CHARACTERIZING INDECOMPOSABLE PLANE CONTINUA FROM THEIR COMPLEMENTS

CLINTON P. CURRY, JOHN C. MAYER, AND E. D. TYMCHATYN

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ABSTRACT. We show that a plane continuum X is indecomposable iff X has a sequence $(U_n)_{n=1}^\infty$ of not necessarily distinct complementary domains satisfying the *double-pass condition*: for any sequence $(A_n)_{n=1}^\infty$ of open arcs, with $A_n \subset U_n$ and $\overline{A_n} \setminus A_n \subset \partial U_n$, there is a sequence of shadows $(S_n)_{n=1}^\infty$, where each S_n is a shadow of A_n , such that $\lim S_n = X$. Such an open arc divides U_n into disjoint subdomains $V_{n,1}$ and $V_{n,2}$, and a *shadow* (of A_n) is one of the sets $\partial V_{n,i} \cap \partial U$.

1. INTRODUCTION

In this paper, a *continuum* is a compact, connected, nonempty metric space. A continuum is *decomposable* if it can be written as the union of two of its proper subcontinua; otherwise, it is *indecomposable*. Let $\mathbb{C}$ denote the complex plane and let $\mathbb{C}_\infty$ denote the Riemann sphere $\mathbb{C} \cup \{\infty\}$. A *plane domain* is a subset of $\mathbb{C}_\infty$ which is conformally isomorphic to the open unit disk $\mathbb{D} \subset \mathbb{C}_\infty$ (which is to say that it is open, connected, simply connected and its boundary is a nondegenerate subcontinuum of $\mathbb{C}_\infty$). If X is a continuum in $\mathbb{C}_\infty$, the components of $\mathbb{C}_\infty \setminus X$ are called *complementary domains* and are plane domains. If $W \subset \mathbb{C}_\infty$, we denote the boundary of W by ∂W . We say that a point x of a continuum is *buried* if it does not lie on the boundary of any complementary domain. The spherical metric on $\mathbb{C}_\infty$ is denoted by d , and H_d denotes the Hausdorff metric on the hyperspace of subcontinua of $\mathbb{C}_\infty$ [18, Section 4.1].

There are several ways of recognizing intrinsically that a continuum X is indecomposable. For instance, X is indecomposable if and only if every proper subcontinuum of X is nowhere dense in X [10]. Also, a continuum X is indecomposable if there are points $a, b, c \in X$ such that no proper subcontinuum of X contains any two of these points. In this paper, we are interested in recognizing indecomposable planar continua not by intrinsic properties, but by the relationship between the continuum and its ambient space.

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Indecomposable continua arise naturally in dynamical systems [8, 7]. However, in specific dynamical systems, it is often difficult to recognize them. In complex analytic dynamics, the *Julia set* of a rational map $f : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ is the set of unstable points under iteration of f (see [15] for definitions). A long-standing question in complex analytic dynamics asks: *Can the Julia set of a rational function be an indecomposable continuum?* Several authors have attacked this question, among them [14, 4] for polynomials, [22] for *bicritical* rational maps (rational maps with exactly two critical points), and [5] for Julia sets of a class of rational functions with no buried points. In this situation, it is much easier to analyze the complement of the Julia set, called the *Fatou set*; this motivates our interest in studying indecomposability from the point of view of a continuum's complement.

The second-named author, with various co-authors [14, 4, 5], investigated the recognition of indecomposable continua from their complements in the case that $\partial U = X$ for some complementary domain U of X . The tool used was prime end theory, and a characterization was obtained in the context of X being a Julia set, making use of the dynamics. The characterization of indecomposable continua from their complements in the current paper primarily addresses the case that X is not the boundary of any of its complementary domains (which would imply that there are infinitely many complementary domains, each having boundaries nowhere dense in X). Even better, this characterization also subsumes the first case and is entirely topological.

To state our characterization theorem, we need some definitions. These concepts are related to those which arose originally in prime end theory.

Definition 1.1. Let U be a plane domain. A *crosscut* of U is an open arc $A = (a, b) \subset U$ such that $\overline{A} = [a, b]$ is a closed arc which meets ∂U exactly in the set $\{a, b\}$. A *generalized crosscut* of U is an open arc $A \subset U$ such that $\overline{A} \setminus A \subset \partial U$.

