AN *EL*-LABELING OF THE SUBGROUP LATTICE

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(Communicated by Jim Haglund)

ABSTRACT. In a 2001 paper, Shareshian conjectured that the subgroup lattice of a finite, solvable group has an *EL*-labeling. We construct such a labeling and verify that our labeling has the expected properties.

1. INTRODUCTION

All groups, posets, and simplicial complexes in this paper are finite. We recall that the *subgroup lattice* $L(G)$ of a group G is the set of all subgroups of the group, ordered under inclusion. $L(G)$ is a lattice, with $H \wedge K = H \cap K$ and $H \vee K = \langle H, K \rangle$.

Any poset P is closely associated with its *order complex* $|P|$, a simplicial complex with faces the chains in P . Considering the order complex allows us to use combinatorial topology definitions and theorems with P . One such definition is that of “shellability.” A “shellable” complex is essentially one where the facets fit nicely together [1, 2, 3, 4]; the precise definition will not be important to us. A shellable poset is one with shellable order complex.

The connection with the subgroup lattice is surprising and beautiful:

Theorem 1.1 (Shareshian [12, Theorem 1.4]). *$L(G)$ is shellable if and only if G is solvable.*

Let us talk about the techniques used to prove the “if” direction of Theorem 1.1. There are two main techniques to show that a bounded poset is shellable, both developed by Björner and Wachs [1, 2, 3, 4]. The first is to label the edges of the Hasse diagram in a manner such that on every interval:

- (1) There is a unique chain where the labels (read from bottom to top) are increasing.
- (2) The unique increasing chain is lexicographically first.

A labeling satisfying these two properties is called an *EL-labeling*.

The second is to label the atoms of the poset. A *recursive atom ordering* of a bounded poset P is an ordering $a_1, a_2, \dots$ of the atoms of P such that

- (1) For any j , the interval $[a_j, \hat{1}]$ has a recursive atom ordering in which the atoms in $[a_j, \hat{1}]$ that are above some a_i for $i < j$ come first.
- (2) For all $i < j$, and x with $a_i, a_j < x$, there is a $k < j$ and an atom $y < x$ of $[a_j, \hat{1}]$ with $a_k < y$.

Received by the editors September 19, 2007.

2000 *Mathematics Subject Classification*. Primary 06A07; Secondary 05E25, 20E15.

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A bounded poset with either an EL -labeling or a recursive atom ordering is shellable. The two are somewhat related: a poset with a recursive atom ordering has a “ CL -labeling”, which is a generalization of the idea of an EL -labeling. As a poset is shellable if and only if its dual is shellable, recursive coatom orderings and dual EL -labelings are also of interest.

Shareshian proved the “if” direction of Theorem 1.1 as follows:

Theorem 1.2 (Shareshian [12, Corollary 4.10]). *If G is solvable, then $L(G)$ has a recursive coatom ordering.*

Note 1.3. Interestingly, the ordering of maximal subgroups (coatoms) that Shareshian used had already been studied by Doerk and Hawkes [6, Chapter A.16, especially Definition 16.5].

An EL -labeling gives useful information about a poset. For example, one of the nicest consequences is that the set of descending chains forms a cohomology basis for $|P|$. Unfortunately, although every poset with a recursive (co-)atom ordering has a (dual) CL -labeling, the construction is complicated enough that nice enumerative results (such as the cohomology basis) coming from EL/CL -labelings are usually difficult or impossible to use.

The topology of the subgroup lattice of a solvable group had been studied before Shareshian. Let G be a solvable group, with chief series $1 = N_0 \subset N_1 \subset \cdots \subset N_k = G$. A *complement* to a subgroup N is a subgroup H with $HN = G$ and $H \cap N = 1$. A *chain of complements* (to the given chief series) is a chain $1 = H_k \subset H_{k-1} \subset \cdots \subset H_0 = G$ where H_i is a complement to N_i (for each i). Then

Theorem 1.4 (Thévenaz [13, Theorem 1.4]). *For any solvable group G , $|L(G)|$ has the homotopy type of a wedge of spheres of dimension $k - 2$, and the spheres are in bijective correspondence with the chains of complements to any given chief series.*

In light of the cohomology basis mentioned above, Theorem 1.4 naturally leads to the following conjecture:

Conjecture 1.5 (Shareshian [12, Conjecture 1.6]). *For any solvable group G , $L(G)$ admits an EL -labeling where the descending chains are the chains of complements to a chief series.*

In the rest of this paper, we will extend the theory of left modular lattices to construct both an EL -labeling and a dual EL -labeling satisfying Conjecture 1.5.

