CONVERGENT MARTINGALES OF OPERATORS AND THE RADON NIKODÝM PROPERTY IN BANACH SPACES

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ABSTRACT. We extend Troitsky's ideas on measure-free martingales on Banach lattices to martingales of operators acting between a Banach lattice and a Banach space. We prove that each norm bounded martingale of cone absolutely summing (c.a.s.) operators (also known as 1-concave operators), from a Banach lattice E to a Banach space Y , can be generated by a single c.a.s. operator. As a consequence, we obtain a characterization of Banach spaces with the Radon Nikodým property in terms of convergence of norm bounded martingales defined on the Chaney-Schaefer l -tensor product $E \widetilde{\otimes}_l Y$. This extends a classical martingale characterization of the Radon Nikodým property, formulated in the Lebesgue-Bochner spaces $L^p(\mu, Y)$ ($1 < p < \infty$).

1. INTRODUCTION

Let (Ω, Σ, μ) be a finite measure space and let Y be a Banach space. For $1 \leq p < \infty$, let $L^p(\mu, Y)$ denote the space of (classes of a.e. equal) Bochner p -integrable functions $f : \Omega \rightarrow Y$ and denote the Bochner norm on $L^p(\mu, Y)$ by Δ_p , i.e.

$$\Delta_p(f) = \left(\int_{\Omega} \|f\|_Y^p d\mu \right)^{1/p}.$$

In the Lebesgue-Bochner spaces, martingale theory provides an important link to the geometric properties of the Banach space Y . For example, Y has the Radon Nikodým property if and only if every martingale $(f_i) \subset L^p(\mu, Y)$, which is uniformly $\|\cdot\|_p$ -bounded, converges in the $\|\cdot\|_p$ -norm for all finite measure spaces (Ω, Σ, μ) and $1 < p < \infty$ (cf. [9]).

It is well known that $L^p(\mu, Y)$ is isometrically isomorphic to the norm completion $L^p(\mu) \widetilde{\otimes}_{\Delta_p} Y$ of $L^p(\mu) \otimes_{\Delta_p} Y$, where Δ_p denotes the induced Bochner norm (cf. [4, 5]). Chaney [1] characterized the Radon Nikodým property on Y in terms of equality of $L^p(\mu) \widetilde{\otimes}_{\Delta_p} Y$ and the space of Dinculeanu's operators from $L^q(\mu)$ to Y ($1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$). These are the operators $T : L^q(\mu) \rightarrow Y$ for which $\infty > \|T\|_q := \sup\{\sum_{i=1}^n \|\alpha_i T(\chi_{E_i})\|_Y\}$, where the supremum is taken over all

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simple functions $f = \sum_{i=1}^n \alpha_i \chi_{E_i}$ with $\chi_{E_1}, \dots, \chi_{E_n} \in L^q(\mu)$, $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ and $\|f\|_q \leq 1$ (cf. [6, 7, 8]).

The aim of this paper is to extend Chaney's characterization by using Troitsky's [18] generalized notion of a martingale in a Banach lattice.

Chaney and Schaefer extended the Bochner norm to the tensor product of a Banach lattice E and a Banach space Y ; the norm $\|\cdot\|_l$, given by

$$\|u\|_l = \inf \left\{ \left\| \sum_{i=1}^n \|y_i\| |x_i| \right\| : u = \sum_{i=1}^n x_i \otimes y_i \right\}$$

for all $u = \sum_{i=1}^n x_i \otimes y_i \in E \otimes Y$, coincides with the Bochner norm on $L^p(\mu) \otimes Y$ for all σ -finite measure spaces (Ω, Σ, μ) and $1 \leq p < \infty$ (see [1], [17] and [11]).

Let E be a Banach lattice and let Y be a Banach space. Recall from [17, Chapter IV, §3] that a linear map $T : E \rightarrow Y$ is called *cone absolutely summing* if for every positive summable sequence (x_n) in E , the sequence (Tx_n) is absolutely summable in Y . Such operators are known, in the terminology of Krivine, as 1-*concave* operators (cf. [15, p. 45] and [13]). The space

$$\mathcal{L}^{\text{cas}}(E, Y) = \{T : E \rightarrow Y : T \text{ is cone absolutely summing}\}$$

is a Banach space with respect to the norm defined by

$$\|T\|_{\text{cas}} = \sup \left\{ \sum_{i=1}^n \|Tx_i\| : x_1, \dots, x_n \in E_+, \left\| \sum_{i=1}^n x_i \right\| = 1, n \in \mathbb{N} \right\}$$

for all $T \in \mathcal{L}^{\text{cas}}(E, Y)$. Cone absolutely summing operators generalize Dinculeanu's operators, mentioned above.

Cone absolutely summing operators extend the Chaney-Schaefer l -tensor product in the following sense: The canonical map $E^* \otimes_l Y \rightarrow \mathcal{L}^{\text{cas}}(E, Y)$ given by $\sum_{i=1}^n x_i^* \otimes y_i =: u \mapsto L_u$, where $L_u x = \sum_{i=1}^n \langle x, x_i^* \rangle y_i$ for all $x \in E$, is an isometry (cf. [17, Chapter IV, §7] and [1, 11, 14]). Here, E^* denotes the continuous dual of E .

2. PRELIMINARIES

We recall the abstract notions in [18] for a filtration and a martingale in a Banach space.

Definition 2.1. Let E be a Banach lattice and Y a Banach space.

- (a) Let (T_i) be a sequence of contractive projections on Y . If $T_{i \wedge j} = T_i T_j$ for each $i, j \in \mathbb{N}$, then (T_i) is called a *filtration* on Y . In the case where (T_i) is a filtration on a Banach lattice, we will always assume each T_i to be positive.
- (b) If (T_i) is a filtration on E with the property that each $\mathcal{R}(T_i)$ is a (closed) Riesz subspace of E , then (T_i) will be called a *BL-filtration* on E (this terminology is also used in [2]).
- (c) If (T_i) is a filtration on E with each T_i strictly positive (i.e., T_i is positive and $\{f \in E : T_i(|f|) = 0\} = \{0\}\}$), then (T_i) is called a *strictly positive filtration*.
- (d) If (T_i) is a filtration on Y and $(f_i) \subset Y$, then the pair (f_i, T_i) is called a *martingale* in Y if $T_i f_j = f_i$ for all $i \leq j$.