Notice that the notion of a generalized crosscut is strictly broader than the notion of a crosscut. It is easy to see that a generalized crosscut of a domain U cuts U into two nonempty disjoint subdomains V_1 and V_2 such that $U = V_1 \cup A \cup V_2$.

Definition 1.2. Let U be a plane domain and A a generalized crosscut of U . We call each component of $U \setminus A$ a *crosscut neighborhood*. If V is a crosscut neighborhood determined by a generalized crosscut A , we call the continuum $S = \partial V \cap \partial U$ a *shadow* of A .

Thus, a generalized crosscut A of a domain U has exactly two crosscut neighborhoods, and consequently two shadows whose union is ∂U . Examples below show that one or both of these shadows can be proper subcontinua of ∂U or, more surprisingly, all of ∂U .

The limits below are interpreted in the metric H_d .

Definition 1.3. A sequence $(U_n)_{n=1}^\infty$ of (not necessarily distinct) complementary domains of a continuum X satisfies the *double-pass condition* if, for any sequence of generalized crosscuts A_n of U_n , there is a sequence of shadows $(S_n)_{n=1}^\infty$ of $(A_n)_{n=1}^\infty$ such that $\lim_{n \rightarrow \infty} S_n = X$.

In Section 3, we prove the following theorem, which is the main theorem of this paper.

Theorem 1.4 (Characterization Theorem). *A planar continuum X is indecomposable if and only if it has a sequence $(U_n)_{n=1}^\infty$ of complementary domains which satisfies the double-pass condition.*

2. PARTIAL AND PRIOR RESULTS

2.1. Brief history. The first partial recognition theorem for indecomposable continua from the complement is that of Kuratowski [13].

Theorem 2.1 (Kuratowski). *If a plane continuum X is the common boundary of three of its complementary domains, then X is either indecomposable or the union of two proper indecomposable subcontinua.*

The following theorem of Rutt seems quite different and is also only applicable if X is the boundary of some complementary domain.

Theorem 2.2 (Rutt, [19]). *If a nondegenerate plane continuum X is the boundary of a complementary domain U and if there is a prime end of U whose impression is $\partial U = X$, then X is either indecomposable or the union of two proper indecomposable subcontinua.*

Without going into detail (but see [4]), the *impression* of a prime end of U is the intersection of the shadows of a sequence $(A_n)_{n=1}^\infty$ of crosscuts of U having the property that for each n , $(A_m)_{m>n}$ is a pairwise closure disjoint null sequence contained in one of the crosscut neighborhoods of A_n .

The connection among the theorems above is made explicit by a technical theorem of Burgess. While the original result is stated in terms of what Burgess calls simple disks, the theorem can be equivalently stated in terms of closed balls. For $a \in \mathbb{C}_\infty$ and $r > 0$, define the *ball* of radius r about a by

$$B_r(a) = \{z \in \mathbb{C}_\infty \mid d(a, z) < r\}.$$

Theorem 2.3 (Burgess, [2, Theorem 9]). *Let H be a closed set and X a continuum in the plane. Suppose X_1 , X_2 , and X_3 are subcontinua of X and D_1 , D_2 , and D_3 are pairwise disjoint closed balls with $D_i \cap H = \emptyset$ and $\emptyset \neq D_i \cap X_i = D_i \cap X$ for each $i \in \{1, 2, 3\}$. Then there do not exist three distinct complementary domains of $X \cup H$ such that each of them intersects each of the balls D_i .*

Using this theorem, Burgess proves the following recognition theorem, which also applies when the continuum is not the union of the boundaries of its complementary domains.

Corollary 2.4 (Burgess, [2, Corollary to Theorem 9]). *If the plane continuum X is the limit of a sequence of distinct complementary domains of X , then either X is indecomposable or there is only one pair of indecomposable continua whose union is X .*

As recognition theorems, the above suffer from the weakness of their conclusion. In [4, 5], dynamical considerations rule out that the Julia set of a polynomial can be the union of two proper indecomposable subcontinua. However, this is under the hypothesis that the Julia set is the boundary of one of its complementary domains. The following definition and recognition theorem appear in [5]. Since it represents a simplification of the proof in [5], we prove Theorem 2.6 making use of Theorem 2.3.