2. LEFT MODULARITY

Our starting point will be left modularity. Let L be any lattice. An element $x \in L$ is *left modular* if for all $y < z$ we have $(y \vee x) \wedge z = y \vee (x \wedge z)$, i.e., if it satisfies one side of the requirement for modularity.

Example 2.1. The Dedekind identity (see for example [11, 1.3.14]) says that $H(N \cap K) = HN \cap K$ for any subgroup N , and subgroups $H \subseteq K$ of a group G . Since a normal subgroup N of G satisfies $HN = NH = \langle H, N \rangle$ for every subgroup H , a normal subgroup is left modular in $L(G)$.

Liu gave a helpful alternative characterization of left modular elements. Let $y < z$ denote a *cover relation*, that is, a pair $y < z$ such that if there is an x with $y \leq x \leq z$, then $x = y$ or $x = z$.

Theorem 2.2 (Liu [8, Theorem 2.1.4], also in [9, Theorem 1.4]). *Let x be an element in a lattice L . The following are equivalent.*

- (1) x is left modular.
- (2) For any $y < z$ we have $x \vee z \neq x \vee y$ or $x \wedge z \neq x \wedge y$.
- (3) For any $y \leq z$ we have $x \vee z = x \vee y$ or $x \wedge z = x \wedge y$, but not both.

Part (3) of Theorem 2.2 leads us to the following definition: let $\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}$ be a (not necessarily maximal) chain with every x_i left modular. Then we say x_{i+1}/x_i *weakly separates* a cover relation $y \leq z$ if $x_i \wedge z = x_i \wedge y$ but $x_{i+1} \vee z = x_{i+1} \vee y$. Any given cover relation is weakly separated by a unique x_{i+1}/x_i in the modular chain.

Then it is natural to consider the labeling

$$\lambda(y \leq z) = i \quad \text{where } x_{i+1}/x_i \text{ weakly separates } y \leq z.$$

Theorem 2.3 (Liu [8, Theorem 3.2.6]). *If the left modular chain $\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}$ is a maximal chain, then λ is an EL -labeling.*

In this situation (where L has a maximal chain of left modular elements) we say that L is *left modular*. Left modular lattices have been studied in several papers [10, 14, 5] in addition to the ones already referenced. Lattices with chains of modular elements were studied in [7].

Motivated by the situation in a solvable group (where the chief series is a left modular chain, but not necessarily a maximal one), we ask what happens with the labeling λ when $\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}$ is not maximal. We don't get an EL -labeling, but we can still say some things about the increasing chains on an interval.

Let $[w, z]$ be an interval in L . Then $w \leq w \vee x_i \wedge z \leq z$ for all i , and we notice that $w \not\leq w \vee x_i \wedge z$ for large enough i (in particular, $i = k$ gives $w \vee \hat{1} \wedge z = z$). So let $c_0 = w$, and inductively construct c_j as follows: let $i(j)$ be the maximal index such that $c_j \vee x_{i(j)} \wedge z = c_j$. Then let

$$c_{j+1} = c_j \vee x_{i(j)+1} \wedge z = w \vee x_{i(j)+1} \wedge z.$$

This gives a chain $\mathbf{c} = \{w = c_0 < c_1 < \cdots < c_m = z\}$ between w and z . Every edge on the interval $[c_j, c_{j+1}]$ receives an $i(j)$ label, since for every y on $[c_j, c_{j+1}]$ we have

$$y \vee x_{i(j)+1} = y \vee (x_{i(j)+1} \wedge z) \vee x_{i(j)+1} = c_{j+1} \vee x_{i(j)+1};$$

while $y \vee x_{i(j)} \wedge z = y$, so that each $y \vee x_{i(j)}$ is distinct.

Lemma 2.4. *A maximal chain on $[w, z]$ is (weakly) increasing if and only if it is an extension of $\mathbf{c}$.*

Proof. Every extension of $[c_j, c_{j+1}]$ has every edge labeled with $i(j)$. Since, by the construction, $i(0) < i(1) < \cdots < i(m-1)$, every maximal extension of $\mathbf{c}$ is (weakly) increasing.

In the other direction, notice that since $w \vee x_{i(0)} \wedge z = w$, but $w \vee x_{i(0)+1} \wedge z \geq w$, there must be an edge $d_j < d_{j+1}$ in any maximal chain $\mathbf{d} = \{w = d_0 < d_1 < \cdots < z\}$ such that $d_j \not\geq w \vee x_{i(0)+1} \wedge z$ but $d_{j+1} \geq w \vee x_{i(0)+1} \wedge z$. Clearly such an edge receives an $i(0)$ label, and since by the definition of the labeling any maximal chain cannot have labels less than $i(0)$, any weakly increasing maximal chain must start with $i(0)$ labels.