- (e) If (f_i, T_i) is a martingale in Y , then (f_i, T_i) is called *fixed* if there exists $f \in Y$ such that $f_i = T_i f$ for all $i \in \mathbb{N}$. In this case, (f_i, T_i) is said to be *fixed on f* .
- (f) If (T_i) is a filtration on E , then (f_i, T_i) is called a *submartingale* in E if $T_i f_j \geq f_i$ for all $i \leq j$.
- (g) Let (T_i) be a filtration on Y . We say that (T_i) is *complemented* in Y if there exists a contractive projection $T_\infty : Y \rightarrow Y$ with $\mathcal{R}(T_\infty) = \overline{\bigcup_{i=1}^\infty \mathcal{R}(T_i)}$ and $T_i T_\infty = T_\infty T_i = T_i$ for all $i \in \mathbb{N}$.

If (Ω, Σ, μ) is a finite measure space, $1 \leq p < \infty$ and (Σ_i) an increasing sequence of σ -subalgebras of Σ , it follows that the sequence of conditional expectations $(\mathbb{E}(\cdot | \Sigma_i))$ on $L^p(\mu)$ satisfies the above definition of a BL-filtration. Moreover, $(\mathbb{E}(\cdot | \Sigma_i))$ is complemented by $\mathbb{E}(\cdot | \bigvee_{i=1}^\infty \Sigma_i) : L^p(\mu) \rightarrow L^p(\mu)$, where $\bigvee_{i=1}^\infty \Sigma_i$ denotes the σ -algebra generated by $\bigcup_{i=1}^\infty \Sigma_i$.

In [17, Chapter III, §11, Proposition 11.5] it is shown that if $T : E \rightarrow E$ is a strictly positive projection on a Banach lattice E , then $\mathcal{R}(T)$ is a Banach sublattice of E . Consequently, every strictly positive filtration on E is also a BL-filtration.

It is easily verified that if (T_i) is a filtration on E , then the sequence of adjoint operators (T_i^*) is a filtration on E^* . However, it is not clear that (T_i^*) is a BL-filtration whenever (T_i) is a BL-filtration. We need an additional definition.

Definition 2.2. Let E be a Banach lattice with a non-empty quasi-interior Q_+ . A filtration (T_i) on E is said to be *quasi-interior preserving* if $T_i Q_+ \subset Q_+$ for each $i \in \mathbb{N}$.

For background reading on quasi-interior points in Banach lattices, we refer the reader to [17, Chapter II, §6].

Lemma 2.3. Let E be a Banach lattice with a non-empty quasi-interior Q_+ . If $T : E \rightarrow E$ is a positive projection, then $TQ_+ \subset Q_+$ if and only if there exists a quasi-interior point $0 < e \in E_+$ such that $Te = e$.

Proof. Note that $T \geq 0$ implies that T is bounded. Suppose there exists $e \in Q_+$ such that $Te = e$ and let $q \in Q_+$. By [17, Chapter II, §6, Theorem 6.3], we have $\lim_{n \rightarrow \infty} T(nq \wedge e) = T(e) = e$. Also, $0 \leq T(nq \wedge e) \leq (nTq) \wedge e \leq e$ with $T(nq \wedge e) \uparrow$. Hence, $\lim_{n \rightarrow \infty} (nTq) \wedge e = e$. Now let $p \in E_+$. Using [17, Chapter II, §6, Theorem 6.3] again, we obtain

$$p = \lim_{m \rightarrow \infty} (me) \wedge p = \lim_{m \rightarrow \infty} (m \lim_{n \rightarrow \infty} (nTq) \wedge e) \wedge p = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} (m(nTq) \wedge e) \wedge p.$$

As $(m(nTq) \wedge e) \wedge p \leq (mnTq) \wedge p \leq p$ for each $n, m \in \mathbb{N}$, it follows that

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} (mnTq) \wedge p = p.$$

Thus, $Tq \in Q_+$ by [17, Chapter II, §6, Theorem 6.3], i.e. $TQ_+ \subset Q_+$. The converse is trivial. $\square$

The spaces $L^p(\mu)$, $1 \leq p < \infty$, have non-empty quasi-interior (e.g., $\mathbf{1}$ is a quasi-interior point of $L^p(\mu)$) and if (Σ_i) is an increasing sequence of σ -subalgebras of Σ , then the sequence of conditional expectations $(\mathbb{E}(\cdot | \Sigma_i))$ is a filtration with $\mathbb{E}(\mathbf{1} | \Sigma_i) = \mathbf{1}$ for each $i \in \mathbb{N}$. By the above lemma, $(\mathbb{E}(\cdot | \Sigma_i))$ is a filtration that is quasi-interior preserving.

Proposition 2.4. *Suppose that E is a Banach lattice possessing a non-empty quasi-interior Q_+ and $T : E \rightarrow E$ is a bounded linear operator. Then $TQ_+ \subset Q_+$ if and only if $T^* : E^* \rightarrow E^*$ is strictly positive.*

Proof. By [17, Chapter II, §6, Theorem 6.3], $q \in Q_+$ if and only if $\langle q, f^* \rangle > 0$ for all $0 < f^* \in E^*$. With this in mind, assume that $TQ_+ \subset Q_+$. Since Q_+ is dense in E_+ , it follows that $T \geq 0$, which implies $T^* \geq 0$. To show strict positivity, suppose that $T^*f^* = 0$ for some $f^* \in E^*$. Then $0 = \langle q, T^*f^* \rangle = \langle Tq, f^* \rangle$ for $q \in Q_+$. Since $Tq \in Q_+$, it follows that $f^* = 0$. Conversely, if T^* is strictly positive, then for all $0 < f^* \in E^*$ we have $T^*f^* > 0$. Thus, for $q \in Q_+$, it follows that $\langle Tq, f^* \rangle = \langle q, T^*f^* \rangle > 0$ for all $f^* \in E^*$. Consequently, $Tq \in Q_+$. $\square$

Corollary 2.5. *Suppose that E is a Banach lattice possessing non-empty quasi-interior. Then for any quasi-interior preserving filtration (T_i) on E , we have that (T_i^*) is a BL-filtration on E^* .*

The following convergence result was shown in [2].

