2}} \int_{\mathbf{D}} \exp \left(-\bar{c} \frac{|x-y|^{\frac{4}{3}}}{s^{\frac{1}{3}}} \right) dy \int_{\mathbf{D}} |y-z|^{-\alpha} \exp \left(-\bar{c} \frac{|y-z|^{\frac{4}{3}}}{s^{\frac{1}{3}}} \right) dz ds. \end{aligned}$$

Let $l = |s^{\frac{1}{4}}(x-y)|$, $v = y-z$. Then

$$\begin{aligned} I(t, x) & \leq \text{const.} \int_0^t s^{\frac{d}{4} - \frac{d}{2}} \int_0^\infty \exp(-\bar{c} l^{\frac{4}{3}}) l^{d-1} dl \int_{B_d(0, R)} \exp \left(-\bar{c} \frac{|v|^{\frac{4}{3}}}{s^{\frac{1}{3}}} \right) |v|^{-\alpha} dv ds \\ & \leq \text{const.} \int_0^t s^{-\frac{d}{4}} \int_{B_d(0, R)} \exp \left(-\bar{c} \frac{|v|^{\frac{4}{3}}}{s^{\frac{1}{3}}} \right) |v|^{-\alpha} dv ds, \end{aligned}$$

where $B_d(0, R)$ is the ball centered at 0 with radius $R > 2\pi$. For $v \neq 0$ fixed, change variable by letting $r = |v|^{\frac{4}{3}} s^{-\frac{1}{3}}$. Then

$$I(t, x) \leq \text{const.} \int_{B_d(0, R)} |v|^{4-d-\alpha} dv \int_{|v|^{\frac{4}{3}} t^{-\frac{1}{3}}}^{\infty} r^{-4+\frac{3}{4}d} \exp(-\bar{c}r) dr.$$

Note that for $d = 4, 5$, $\frac{3}{4}d - 4 < 0$. Hence

$$\begin{aligned} I(t, x) &\leq \text{const.} \int_{B_d(0, R)} |v|^{4-d-\alpha} (|v|^{\frac{4}{3}} t^{-\frac{1}{3}})^{-4+\frac{3}{4}d} dv \\ &= \text{const.} \ t^{\frac{4}{3}-\frac{d}{4}} \int_{B_d(0, R)} |v|^{-\frac{4}{3}-\alpha} dv \\ &\leq \text{const.} \ t^{\frac{4}{3}-\frac{d}{4}}, \end{aligned}$$

for any $\alpha \in (0, d - \frac{4}{3})$. Thus the proof of the lemma is completed. $\square$

In what follows, we proceed to the proof of Theorem 3.1.

Proof of Theorem 3.1. Applying the same iterative scheme as in Theorem 2.1, we have

$$(3.6) \quad u(t, x) = \sum_{i=0}^{\infty} \tilde{H}_i(t, x)$$

and

$$(3.7) \quad E[|u(t, x)|^2] = \sum_{i=0}^{\infty} \tilde{I}_i(t, x).$$

Next we estimate the term $\tilde{I}_i$ defined by (3.3) for $i \geq 1$. By Lemma 3.1, for all $(t, x) \in [0, \infty) \times \mathbf{D}$,

$$\begin{aligned} \tilde{I}_i(t, x) &\leq \text{const.} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{i-1}} |s_{i-1} - s_i|^{\frac{4}{3}-\frac{d}{4}} \times |s_{i-2} - s_{i-1}|^{\frac{4}{3}-\frac{d}{4}} \\ &\quad \times \cdots \times |t - s_1|^{\frac{4}{3}-\frac{d}{4}} ds_i \cdots ds_1 \\ (3.8) \quad &= \text{const.} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{i-2}} \frac{s_{i-1}^{(\frac{4}{3}-\frac{d}{4}+1)}}{\frac{4}{3}-\frac{d}{4}+1} \\ &\quad \times |s_{i-2} - s_{i-1}|^{\frac{4}{3}-\frac{d}{4}} \times \cdots \times |t - s_1|^{\frac{4}{3}-\frac{d}{4}} ds_{i-1} \cdots ds_1. \end{aligned}$$

By a similar argument as in the proof of Theorem 2.1, we get

$$\begin{aligned} \tilde{I}_i(t, x) &\leq \text{const.} \ t^{i(1+\frac{4}{3}-\frac{d}{4})} \prod_{k=1}^{i-1} \int_0^1 a^{k(\frac{7}{3}-\frac{d}{4})} (1-a)^{\frac{4}{3}-\frac{d}{4}} da \\ (3.9) \quad &\leq \text{const.} \ t^{i(1+\frac{4}{3}-\frac{d}{4})} \prod_{k=1}^{i-1} \frac{1}{k(\frac{7}{3}-\frac{d}{4})+1} \\ &\leq \text{const.} \ \frac{t^{i(1+(\frac{4}{3}-\frac{d}{4}))}}{(\frac{7}{3}-\frac{d}{4})^{i-1}} \times \frac{1}{(i-1)!}. \end{aligned}$$

By the boundness of u_0 , for $t > 0$ large enough,

$$\begin{aligned} E[|u(t, x)|^2] &\leq \text{const. } t^{1+(\frac{4}{3}-\frac{d}{4})} \sum_{i=1}^{\infty} \left[\frac{t^{1+(\frac{4}{3}-\frac{d}{4})}}{\frac{7}{3}-\frac{d}{4}} \right]^{i-1} \times \frac{1}{(i-1)!} \\ &= \text{const. } t^{1+(\frac{4}{3}-\frac{d}{4})} \exp \left[\frac{t^{1+(\frac{4}{3}-\frac{d}{4})}}{\frac{7}{3}-\frac{d}{4}} \right], \end{aligned}$$

which implies

$$\limsup_{t \rightarrow \infty} \frac{\log \left\{ \sup_{x \in \mathbf{D}} E[|u(t, x)|^2] \right\}}{t^{1+(\frac{4}{3}-\frac{d}{4})}} \leq \frac{1}{\frac{7}{3}-\frac{d}{4}}.$$

This proves the theorem. $\square$

Finally, by a similar way as in Section 2, we also get the convergence estimate for the series solution of Equation (1.4) when $d = 4, 5$.

Proposition 3.1. *Let $d = 4, 5$ and $N \in \mathbf{N}$. Define the first N -finite chaos terms of the solution of (1.4) by*

$$\tilde{u}_N(t, x) = \sum_{i=0}^N \tilde{H}_i(t, x).$$

Then for any $T > 0$, there exists a constant $\tilde{c} > 0$ such that, for all $(t, x) \in [0, T] \times \mathbf{D}$,

$$(3.10) \quad E[|u(t, x) - \tilde{u}_N(t, x)|^2] \leq \tilde{c}b(d, N),$$

with

$$b(d, N) := \frac{T^{1+(\frac{4}{3}-\frac{d}{4})}}{N!} \left[\frac{T^{1+(\frac{4}{3}-\frac{d}{4})}}{\frac{7}{3}-\frac{d}{4}} \right]^N$$

and $\lim_{N \rightarrow \infty} b(d, N) = 0$.

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