Definition 2.5. An *antichain* of crosscuts of a plane domain U is a sequence $(H_n)_{n=1}^\infty$ of distinct pairwise closure disjoint crosscuts of U such that, for each m , one crosscut neighborhood of H_m contains all the crosscuts $(H_n)_{n \neq m}$.

Theorem 2.6. Let $U \subset \mathbb{C}_\infty$ be a plane domain. Let $z \in U$. Suppose there exists an antichain $(H_n)_{n=1}^\infty$ of crosscuts of U such that $\lim_{n \rightarrow \infty} \text{Sh}(H_n) = \partial U$, where $\text{Sh}(H_n)$ is the shadow of the crosscut neighborhood W_n of $U \setminus H_n$ which misses z . Then ∂U is indecomposable.

Proof. For a contradiction, suppose ∂U satisfies the hypotheses of the theorem but may be written as the union of proper subcontinua X_1 and X_2 . By passing to a subsequence, we may assume that $(H_n)_{n=1}^\infty$ converges to a point of ∂U . Choose disjoint closed balls D_1 and D_2 such that

- (1) $z \notin D_i$,
- (2) $D_i \cap H_n = \emptyset$ for all $i \in \{1, 2\}, n \in \mathbb{N}$,
- (3) X_i intersects the interior of D_i , and
- (4) $D_i \cap X_j = \emptyset$ if $i \neq j$.

Choose three crosscuts H_1, H_2 , and H_3 so that the component W_i of $\mathbb{C}_\infty \setminus (\partial U \cup H_i)$ missing z hits both D_1 and D_2 . Notice that since the crosscuts are members of an antichain, $W_i \cap W_j = \emptyset$ for distinct i and j in $\{1, 2, 3\}$. Let R_1, R_2 , and R_3 be arcs from z to ∂U disjoint (except for z) from each other and from $D_1 \cup D_2$, and such that each H_i lies in a different complementary component U_i in U of

$$X = \partial U \cup R_1 \cup R_2 \cup R_3.$$

Notice that $W_i \subset U_i$. Define $X_3 = R_1 \cup R_2 \cup R_3$, and let $D_3 \subset U$ be a closed ball about z which is disjoint from D_1 and D_2 . Thus, each U_i intersects each of D_1, D_2 , and D_3 ; this contradicts Theorem 2.3, with $H = \emptyset$ in the statement. $\square$

2.2. Necessary condition. In this section we show that for a plane continuum X to be indecomposable, it is necessary that X have a sequence of complementary domains whose boundaries converge to X . The proof requires a few additional facts and definitions.

Definition 2.7. The *composant*, denoted $C(p)$, of a point p in a continuum X is the union of all the proper subcontinua of X that contain p .

Theorem 2.8 ([10]). Let X be a nondegenerate indecomposable continuum. Then the following hold:

- (1) X has c pairwise disjoint composants.
- (2) Each composant is dense in X .
- (3) Each composant can be written as a countable increasing union of nowhere dense proper subcontinua of X , converging to X in the Hausdorff metric.

Definition 2.9. A connected topological space X is said to be *unicoherent* if, for any pair A and B of closed, connected subsets such that $A \cup B = X$, the intersection $A \cap B$ is connected.

Note that the plane itself, an open ball, or a closed ball in the plane are each unicoherent [20]. Recall that $B_r(a)$ denotes the open ball of radius $r > 0$ about center a .

Theorem 2.10. *Let X be an indecomposable plane continuum. Then there is a sequence $(U_n)_{n=1}^\infty$ of (not necessarily distinct) complementary domains of X such that $\lim \partial U_n = X$.*

Proof. This is clear if X is a point, so assume X is a nondegenerate indecomposable continuum. Take $p, q, r \in X$, each in a different composant of X . For each $n \in \mathbb{N}$, define

Q_n = the component of $X \setminus B_{1/n}(p)$ containing q ,

R_n = the component of $X \setminus B_{1/n}(p)$ containing r .