Proposition 2.6. *Let Y be a Banach space and (T_i) a filtration on Y . Then the following statements hold:*

- (a) $f \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ if and only if $\|T_i f - f\| \rightarrow 0$ as $i \rightarrow \infty$.
- (b) If (f_i) is a martingale relative to (T_i) , then (f_i) converges to f if and only if $f \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ and $f_i = T_i f$ for all $i \in \mathbb{N}$.

Consider the case where a filtration (T_i) on a Banach lattice E is complemented in E by a contractive projection $T_\infty : E \rightarrow E$. Since each T_i is assumed to be positive, it follows that T_∞ is also positive. Indeed, if $f \in E_+$, then $T_i f \in E_+$ for each $i \in \mathbb{N}$. Thus, Proposition 2.6 implies $\lim_{i \rightarrow \infty} T_i f = \lim_{i \rightarrow \infty} T_i T_\infty f = T_\infty f \geq 0$.

Definition 2.7. Let Y be a Banach space and (T_i) a filtration on Y .

- (a) Define the *space of norm bounded martingales* as

$$\mathcal{M}(Y, T_i) = \{(f_i, T_i) \text{ a martingale in } Y : \sup_i \|f_i\| < \infty\},$$

together with the norm defined by $\|(f_i, T_i)\| = \sup_i \|f_i\|$ for all $(f_i, T_i) \in \mathcal{M}(Y, T_i)$.

- (b) Define the *space of norm convergent martingales* by

$$\mathcal{M}_{\text{nc}}(Y, T_i) = \{(f_i, T_i) \in \mathcal{M}(Y, T_i) : (f_i) \text{ is norm convergent in } Y\}.$$

- (c) Define the *space of fixed martingales* by

$$\mathcal{M}_{\text{f}}(Y, T_i) = \{(f_i, T_i) \in \mathcal{M}(Y, T_i) : \exists f \in Y \text{ so that } T_i f = f_i \forall i \in \mathbb{N}\}.$$

It is easily shown that $\mathcal{M}(Y, T_i)$ and $\mathcal{M}_{\text{nc}}(Y, T_i)$ are Banach spaces. By Proposition 2.6(b), we have the inclusions $\mathcal{M}_{\text{nc}}(Y, T_i) \subset \mathcal{M}_{\text{f}}(Y, T_i) \subset \mathcal{M}(Y, T_i)$. Note that if (T_i) is a complemented filtration on Y , then $\mathcal{M}_{\text{nc}}(Y, T_i) = \mathcal{M}_{\text{f}}(Y, T_i)$. If (f_i, T_i) is fixed, then it is also fixed on an element in $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ and thus convergent by Proposition 2.6.

In the case where Y is reflexive, Troitsky showed in [18, Corollary 18] that $\mathcal{M}_{\text{f}}(Y, T_i) = \mathcal{M}(Y, T_i)$. If (T_i) is also complemented in Y , then $\mathcal{M}_{\text{nc}}(Y, T_i) = \mathcal{M}_{\text{f}}(Y, T_i) = \mathcal{M}(Y, T_i)$.

For a BL-filtration (T_i) on a Banach lattice E , [2, Proposition 3.7] implies that $\mathcal{M}_{\text{nc}}(E, T_i)$ is a Banach lattice with respect to the ordering defined by $(f_i, T_i) \geq$

$0 \Leftrightarrow f_i \geq 0$ for all $i \in \mathbb{N}$. Moreover, it is shown that $\mathcal{M}_{\text{nc}}(E, T_i)$ is Riesz isometric to $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$.

In the case where E is a KB-space and (T_i) is a filtration on E , Troitsky showed in [18, Theorem 7] that $\mathcal{M}(E, T_i)$ is also a Banach lattice under this ordering. Moreover, Troitsky showed the following, which is key to our main result (cf. [18, p. 446]).

Proposition 2.8. *Let E be a KB-space and (T_i) a filtration on E . If (s_i, T_i) is a norm bounded submartingale in E , then there exists a unique least martingale $(f_i, T_i) \in \mathcal{M}(E, T_i)$ such that $s_i \leq f_i$ for each $i \in \mathbb{N}$. Moreover, $\|(f_i, T_i)\| \leq \|(s_i, T_i)\|$.*

The notion of a filtration (T_i) on a Banach lattice E can be extended to the l -tensor product $E \widetilde{\otimes}_l Y$ for any Banach space Y . Indeed, it follows from the proof of [2, Lemma 4.2] that $(T_i \otimes_l \text{id}_Y)$ is a sequence of commuting contractive projections on $E \widetilde{\otimes}_l Y$ with increasing range. This extension is consistent with a classical filtration on the Lebesgue-Bochner spaces.

3. CONE ABSOLUTELY SUMMING MARTINGALES

To characterize the Radon Nikodým property, we require a fair amount of preparation. We first consider a canonical filtration on the space of cone absolutely summing operators from a Banach lattice E to a Banach space Y .

Proposition 3.1. *Let E be a Banach lattice and Y a Banach space. Suppose that (T_i) is a BL-filtration on E . Then the sequence $(\widehat{T}_i)$ of maps $\widehat{T}_i : \mathcal{L}^{\text{cas}}(E, Y) \rightarrow \mathcal{L}^{\text{cas}}(E, Y)$, defined by $\widehat{T}_i F = F \circ T_i$ for each $F \in \mathcal{L}^{\text{cas}}(E, Y)$ and $i \in \mathbb{N}$, is a sequence of contractive projections on $\mathcal{L}^{\text{cas}}(E, Y)$ with $\widehat{T}_{i \wedge j} = \widehat{T}_i \widehat{T}_j$.*