The first edge of $\mathbf{d}$ receives the label $i(0)$ only if $d_0 \vee x_{i(0)+1} = d_1 \vee x_{i(0)+1}$; thus,

$$d_1 \leq d_1 \vee x_{i(0)+1} \wedge z = d_0 \vee x_{i(0)+1} \wedge z = c_1,$$

and so the first edge of $\mathbf{d}$ is in $[c_0, c_1]$. Repeating this argument inductively on $[d_1, z]$ gives that $\mathbf{d}$ is an extension of $\mathbf{c}$, as desired. $\square$

Corollary 2.5. *A maximal chain on $[w, z]$ is (tied for) lexicographically first if and only if it is an extension of $\mathbf{c}$.*

Note 2.6. There is not in general a unique lexicographically first or increasing chain, as $\mathbf{c}$ may have many extensions.

Note 2.7. We use the term “weakly separated” to highlight that a maximal chain might have multiple i labels. One might say that $y \prec z$ was *separated* by x_{i+1}/x_i if the edge was weakly separated and also $x_{i+1} \wedge y = x_i \wedge y$ and $x_{i+1} \vee z = x_i \vee z$ (but we will not use this).

In Section 3, we will show that intervals in $L(G)$ with repeated i labels are isomorphic to certain sublattices of $[N_i, N_{i+1}]$, and in Section 4 we will use this isomorphism to refine λ to an EL -labeling in the subgroup lattice (of a solvable group).

3. PROJECTING INTO $[N_i, N_{i+1}]$

Let G be a solvable group with a chief series $1 = N_0 \subset N_1 \subset \cdots \subset N_k = G$, and let H be any subgroup. The subgroups of $L(G)$ that are normalized by H form a sublattice $L_H(G)$. In this section we will relate certain sections

$$\mathcal{N}_i(H) \triangleq [N_i, N_{i+1}] \cap L_H(G)$$

of this lattice to weak separation by the chief series. First:

Lemma 3.1. *For any H , $\mathcal{N}_i(H)$ is a modular lattice.*

Proof. $\mathcal{N}_i(H)$ is closed under intersection and join, so it is a sublattice of $[N_i, N_{i+1}]$. By the Correspondence Theorem [11, 1.4.6], we have that $[N_i, N_{i+1}] \cong L(N_{i+1}/N_i)$. Since N_{i+1}/N_i is abelian, $[N_i, N_{i+1}]$ is a modular lattice, and sublattices of a modular lattice are modular. $\square$

Second, we have a relationship between weak separation of an edge in $L(G)$ and $\mathcal{N}_i$.

Lemma 3.2. *If $E \subset F$ is weakly separated by N_{i+1}/N_i , then $\mathcal{N}_i(E) = \mathcal{N}_i(F)$.*

Proof. $N_{i+1}E = N_{i+1}F$, so $F \subseteq EN_{i+1}$. Since every subgroup N in the interval $[N_i, N_{i+1}]$ is normalized by N_{i+1} , we see that if E normalizes N , then so does F . The converse is immediate. $\square$

Note 3.3. When we are looking at an edge or chain(s) of edges that are weakly separated by N_{i+1}/N_i , we will often simply write $\mathcal{N}_i$ to mean $\mathcal{N}_i(E) = \mathcal{N}_i(F) = \dots$. Lemma 3.2 tells us that this notation makes sense.

Finally, we construct a projection map from $L(G)$ to $[N_i, N_{i+1}]$. Let

$$\rho_i(H) = N_i \vee H \wedge N_{i+1} = N_i H \cap N_{i+1}.$$

It is clear that this is really in $[N_i, N_{i+1}]$. In fact, $\rho_i(H)$ is in $\mathcal{N}_i(H)$ (since N_i , N_{i+1} , and H are all normalized by H). Much more is true. Let $[W, Z]_S$ denote the

interval $[W, Z]$ in the sublattice $\mathcal{S}$ of $L(G)$; that is, let $[W, Z]_{\mathcal{S}}$ consist of all $H \in \mathcal{S}$ that are between W and Z .