Notice that $\lim_{n \rightarrow \infty} Q_n = \lim_{n \rightarrow \infty} R_n = X$, by Theorem 2.8. Since Q_n and R_n are different components of $X \setminus B_{1/n}(p)$, they are separated in $\mathbb{C}_\infty \setminus B_{1/n}(p)$ by $\mathbb{C}_\infty \setminus (B_{1/n}(p) \cup X)$. Also, Q_n and R_n are closed in the normal space $\mathbb{C}_\infty \setminus B_{1/n}(p)$, so there is a subset K_n , closed in $\mathbb{C}_\infty \setminus B_{1/n}(p)$, of $\mathbb{C}_\infty \setminus (B_{1/n}(p) \cup X)$ which separates Q_n and R_n . Since $\mathbb{C}_\infty \setminus B_{1/n}(p)$ is homeomorphic to the closed unit disk in the plane, it is unicoherent, so a component L_n of K_n is a closed (in $\mathbb{C}_\infty \setminus B_{1/n}(p)$) and connected subset of $\mathbb{C}_\infty \setminus (B_{1/n}(p) \cup X)$ which separates Q_n and R_n in $\mathbb{C}_\infty \setminus B_{1/n}(p)$ [20]. Moreover, since $L_n \subset \mathbb{C}_\infty \setminus X$, it lies in a single complementary domain U_n of X . The sequence $(U_n)_{n=1}^\infty$ formed in this way is the required sequence of complementary domains.

It is evident that $\lim_{n \rightarrow \infty} \partial U_n \subset X$; we aim to show that $X \subset \lim_{n \rightarrow \infty} \partial U_n$. Choose $\epsilon > 0$, and $x \in X$. Let $N \in \mathbb{N}$ such that, for all $n \geq N$,

- (1) $Q_n \cap B_\epsilon(x) \neq \emptyset$,
- (2) $R_n \cap B_\epsilon(x) \neq \emptyset$, and
- (3) $B_{1/n}(p) \cap B_\epsilon(x) = \emptyset$.

For $n \geq N$, choose $q_n \in Q_n \cap B_\epsilon(x)$ and $r_n \in R_n \cap B_\epsilon(x)$ for $n \geq N$. The straight line segment A_n from q_n to r_n is a subset of $B_\epsilon(x)$ and, hence, of $\mathbb{C}_\infty \setminus B_{1/n}(p)$, so A_n must meet L_n , since L_n separates q_n from r_n in $\mathbb{C}_\infty \setminus B_{1/n}(p)$. Since $L_n \subset U_n$, A_n intersects $U_n \cap B_\epsilon(x)$. Since q_n, r_n are not in U_n (they lie in X), A_n intersects ∂U_n , and $\partial U_n \cap B_\epsilon(x) \neq \emptyset$. This is true for all $n \geq N$, so $x \in \liminf_{n \rightarrow \infty} \partial U_n \subset \lim_{n \rightarrow \infty} \partial U_n$. This completes the proof. $\square$

3. THE CHARACTERIZATION THEOREM

We saw in Subsection 2.2 that having a sequence of complementary domains whose boundaries converge to X is a necessary condition for the plane continuum X to be indecomposable. Example 3.2 below shows that this condition is not sufficient, even if the domains are distinct, and suggests that we must find a way to rule out that the sequence of complementary domains “splits” into “halves”, each of which converge to proper indecomposable subcontinua.

3.1. Examples.

Example 3.1. The Knaster buckethandle continuum, depicted on the left in Figure 1, is a standard example of an indecomposable continuum. It can be viewed as a disk from which successively deeper and denser fjords are dug. Notice that for any generalized crosscut drawn in its single complementary domain, infinitely many fjords lie in one crosscut neighborhood or the other, so one shadow is dense. Notice that one composant of X is the one-to-one continuous image of the half-line $[0, \infty)$. The point of X corresponding to the point 0 of $[0, \infty)$ is called the *endpoint* of X . For a more precise construction, see [13, Vol. II, p. 205].

FIGURE 1. The Knaster buckethandle continuum (left); a union of two Knaster continua (right) meeting at a sequence of points converging to their common endpoint.

FIGURE 2. Two Knaster continua meeting on an end interval of each.