Proof. Since (T_i) is a filtration, $F \circ T_i \in \mathcal{L}^{\text{cas}}(E, Y)$ and $\widehat{T}_i$ is a well-defined, linear projection for each $i \in \mathbb{N}$. It also follows from

$$\begin{aligned} \|\widehat{T}_i F\|_{\text{cas}} &= \sup \left\{ \sum_{j=1}^n \|F T_i x_j\| : (x_j)_{j=1}^n \subset E_+, \left\| \sum_{j=1}^n x_j \right\| \leq 1 \right\} \\ &= \sup \left\{ \sum_{j=1}^n \|F x_j\| : (x_j)_{j=1}^n \subset \mathcal{R}(T_i)_+, \left\| \sum_{j=1}^n x_j \right\| \leq 1 \right\} \\ &\leq \sup \left\{ \sum_{j=1}^n \|F x_j\| : (x_j)_{j=1}^n \subset E_+, \left\| \sum_{j=1}^n x_j \right\| \leq 1 \right\} \\ &= \|F\|_{\text{cas}} \end{aligned}$$

that each $\widehat{T}_i$ is bounded and $\sup_{i \in \mathbb{N}} \|\widehat{T}_i\| = 1$. Moreover,

$$\widehat{T}_i \widehat{T}_j F = F \circ T_j \circ T_i = F \circ T_{i \wedge j} = \widehat{T}_{i \wedge j} F$$

for each $F \in \mathcal{L}^{\text{cas}}(E, Y)$ and $i, j \in \mathbb{N}$. This completes the proof. $\square$

In view of the above proposition, we are justified in making the following definition.

Definition 3.2. Let E be a Banach lattice and Y a Banach space. Suppose that (T_i) is a BL-filtration on E . Then $(\widehat{T}_i)$, as defined in Proposition 3.1, is called the *filtration on $\mathcal{L}^{\text{cas}}(E, Y)$ induced by (T_i)* .

We exhibit a known characterization of cone absolutely summing operators.

Lemma 3.3. *Let E be a Banach lattice, Y a Banach space and $l > 0$. For any bounded operator $T : E \rightarrow Y$ the following statements are equivalent:*

- (a) *T is cone absolutely summing with $\|T\|_{\text{cas}} \leq l$.*
- (b) *There exists $x_T^* \in E_+^*$ so that $\|x_T^*\| \leq l$ and $\|Tx\| \leq \langle |x|, x_T^* \rangle$ for all $x \in E$.*
- (c) *There exist an AL-space L , $0 \leq T_1 \in \mathcal{L}(E, L)$ and $T_2 \in \mathcal{L}(L, Y)$ such that $T = T_2 \circ T_1$, where $\|T_1\| \leq l$ and $\|T_2\| \leq 1$.*

In the case where E is separable, we may take $L = L^1(\mu)$ in (c), where (Ω, Σ, μ) is a finite measure space.

The proof of the equivalence of (a), (b) and (c) in the above lemma may be found in [17, Chapter IV, §3, Proposition 3.3]. However, the last part of the lemma requires a proof: Assume E is separable. Then, by [17, Chapter II, §6, Proposition 6.2], there exists a quasi-interior point $0 \leq e \in E$. By construction, the map $T_1 : E \rightarrow L$ is a Riesz homomorphism with dense range (see [17, p. 243, §3]). Thus, it follows by [17, Chapter II, §6, Proposition 6.4] that $T_1 e$ is a quasi-interior point of L . Hence, $T_1 e$ is also a weak order unit of L . It follows by Kakutani's representation theorem for AL-spaces (cf. [12] or [15, Theorem 1.b.2]) that L is Riesz and isometrically isomorphic to $L^1(\mu)$, where (Ω, Σ, μ) may be chosen to be finite.

For our next result, we recall that the functional $x_T^* \in E_+^*$ in Lemma 3.3(b) is the extension of the additive map $\rho_T : E_+ \rightarrow E_+$, defined by

$$\rho_T(x) = \sup \left\{ \sum_{i=1}^{\infty} \|Tx_i\| : (x_i) \in (\ell^1 \widetilde{\otimes}_{\varepsilon} E)_+, \sum_{i=1}^{\infty} x_i = x \right\}$$

for each $x \in E_+$ (cf. [17, Chapter IV, §2, Theorem 2.7]). Here, $\ell^1 \widetilde{\otimes}_{\varepsilon} E$ denotes the space of unconditionally summable sequences in E .

Proposition 3.4. *Let E be a Banach lattice with order continuous dual and Y a Banach space. Suppose that (T_i) is a BL-filtration on E and $(\widehat{T}_i)$ is the filtration on $\mathcal{L}^{\text{cas}}(E, Y)$ induced by (T_i) . If $(F_i, \widehat{T}_i) \in \mathcal{M}(\mathcal{L}^{\text{cas}}(E, Y), \widehat{T}_i)$, then there exists $0 \leq (f_i^*, T_i^*) \in \mathcal{M}(E^*, T_i^*)$ such that $\|F_i x\| \leq \langle |x|, f_i^* \rangle$ for each $x \in E$ and $i \in \mathbb{N}$. Moreover, $\sup_{i \in \mathbb{N}} \|f_i^*\| \leq \sup_{i \in \mathbb{N}} \|F_i\|_{\text{cas}}$.*

Proof. By Lemma 3.3 there exists, for each F_i , a positive functional $x_{F_i}^* \in E^*$ with $\|x_{F_i}^*\| \leq \sup_{i \in \mathbb{N}} \|F_i\|_{\text{cas}} := l$ and $\|F_i x\| \leq \langle |x|, x_{F_i}^* \rangle$ for each $x \in E$ and $i \in \mathbb{N}$. Define $s_i^* \in E_+^*$ by $\langle x, s_i^* \rangle = \langle T_i x, x_{F_i}^* \rangle$ for each $x \in E$ and $i \in \mathbb{N}$. Then, $\sup_{i \in \mathbb{N}} \|s_i^*\| \leq l$ and, since $x_{F_i}^* \geq 0$, we get

$$\|F_i x\| = \|\widehat{T}_i F_i x\| = \|F_i T_i x\| \leq \langle |T_i x|, x_{F_i}^* \rangle \leq \langle T_i |x|, x_{F_i}^* \rangle = \langle |x|, s_i^* \rangle$$

for all $x \in E$ and $i \in \mathbb{N}$. We now show that (s_i^*, T_i^*) is a submartingale. Let $i \leq j$ and $x \in E_+$. Then,

$$\begin{aligned} \langle x, T_i^* s_j^* \rangle &= \langle T_i x, s_j^* \rangle = \langle T_j T_i x, x_{F_j}^* \rangle = \langle T_i x, x_{F_j}^* \rangle \\ &= \sup \left\{ \sum_{n=1}^{\infty} \|F_j x_n\| : (x_n) \in (\ell^1 \tilde{\otimes}_{\varepsilon} E)_+, \sum_{n=1}^{\infty} x_n = T_i x \right\} \\ &\geq \sup \left\{ \sum_{n=1}^{\infty} \|F_j x_n\| : (x_n) \in (\ell^1 \tilde{\otimes}_{\varepsilon} \mathcal{R}(T_i))_+, \sum_{n=1}^{\infty} x_n = T_i x \right\} \end{aligned}$$

as $(\ell^1 \tilde{\otimes}_{\varepsilon} \mathcal{R}(T_i))_+ \subset (\ell^1 \tilde{\otimes}_{\varepsilon} E)_+$. Since T_i is a projection, it follows for $x \in E$ that $x \in \mathcal{R}(T_i)$ if and only if $x = T_i x$. In addition, the positivity of T_i implies $(T_i x_n) \in (\ell^1 \tilde{\otimes}_{\varepsilon} \mathcal{R}(T_i))_+$ for all $(x_i) \in (\ell^1 \tilde{\otimes}_{\varepsilon} E)_+$. Consequently, for $(x_n) \in (\ell^1 \tilde{\otimes}_{\varepsilon} E)_+$, we have $(x_n) \in (\ell^1 \tilde{\otimes}_{\varepsilon} \mathcal{R}(T_i))_+$ if and only if $(x_n) = (T_i x_n)$. Thus,

$$\begin{aligned} &\sup \left\{ \sum_{n=1}^{\infty} \|F_j x_n\| : (x_n) \in (\ell^1 \tilde{\otimes}_{\varepsilon} \mathcal{R}(T_i))_+, \sum_{n=1}^{\infty} x_n = T_i x \right\} \\ &= \sup \left\{ \sum_{n=1}^{\infty} \|F_j T_i x_n\| : (x_n) \in (\ell^1 \tilde{\otimes}_{\varepsilon} E)_+, \sum_{n=1}^{\infty} x_n = T_i x \right\} \\ &= \sup \left\{ \sum_{n=1}^{\infty} \|\hat{T}_i F_j x_n\| : (x_n) \in (\ell^1 \tilde{\otimes}_{\varepsilon} E)_+, \sum_{n=1}^{\infty} x_n = T_i x \right\} \\ &= \sup \left\{ \sum_{n=1}^{\infty} \|F_i x_n\| : (x_n) \in (\ell^1 \tilde{\otimes}_{\varepsilon} E)_+, \sum_{n=1}^{\infty} x_n = T_i x \right\} \\ &= \langle T_i x, x_{F_i}^* \rangle = \langle x, s_i^* \rangle. \end{aligned}$$

Since $s_i^*(x) \leq T_i^* s_j^*(x)$ for all $x \in E_+$, it follows that $s_i^* \leq T_i^* s_j^*$. Consequently, (s_i^*, T_i^*) is a submartingale. Since E^* is order continuous, it follows that E^* is a KB-space (cf. [16, Theorem 2.4.14]). Thus, by Proposition 2.8, there exists a unique least martingale $0 \leq (f_i^*, T_i^*) \in \mathcal{M}(E^*, T_i^*)$ that dominates the submartingale (s_i^*, T_i^*) , with $\sup_{i \in \mathbb{N}} \|f_i^*\| \leq \sup_{i \in \mathbb{N}} \|s_i^*\| \leq l = \sup_{i \in \mathbb{N}} \|F_i\|_{\text{cas}}$. Hence,

$$\|F_i x\| \leq \langle |x|, s_i^* \rangle \leq \langle |x|, f_i^* \rangle$$

for all $x \in E$, and the proof is complete. $\square$

Theorem 3.5. *Let E be a Banach lattice with order continuous dual and Y a Banach space. Suppose that (T_i) is a BL-filtration on E and $(\hat{T}_i)$ is the filtration on $\mathcal{L}^{\text{cas}}(E, Y)$ induced by (T_i) . Then $\mathcal{M}_f(\mathcal{L}^{\text{cas}}(E, Y), \hat{T}_i) = \mathcal{M}(\mathcal{L}^{\text{cas}}(E, Y), \hat{T}_i)$.*

Proof. The inclusion $\mathcal{M}_f(\mathcal{L}^{\text{cas}}(E, Y), \hat{T}_i) \subset \mathcal{M}(\mathcal{L}^{\text{cas}}(E, Y), \hat{T}_i)$ is obvious. For the reverse inclusion, let $(F_i, \hat{T}_i) \in \mathcal{M}(\mathcal{L}^{\text{cas}}(E, Y), \hat{T}_i)$. By Proposition 3.4, there exists $0 \leq (f_i^*, T_i^*) \in \mathcal{M}(E^*, T_i^*)$ such that $\sup_{i \in \mathbb{N}} \|f_i^*\| \leq \sup_{i \in \mathbb{N}} \|F_i\|_{\text{cas}}$ and $\|F_i x\| \leq \langle |x|, f_i^* \rangle$ for each $x \in E$ and $i \in \mathbb{N}$. Let $\sup_{i \in \mathbb{N}} \|f_i^*\| := K$ and define $f^* : \bigcup_{i=1}^{\infty} \mathcal{R}(T_i) \rightarrow \mathbb{R}$ by $\langle x, f^* \rangle = \lim_{i \rightarrow \infty} \langle x, f_i^* \rangle$ for each $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$. Observe that f^* is well defined, as for $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$ there exists $i \in \mathbb{N}$ such that $x \in \mathcal{R}(T_i)$. Consequently, $i \leq j$ implies $\langle x, f_i^* \rangle = \langle x, T_i^* f_j^* \rangle = \langle T_i x, f_j^* \rangle = \langle x, f_j^* \rangle$. Thus, $\langle x, f^* \rangle = \lim_{i \rightarrow \infty} \langle x, f_i^* \rangle$ exists for each $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$. Evidently, f^* is positive, linear and the inequality $|\langle x, f^* \rangle| = \lim_{i \rightarrow \infty} |\langle x, f_i^* \rangle| \leq \lim_{i \rightarrow \infty} \|f_i^*\| \|x\| = K \|x\|$ shows that f^* is also bounded with norm $\|f^*\| \leq K$.