Example 3.2. The continuum X depicted on the right in Figure 1 is an example of a continuum which satisfies the hypotheses of Theorem 2.4 without being indecomposable. It is a symmetric union of two Knaster buckethandle continua X_1 and X_2 intersecting in a countable set which lies on a vertical line. The continuum has infinitely many complementary domains $(U_i)_{i=1}^{\infty}$. The decomposability of X can be detected from the complementary domains as follows: to its complementary domains $(U_i)_{i=1}^{\infty}$, associate the collection of crosscuts $(K_i)_{i=1}^{\infty}$, where K_i lies in U_i on the vertical axis of symmetry. Each crosscut has one shadow which is a subcontinuum of X_1 and another which is a subcontinuum of X_2 . Therefore, any convergent sequence of shadows must converge in the limit to a proper subcontinuum of X .

Example 3.3. The continuum in Figure 2 is the union of a pair of Knaster continua X_1 and X_2 with distinct endpoints such that $X_1 \cap X_2$ is the horizontal arc A between the endpoints of X_1 and X_2 . This continuum, like Example 3.1, has the property that every crosscut has a dense shadow, despite the continuum's decomposability. Let K be a generalized crosscut such that one end lands on a point of $X \setminus A$ and the other end wiggles between the Knaster continua and compactifies on A . Neither shadow of this generalized crosscut is dense, so the constant sequence consisting of this crosscut fails the double pass condition.

3.2. Proof of characterization theorem. In Definition 1.3, we defined the double-pass condition on a sequence of generalized crosscuts in a sequence of complementary domains which is motivated by Example 3.2 and by the similarly functioning condition of Cook and Ingram [6] introduced for recognizing indecomposable (chainable) continua in terms of refining open covers. Here we prove our main theorem: the existence of a sequence of complementary domains satisfying our double-pass condition is equivalent to indecomposability.

The following lemma follows from [23, (A1.4)], but we include a self-contained proof here for convenience.

Lemma 3.4. *Suppose that $\phi : U \rightarrow \mathbb{D}$ is a conformal isomorphism, where U is a plane domain. Then the image of a null sequence $(K_n)_{n=1}^\infty$ of crosscuts of U is a null sequence of crosscuts of $\mathbb{D}$.*

Proof. By way of contradiction, let $(K_n)_{n=1}^\infty$ be a null sequence of crosscuts of U such that the image sequence $(A_n)_{n=1}^\infty = (\phi(K_n))_{n=1}^\infty$ consists of crosscuts whose diameters are bounded away from zero. Then, by passing to a subsequence, we may assume that the image sequence accumulates on a nondegenerate continuum $L \subset \overline{\mathbb{D}}$. Since $(K_n)_{n=1}^\infty$ does not accumulate on a subset of U , we see that $L \subset \partial\mathbb{D}$. Also, we may assume without loss of generality that $(K_n)_{n=1}^\infty$ converges to a point of $x \in \partial U$.

Let $t \in L$. There exists a chain of crosscuts $(A'_n)_{n=1}^\infty$ of $\mathbb{D}$ converging to t which maps to a null sequence $(K'_n)_{n=1}^\infty$ of crosscuts of U by ϕ^{-1} (see [15, Lemma 17.9]). We may assume that the sequence $(K'_n)_{n=1}^\infty$ converges to a point of ∂U by passing to a subsequence. Since $(A_n)_{n=1}^\infty$ accumulate on t , all but finitely many A_n intersect the crosscut neighborhood of A'_m corresponding to t . Also, since $(K_n)_{n=1}^\infty$ forms a null sequence in U , we see that all but finitely many K_n (thus A_n) lie entirely within the crosscut neighborhood of K'_m (thus A'_m) corresponding to t . However, the crosscut neighborhoods of A'_m form a null sequence, so $(A_n)_{n=1}^\infty$ form a null sequence. $\square$

We say that a pair of subsets E_1 and E_2 of $\partial\mathbb{D}$ are *unlinked* if there exist intervals $I_1 \supset E_1$ and $I_2 \supset E_2$ such that $|I_1 \cap I_2| \leq 2$.

Lemma 3.5. *Suppose $U \subset \mathbb{C}_\infty$ is a plane domain, and let $\phi : U \rightarrow \mathbb{D}$ be a conformal isomorphism. Let B_1 and B_2 be disjoint closed balls meeting ∂U in their interiors, and let $E_i \subset \partial\mathbb{D}$ denote the endpoints of all of the crosscuts of $\mathbb{D}$ which constitute $\phi((\partial B_i) \cap U)$. If E_1 and E_2 are unlinked, then there is a generalized crosscut K of U which separates $B_1 \cap U$ from $B_2 \cap U$ in U . Moreover, if ∂U is locally connected, K is a crosscut of U .*

Proof. Let I_1 and I_2 be minimal closed intervals of $\partial\mathbb{D}$ such that $I_i \supset E_i$ for $i \in \{1, 2\}$ and $|I_1 \cap I_2| \leq 2$. Note that $(\partial B_i) \cap U$ is the union of a null sequence of crosscuts of U , so $\phi((\partial B_i) \cap U)$ is the union of a null sequence of crosscuts of $\mathbb{D}$ by Lemma 3.4. I_1 and I_2 being unlinked implies that neither $\phi(B_1 \cap U)$ nor $\phi(B_2 \cap U)$ separates the other in $\mathbb{D}$. By connectedness of $\mathbb{D}$, there is then a unique component D of $\mathbb{D} \setminus \phi((B_1 \cup B_2) \cap U)$ which meets both $\phi(\partial B_1 \cap U)$ and $\phi(\partial B_2 \cap U)$. The crosscuts on ∂D form a null sequence, so it is not difficult to show that $\overline{D} \subset \mathbb{D}$ is locally connected.

The endpoints of I_1 and I_2 are each on ∂D . Let K be a crosscut joining an endpoint of I_1 to an endpoint of I_2 such that K separates the interiors of I_1 and

I_2 , in $\partial\mathbb{D}$, and thus $\phi(B_1)$ and $\phi(B_2)$ in $\mathbb{D}$. Then $\phi^{-1}(K)$ is a generalized crosscut of U which separates $B_1 \cap U$ from $B_2 \cap U$. Further, if ∂U is locally connected, ϕ^{-1} extends to a continuous function $\overline{\phi}^{-1} : \overline{\mathbb{D}} \rightarrow \overline{U}$ [15, Theorem 17.14]. In this case, $\phi^{-1}(K)$ is a true crosscut of U , as $\overline{\phi^{-1}(K)} = \overline{\phi}^{-1}(\overline{K})$ is an arc. $\square$

Now we have the tools to prove our Characterization Theorem 1.4.

Proof of Theorem 1.4. First, suppose that X is indecomposable. We show X satisfies the double-pass condition. By Theorem 2.10, there exists a sequence $(U_n)_{n=1}^\infty$ of complementary domains of X such that $\lim_{n \rightarrow \infty} \partial U_n = X$. Let $(K_n)_{n=1}^\infty$ be a sequence of generalized crosscuts, with K_n in U_n for each $n \in \mathbb{N}$. Let A_n and B_n be the shadows of K_n , with $H_d(A_n, X) \leq H_d(B_n, X)$, where H_d denotes the Hausdorff metric.

We claim that $\lim_{n \rightarrow \infty} A_n = X$. Since the hyperspace of subcontinua of X is a compact metric space, it is sufficient to show that every convergent subsequence of $(A_n)_{n=1}^\infty$ converges to X . Let $(n_i)_{i=1}^\infty$ be such that $(A_{n_i})_{i=1}^\infty$ converges to a continuum $A \subset X$. By passing to a subsequence, we may assume $(B_{n_i})_{i=1}^\infty$ also converges to a continuum $B \subset X$. Since $\lim_{i \rightarrow \infty} \partial U_{n_i} = X$ and $A_{n_i} \cup B_{n_i} = \partial U_{n_i}$, we have that $A \cup B = X$. Since X is indecomposable, A and B may not both be proper subcontinua of X , so $A = X$ or $B = X$. Since $H_d(A_{n_i}, X) \leq H_d(B_{n_i}, X)$ for all i , we have $A = X$. This concludes the proof of this implication.

Now we prove the converse. Let X be a continuum with a sequence $(U_n)_{n=1}^\infty$ of complementary domains satisfying the double-pass condition. Suppose, by way of contradiction, that $X = X_1 \cup X_2$, where X_1 and X_2 are proper subcontinua of X . We can then find open balls B_1 and B_2 such that

- (1) $\overline{B_1} \cap \overline{B_2} = \emptyset$,
- (2) $B_i \cap X_i \neq \emptyset$ for $i = 1, 2$, and
- (3) $\overline{B_i} \cap X_j = \emptyset$ for $i \neq j$.