Now define a map $F : \bigcup_{i=1}^{\infty} \mathcal{R}(T_i) \rightarrow Y$ by $Fx = \lim_{i \rightarrow \infty} F_i x$ for each $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$. The map F is well defined because, for each $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$, there is some $i \in \mathbb{N}$ for which $x \in \mathcal{R}(T_i)$. Thus, $i \leq j$ implies $F_i x = \widehat{T}_i F_j x = F_j T_i x = F_j x$ so that $Fx = \lim_{i \rightarrow \infty} F_i x$ exists for each $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$. It is now evident that F is linear. Moreover, since $\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$ is a Riesz subspace of E , we have

$$\|Fx\| = \lim_{i \rightarrow \infty} \|F_i x\| \leq \lim_{i \rightarrow \infty} \langle |x|, f_i^* \rangle = \langle |x|, f^* \rangle \leq K\|x\|$$

for all $x \in \bigcup_{i=1}^{\infty} \mathcal{R}(T_i)$. Thus, F is bounded. Let $\overline{f}^*$ and $\overline{F}$ denote the unique continuous extensions of f^* and F respectively to the Banach sublattice $\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$ of E . Then we have $\|\overline{F}x\| \leq \langle |x|, \overline{f}^* \rangle$ for all $x \in \overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}$. Consequently, Lemma 3.3 implies $\overline{F} \in \mathcal{L}^{\text{cas}}(\overline{\bigcup_{i=1}^{\infty} \mathcal{R}(T_i)}, Y)$. By [17, Chapter IV, §3, Proposition 3.9], $\overline{F}$ possesses an extension $\overline{F}_{\infty} \in \mathcal{L}^{\text{cas}}(E, Y)$ with $\|\overline{F}\|_{\text{cas}} = \|\overline{F}_{\infty}\|_{\text{cas}}$. Finally, $\widehat{T}_i \overline{F}_{\infty} x = F T_i x = \lim_{j \rightarrow \infty} F_j T_i x = \lim_{j \rightarrow \infty} \widehat{T}_i F_j x = F_i x$ for all $x \in E$ and $i \in \mathbb{N}$. Thus, $(F_i, \widehat{T}_i) \in \mathcal{M}_f(\mathcal{L}^{\text{cas}}(E, Y), \widehat{T}_i)$. $\square$

We continue our preparations with the next lemma, which is a simple restatement of well-known facts about order continuity of the norm in dual Banach lattices.

Lemma 3.6. *Let E be a Banach lattice such that E^* has order continuous norm. If $T : E \rightarrow \ell^1$ is a positive linear operator, then T is compact.*

Proof. Let $T : E \rightarrow \ell^1$ be a positive operator. Denote the restriction of T^* to c_0 by $T^*|_{c_0}$. Then $T^*|_{c_0} : c_0 \rightarrow E^*$ is positive. But E^* is a KB-space by [16, Theorem 2.4.14]; thus, $T^*|_{c_0}$ is weakly compact (cf. [17, Chapter II, §5, Proposition 5.15]). Consequently, $(T^*|_{c_0})^* : E^{**} \rightarrow \ell^1$ is compact because ℓ^1 has the Schur property. Hence, $T = (T^*|_{c_0})^*|_E$ is compact. $\square$

Lastly, we need the following characterization of the l -tensor product, which is shown in [14, Theorem 5.2].

Theorem 3.7. *Let E be a Banach lattice, Y a Banach space and $T \in \mathcal{L}(E, Y)$. Then $T \in E^* \widetilde{\otimes}_l Y$ if and only if there exist $0 \leq S \in \mathcal{L}(E, \ell^1)$ and $R \in \mathcal{L}(\ell^1, Y)$ such that S is compact and $T = R \circ S$. Further, $\|T\|_{\text{cas}} = \inf \|R\| \|S\|_r$, where the infimum is taken over all such factorizations of T .*

We are now prepared to characterize the Radon Nikodým property:

Theorem 3.8. *Let Y be a Banach space. Then the following statements are equivalent:*

- (a) Y has the Radon Nikodým property.
- (b) $E^* \widetilde{\otimes}_l Y = \mathcal{L}^{\text{cas}}(E, Y)$ for all separable Banach lattices E with order continuous dual.
- (c) $\mathcal{M}(E^* \widetilde{\otimes}_l Y, T_i^* \otimes_l \text{id}_Y) = \mathcal{M}_f(E^* \widetilde{\otimes}_l Y, T_i^* \otimes_l \text{id}_Y)$ for all separable Banach lattices E with order continuous dual and all BL-filtrations (T_i) on E .
- (d) $\mathcal{M}(E \widetilde{\otimes}_l Y, T_i \otimes_l \text{id}_Y) = \mathcal{M}_{\text{nc}}(E \widetilde{\otimes}_l Y, T_i \otimes_l \text{id}_Y)$ for all separable reflexive Banach lattices E and all complemented, quasi-interior preserving BL-filtrations (T_i) on E .
- (e) $\mathcal{M}(E \widetilde{\otimes}_l Y, T_i \otimes_l \text{id}_Y) = \mathcal{M}(E, T_i) \widetilde{\otimes}_l Y$ for all separable reflexive Banach lattices E and all complemented, quasi-interior preserving BL-filtrations (T_i) on E .

Proof. (a) $\Rightarrow$ (b) Let E be a separable Banach lattice with order continuous dual. Let $T \in \mathcal{L}^{\text{cas}}(E, Y)$. By Lemma 3.3, there exist a finite measure space (Ω, Σ, μ) and operators $0 \leq T_1 \in \mathcal{L}(E, L^1(\mu))$ and $T_2 \in \mathcal{L}(L^1(\mu), Y)$ such that $T = T_2 \circ T_1$ where $\|T_1\| \leq \|T\|_{\text{cas}}$ and $\|T_2\| \leq 1$. Since Y has the Radon Nikodým property, the Lewis-Stegall Theorem (cf. [5, Chapter III, §1, Theorem 8]) guarantees the existence of operators $0 \leq S_1 \in \mathcal{L}(L^1(\mu), \ell^1)$ and $S_2 \in \mathcal{L}(\ell^1, Y)$ such that $T_2 = S_2 \circ S_1$. Since E^* is order continuous, the positive operator $S_1 \circ T_1 : E \rightarrow \ell^1$ is compact by Lemma 3.6. Hence, $T \in E^* \widetilde{\otimes}_l Y$ by Theorem 3.7.

(b) $\Rightarrow$ (c) Suppose E is a separable Banach lattice with order continuous dual and (T_i) is a filtration on E . Let $(f_i, T_i^* \otimes_l \text{id}_Y) \in \mathcal{M}(E^* \widetilde{\otimes}_l Y, T_i^* \otimes_l \text{id}_Y)$. By (b), $E^* \widetilde{\otimes}_l Y$ is isometric to $\mathcal{L}^{\text{cas}}(E, Y)$ under the continuous extension of the canonical isometry $E^* \otimes_l Y \rightarrow \mathcal{L}^{\text{cas}}(E, Y)$, given by $u \mapsto L_u$, where $L_u = \sum_{i=1}^n \langle \cdot, x_i^* \rangle y_i$ for $u = \sum_{i=1}^n x_i^* \otimes y_i$. Let $(F_i) \subset \mathcal{L}^{\text{cas}}(E, Y)$ be the sequence corresponding to the martingale $(f_i, T_i^* \otimes_l \text{id}_Y)$. Observe that, for $u = \sum_{k=1}^n x_k^* \otimes y_k \in E^* \otimes_l Y$, we have

$$(T_i^* \otimes_l \text{id}_Y)u \mapsto \sum_{k=1}^n \langle \cdot, T_i^* x_k^* \rangle y_k = L_u \circ T_i = \widehat{T}_i L_u$$

for all $i \in \mathbb{N}$. It follows that $(F_i, \widehat{T}_i) \in \mathcal{M}(\mathcal{L}^{\text{cas}}(E, Y), \widehat{T}_i)$. By Theorem 3.5, there exists $F_\infty \in \mathcal{L}^{\text{cas}}(E, Y)$ such that $\widehat{T}_i F_\infty = F_i$ for each $i \in \mathbb{N}$. Consequently, $(f_i, T_i^* \otimes_l \text{id}_Y) \in \mathcal{M}_f(E^* \widetilde{\otimes}_l Y, T_i^* \otimes_l \text{id}_Y)$.

(c) $\Rightarrow$ (d) Since E is a separable reflexive Banach lattice, E has non-empty quasi-interior, E^{**} is order continuous and E^* is separable (cf. [17, Chapter II, §5, Theorem 5.16]). Moreover, by Corollary 2.5, (T_i^*) is a BL-filtration on E^* . Consequently, by (c) and [14, Theorem 6.1], we have

$$\begin{aligned} \mathcal{M}(E \widetilde{\otimes}_l Y, T_i \otimes_l \text{id}_Y) &= \mathcal{M}(E^{**} \widetilde{\otimes}_l Y, T_i^{**} \otimes_l \text{id}_Y) \\ &= \mathcal{M}_f(E^{**} \widetilde{\otimes}_l Y, T_i^{**} \otimes_l \text{id}_Y) = \mathcal{M}_f(E \widetilde{\otimes}_l Y, T_i \otimes_l \text{id}_Y). \end{aligned}$$

Since the BL-filtration (T_i) is complemented by a (positive) contractive projection $T_\infty : E \rightarrow E$, we have by [2, Lemma 4.2 and Lemma 5.1] that $\mathcal{R}(T_\infty \otimes_l \text{id}_Y) = \mathcal{R}(T_\infty) \widetilde{\otimes}_l Y = \bigcup_{i=1}^\infty \mathcal{R}(T_i) \widetilde{\otimes}_l Y = \bigcup_{i=1}^\infty \mathcal{R}(T_i \otimes_l \text{id}_Y)$. It now follows by a continuity argument that $(T_i \otimes_l \text{id}_Y)$ is a filtration on $E \widetilde{\otimes}_l Y$ complemented by $T_\infty \otimes_l \text{id}_Y$. Thus, $\mathcal{M}(E \widetilde{\otimes}_l Y, T_i \otimes_l \text{id}_Y) = \mathcal{M}_{\text{nc}}(E \widetilde{\otimes}_l Y, T_i \otimes_l \text{id}_Y)$, as required.

(d) $\Rightarrow$ (a) For all finite measure spaces (Ω, Σ, μ) and $1 < p < \infty$, the Banach lattice $L^p(\mu)$ is separable and reflexive. By (d), it follows that $\mathcal{M}(L^p(\mu, Y), \Sigma_i) = \mathcal{M}_{\text{nc}}(L^p(\mu, Y), \Sigma_i)$ for every filtration (Σ_i) . Thus, Y has the Radon Nikodým property by [9, Theorem II.2.2.2].

(d) $\Leftrightarrow$ (e) Suppose $\mathcal{M}(E \widetilde{\otimes}_l Y, T_i \otimes_l \text{id}_Y) = \mathcal{M}_{\text{nc}}(E \widetilde{\otimes}_l Y, T_i \otimes_l \text{id}_Y)$. Since (T_i) is a BL-filtration, it follows that $\mathcal{M}(E \widetilde{\otimes}_l Y, T_i \otimes_l \text{id}_Y) = \mathcal{M}_{\text{nc}}(E, T_i) \widetilde{\otimes}_l Y$ by [2, Corollary 5.2]. Since E is reflexive, $\mathcal{M}_{\text{nc}}(E, T_i)$ is Riesz and isometrically isomorphic to $\mathcal{M}(E, T_i)$. Thus, $\mathcal{M}(E \widetilde{\otimes}_l Y, T_i \otimes_l \text{id}_Y) = \mathcal{M}(E, T_i) \widetilde{\otimes}_l Y$ by [14, Theorem 6.1]. Conversely, using [2, Corollary 5.2] and [14, Theorem 6.1] again, we obtain $\mathcal{M}(E \widetilde{\otimes}_l Y, T_i \otimes_l \text{id}_Y) = \mathcal{M}(E, T_i) \widetilde{\otimes}_l Y = \mathcal{M}_{\text{nc}}(E, T_i) \widetilde{\otimes}_l Y = \mathcal{M}_{\text{nc}}(E \widetilde{\otimes}_l Y, T_i \otimes_l \text{id}_Y)$, as required. $\square$

The above theorem allows us to generalize [5, Chapter IV, §1, Theorem 1], which characterizes the “Asplund spaces”. A Banach space Y is called an *Asplund space* if Y^* has the Radon Nikodým property.