Since $(U_n)_{n=1}^\infty$ satisfies the double-pass condition, there exists a particular $N \in \mathbb{N}$ such that, for any generalized crosscut K of U_N , one shadow of K intersects both B_1 and B_2 . We fix $U = U_N$, and let $\phi : U \rightarrow \mathbb{D}$ be a conformal isomorphism. Define E_1 and E_2 as in Lemma 3.5 to be the sets of endpoints of the crosscuts comprising $\phi((\partial B_1) \cap U)$ and $\phi((\partial B_2) \cap U)$, respectively. There are two cases: Either E_1 and E_2 are linked or they are not. The second case cannot occur, as Lemma 3.5 asserts the existence of a generalized crosscut K_0 of U separating $B_1 \cap U$ and $B_2 \cap U$, contrary to our assumption.

Thus, E_1 and E_2 are linked. Note that each of $\phi((\partial B_1) \cap U)$ and $\phi((\partial B_2) \cap U)$ consists of crosscuts of $\mathbb{D}$ with endpoints in E_1 and E_2 , respectively. There are two cases: (1) either one of $\phi((\partial B_1) \cap U)$ or $\phi((\partial B_2) \cap U)$ separates the other in $\mathbb{D}$ or (2) neither $\phi((\partial B_1) \cap U)$ nor $\phi((\partial B_2) \cap U)$ separates the other in $\mathbb{D}$. In each case, we construct a crosscut $A \subset \mathbb{D}$. This crosscut will have the property that $\phi^{-1}(A)$ is a crosscut of U , which we show leads to a separation of one of X_1 or X_2 , a contradiction.

In case (1), without loss of generality, $\phi((\partial B_1) \cap U)$ separates $\phi((\partial B_2) \cap U)$ in $\mathbb{D}$. Since $\mathbb{D}$ is unicoherent, a component of $\phi((\partial B_1) \cap U)$ also separates $\phi((\partial B_2) \cap U)$, so a crosscut in $\phi((\partial B_1) \cap U)$ does. Let A be this crosscut. Then $\phi^{-1}(A)$ is a crosscut of U separating $B_2 \cap U$ in U .

For case (2), we suppose that neither $\phi((\partial B_1) \cap U)$ nor $\phi((\partial B_2) \cap U)$ separates the other in $\mathbb{D}$. Since E_1 and E_2 are linked, let e_1 and e'_1 be points of E_1 separated

in $\partial\mathbb{D}$ by points of E_2 . Let K_1 and K'_1 be crosscuts of $\mathbb{D}$ in $\phi((\partial B_1) \cap U)$ that have e_1 and e'_1 , respectively, as endpoints. Since $\phi((\partial B_2) \cap U)$ does not separate K_1 from K'_1 in $\mathbb{D}$, there is an arc C from a point of K_1 to a point of K'_1 in $\mathbb{D} \setminus \phi(\overline{B_2} \cap U)$. Let $A \subset K_1 \cup K'_1 \cup C$ be a crosscut of $\mathbb{D}$ from e_1 to e'_1 which then separates $\phi(B_2 \cap U)$ in $\mathbb{D}$. Because $\phi^{-1}(K_1)$ and $\phi^{-1}(K'_1)$ are crosscuts of U , we see that $\phi^{-1}(A)$ is a crosscut of U . Moreover, $\phi^{-1}(A)$ separates $B_2 \cap U$ in U .

The proofs in cases (1) and (2) now proceed together. Let S_1 be an irreducible arc which joins points of $\phi((\partial B_2) \cap U)$ which are separated by A ; we may stipulate that S_1 intersects A exactly once, transversely. By applying ϕ^{-1} to both A and S_1 , we obtain a crosscut A' of U and a compact arc $S'_1 \subset U$ between points of ∂B_2 which intersects A' once transversely. Let S'_2 be a compact arc in $\overline{B_2}$ which joins the endpoints of S'_1 . Then $S'_1 \cup S'_2 = S$ is a simple closed curve. Observe that $S \cap X_1 = \emptyset$, since $S'_1 \subset U$ and $S'_2 \subset \overline{B_2}$. However, the compact arc $\overline{A'}$ joins points of X_1 and intersects S exactly once, transversely. Thus, some point of X_1 lies inside and another point lies outside of S , while $X_1 \cap S = \emptyset$, contradicting the connectedness of X_1 . $\square$

Examination of the proof of Theorem 1.4 gives a stronger theorem for continua whose complementary domains have locally connected boundaries.