Corollary 3.9. *Let Y be a Banach space. Then Y is an Asplund space if and only if $E^* \widetilde{\otimes}_l Y^* = (E \widetilde{\otimes}_l Y)^*$ for all separable Banach lattices E with order continuous dual.*

Proof. By Theorem 3.8, Y^* has the Radon Nikodým property if and only if $E^* \widetilde{\otimes}_l Y^* = \mathcal{L}^{\text{cas}}(E, Y^*)$ for all separable Banach lattices E with order continuous dual. But, by a theorem of Jacobs, we have $\mathcal{L}^{\text{cas}}(E, Y^*) = (E \widetilde{\otimes}_l Y)^*$ (cf. [10], [17, Chapter IV, §7, Theorem 7.4] and [3]). Thus, $E^* \widetilde{\otimes}_l Y^* = \mathcal{L}^{\text{cas}}(E, Y^*) = (E \widetilde{\otimes}_l Y)^*$, which completes the proof. $\square$

It is important to note that the above theorem does not include the case $E = L^1(\mu)$. However, by [5, Chapter IV, §1, Theorem 1], Y is an Asplund space if and only if $L^1(\mu, Y)^* = L^\infty(\mu, Y^*)$ for all finite measure spaces (Ω, Σ, μ) .

Combining Theorem 3.8 with [2, Theorem 5.3] yields another corollary:

Corollary 3.10. *Let Y be a Banach space. Then the following conditions are equivalent:*

- (a) *Y has the Radon Nikodým property.*
- (b) *For every separable reflexive Banach lattice E and every complemented, quasi-interior preserving BL-filtration (T_i) on E , we have $(f_n) \in \mathcal{M}(E \widetilde{\otimes}_l Y, T_i \otimes_l \text{id}_Y)$ if and only if for each $i \in \mathbb{N}$, there exist $(x_i^{(n)}, T_n)_{n=1}^\infty \in \mathcal{M}_{\text{nc}}(E, T_i)$ and $y_i \in Y$ such that, for each $n \in \mathbb{N}$, we have $f_n = \sum_{i=1}^\infty x_i^{(n)} \otimes y_i$, where $\|\sum_{i=1}^\infty |\lim_{n \rightarrow \infty} x_i^{(n)}|\| < \infty$ and $\lim_{i \rightarrow \infty} \|y_i\| = 0$.*

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REFERENCES

1. J. Chaney, *Banach lattices of compact maps*, Math. Z. **129** (1972), 1–19. MR0312329 (47:891)
2. S.F. Cullender and C.C.A. Labuschagne, *A description of norm-convergent martingales on vector-valued L^p -spaces*, J. Math. Anal. Appl. **323** (2006), 119–130. MR2261155 (2007j:60067)
3. ———, *A note on the M -norm of Chaney-Schaefer*, Quaestiones Math. **30** (2007), no. 2, 151–158. MR2337359
4. A. Defant and K. Floret, *Tensor norms and operator ideals*, Mathematics Studies, no. 176, North-Holland, Amsterdam, London, New York, Tokyo, 1993. MR1209438 (94e:46130)
5. J. Diestel and J.J. Uhl, *Vector measures*, Mathematical Surveys, no. 15, A.M.S., Providence, Rhode Island, 1977. MR0453964 (56:12216)
6. N. Dinculeanu, *Integral representation of linear operators. I*, Stud. Cerc. Mat. **18** (1966), 349–385. MR0213504 (35:4366a)
7. ———, *Integral representation of linear operators II*, Stud. Cerc. Mat. **18** (1966), 483–536. MR0213505 (35:4366b)
8. ———, *Linear operations on L^p -spaces*, Vector and Operator Valued Measures and Applications, Academic Press, New York, 1973, pp. 109–124. MR0341066 (49:5816)
9. L. Egghé, *Stopping time techniques for analysts and probabilists*, London Mathematical Society Lecture Note Series, no. 100, Cambridge University Press, London, New York, New Rochelle, Melbourne, Sydney, 1984. MR808582 (87j:60070)
10. H. Jacobs, *Ordered topological tensor products*, Ph.D. thesis, University of Illinois, 1969.

11. G.A.M. Jeurnink, *Integration of functions with values in a Banach lattice*, Ph.D. thesis, University of Nijmegen, The Netherlands, 1982.
12. S. Kakutani, *Concrete representation of abstract L -spaces and the mean ergodic theorem*, Ann. of Math. **42** (1941), 523–537. MR0004095 (2:318d)
13. J.L. Krivine, *Théorèmes de factorisation dans les espaces réticulés*, Séminaire Maurey-Schwartz 1973–1974 (École Polytechnique, Paris), Exposés 22–23, 1974. MR0440334 (55:13209)
14. C.C.A. Labuschagne, *Characterizing the one-sided tensor norms Δ_p and ${}^t\Delta_p$* , Quaestiones Math. **27** (2004), 339–363. MR2125036 (2005m:46036)
15. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces II*, A Series of Modern Surveys in Mathematics, no. 97, Springer-Verlag, Berlin, Heidelberg, New York, 1979. MR540367 (81c:46001)
16. P. Meyer-Nieberg, *Banach lattices*, Springer-Verlag, Heidelberg, Berlin, New York, 1991. MR1128093 (93f:46025)
17. H.H. Schaefer, *Banach lattices and positive operators*, Die Grundlehren der Mathematischen Wissenschaften, no. 215, Springer-Verlag, Berlin, Heidelberg, New York, 1974. MR0423039 (54:11023)
18. V. Troitsky, *Martingales in Banach lattices*, Positivity **9** (2005), 437–456. MR2188530 (2006j:60045)

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