Definition 3.6. A sequence of complementary domains $(U_n)_{n=1}^\infty$ satisfies the *crosscut condition* if, for every sequence of crosscuts $(A_n)_{n=1}^\infty$, $A_n \subset U_n$, there exists a choice of shadows S_n of A_n such that $\lim S_n = X$.

Example 3.3 showed that this is a strictly weaker condition than the double-pass condition. However, the following shows that for a certain class of continua, the two notions are equivalent.

Corollary 3.7. A planar continuum X whose complementary domains have locally connected boundaries is indecomposable if and only if it has a sequence $(U_n)_{n=1}^\infty$ of complementary domains which satisfies the crosscut condition.

This follows from the proof of Theorem 1.4, since the generalized crosscut of U constructed in the proof with Lemma 3.5 is a crosscut if ∂U is locally connected.

4. QUESTIONS AND FURTHER RESULTS

We close with a question about rational Julia sets for which our Characterization Theorem may prove useful and two theorems by the first author that will appear in a subsequent paper extending our results to surfaces.

Question 4.1. Let $J = J(R)$ be the Julia set of a rational function $R : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ and suppose that J has buried points. Can J be the union of two proper indecomposable subcontinua? In particular, can J contain a proper indecomposable subcontinuum with interior in J ?

Definition 4.2. A *surface* is a connected Hausdorff space with a countable basis, each point of which has a neighborhood homeomorphic to an open ball in the plane. Let X be a continuum in the surface S . As before, a component of $S \setminus X$ is called a *complementary domain*.

Definition 4.3. A connected topological space X is *multicoherent* of degree k if, for any pair of closed, connected sets A and B such that $A \cup B = X$, the intersection $A \cap B$ consists of at most k components.

A complementary domain in a surface, unlike in the planar case, need not be simply connected. Using the notion of multicoherence and its consequences (see [21, Theorem 1] for the relevant extension of the Phragmén-Brouwer theorem), we can prove the following theorem. We omit the proof, which is similar to the proof of Theorem 2.10.

Theorem 4.4. *Let S be a compact surface and X an indecomposable subcontinuum of S . Then there is a sequence $(U_n)_{n=1}^\infty$ of complementary domains of X such that $\lim \partial U_n = X$.*

We claim in Theorem 4.4 that having a sequence of complementary domains converging to X is a necessary condition for a continuum X contained in a surface S to be indecomposable. We saw in the plane a partial converse: given a sequence of distinct complementary domains $(U_n)_{n=1}^\infty$ such that $\lim \partial U_n = X$, it follows that X is either indecomposable or the union of two proper indecomposable subcontinua (Theorem 2.4). In this connection, we close with the following two theorems generalizing Burgess's Theorem 2.4 and our Characterization Theorem 1.4 to continua in surfaces, proofs of which will appear subsequently in a paper by the first-named author.

Theorem 4.5. *Let S be a compact surface. Suppose a continuum $X \subset S$ has a sequence $(U_n)_{n=1}^\infty$ of distinct complementary domains with $\lim \partial U_n = X$. Then either X is indecomposable or there is only one pair of indecomposable subcontinua whose union is X .*

Theorem 4.6. *Let S be a compact surface. Suppose the continuum $X \subset S$ has buried points. Then X is indecomposable iff X has a sequence $(U_n)_{n=1}^\infty$ of distinct complementary domains satisfying the double-pass condition: for any sequence $(A_n)_{n=1}^\infty$ of generalized crosscuts (suitably defined), with $A_n \subset U_n$, there is a sequence of shadows $(S_n)_{n=1}^\infty$, where each S_n is a shadow of A_n , such that $\lim S_n = X$.*

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA AT BIRMINGHAM, BIRMINGHAM,
ALABAMA 35294-1170
E-mail address: `clintonc@uab.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA AT BIRMINGHAM, BIRMINGHAM,
ALABAMA 35294-1170
E-mail address: `mayer@math.uab.edu`

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SASKATCHEWAN, SASKATOON,
SASKATCHEWAN, CANADA S7N 0W0
E-mail address: `tymchat@math.usask.ca`