Proposition 3.4. *If there is a chain on the interval $[W, Z]$ with every edge weakly separated by N_{i+1}/N_i , then ρ_i on $[W, Z]$ gives a poset isomorphism*

$$[W, Z]_{L(G)} \cong [\rho_i(W), \rho_i(Z)]_{\mathcal{N}_i}.$$

Example 3.5. Consider the alternating group on 4 elements with the normal series $N_0 = 1$, N_1 the Klein 4 subgroup, and $N_2 = A_4$. Then $\langle(1\ 2\ 3)\rangle \subset A_4$ is weakly separated by N_1/N_0 , and it projects to $N_0 \subset_{\mathcal{N}_1} N_1$, an edge in the sublattice $\mathcal{N}_i = \mathcal{N}_i(A_4)$. Notice that, although $N_0 \subset N_1$ is a cover relation in $\mathcal{N}_i$, it is not a cover relation in $L(G)$, as $N_0 = 1 \subset \langle(1\ 2)(3\ 4)\rangle \subset N_1$.

Proof. (of Proposition 3.4) It is immediate from the definition that ρ_i is a poset map, so it suffices to produce an inverse map. Let ϕ_i be the map $N \mapsto WN \cap Z$. Since there is a chain with every edge weakly separated by i , $N_i \cap W = N_i \cap Z$ and $N_{i+1}W = N_{i+1}Z$.

Then for H on $[W, Z]$ we have (by repeated application of the Dedekind identity)

$$\begin{aligned} \phi_i \rho_i(H) &= W(N_i H \cap N_{i+1}) \cap Z = N_i H \cap N_{i+1} W \cap Z \\ &= N_i H \cap N_{i+1} Z \cap Z = N_i H \cap Z = H(N_i \cap Z) \\ &= H(N_i \cap W) = H, \end{aligned}$$

while for N in $\mathcal{N}_i$ we get

$$\begin{aligned} \rho_i \phi_i(N) &= N_i(WN \cap Z) \cap N_{i+1} = WN \cap ZN_i \cap N_{i+1} \\ &= WN \cap N_{i+1} \cap \rho_i(Z) = N(N_i W \cap N_{i+1}) \cap \rho_i(Z) \\ &= \rho_i(W)N \cap \rho_i(Z), \end{aligned}$$

and for N between $\rho_i(W)$ and $\rho_i(Z)$ we have $\rho_i \phi_i(N) = N$. $\square$

Note 3.6. Our use of the fact that N is in $\mathcal{N}_i$ in the proof of Proposition 3.4 is somewhat subtle: it comes in when we assume that WN is a subgroup. (Otherwise, $\phi_i(N)$ is not necessarily in $L(G)$.)

Corollary 3.7. *If $E \subset F$ is a cover relation in $L(G)$, then $\rho_i(E) \subset \rho_i(F)$ is a cover relation in $\mathcal{N}_i$.*

4. LABELING $L(G)$

Proposition 3.4 and Corollary 3.7 make it clear how to construct an EL -labeling of $L(G)$: label first by the weak separation labeling, then refine by the modular labeling in the projection to $\mathcal{N}_i$.

More precisely, for each distinct $\mathcal{N}_i = \mathcal{N}_i(H)$, let $\lambda^{\mathcal{N}_i}$ be the modular EL -labeling of $\mathcal{N}_i$. Suppose that $E \subset F$ is an edge in $L(G)$, weakly separated by N_{i+1}/N_i .

Then label the edge with the pair

$$\lambda(E \subset F) = (i, \lambda^{\mathcal{N}_i}(\rho_i(E) \subset \rho_i(F))).$$

As is usual, pairs (i, j) are ordered lexicographically.

Theorem 4.1. λ is an EL -labeling of $L(G)$.

Proof. Lemma 2.4 and Corollary 2.5 tell us that any increasing (lexicographically first) chain on $[W, Z]$ is an extension of the chain $\mathbf{c} = \{C_0 \subseteq C_1 \subseteq \dots \subseteq C_m\}$, inductively obtained by taking $C_0 = W$, and $C_{j+1} = N_{i(j)+1}C_j \cap Z$, where $i(j)$ is the maximal index such that $N_{i(j)}C_j \cap Z = C_j$.

Every edge on the interval $[C_j, C_{j+1}]$ is weakly separated by $N_{i(j)+1}/N_{i(j)}$, so projects to the same $\mathcal{N}_i$, and the modular EL -labeling on $\mathcal{N}_i$ gives a unique increasing (lexicographically first) chain on $[C_j, C_{j+1}]$, hence a unique increasing (lexicographically first) extension of $\mathbf{c}$. $\square$

Note 4.2. A left modular element in L is also left modular in the dual lattice L^* , and Lemma 3.2 and Proposition 3.4 say the same thing in L^* as in L . Thus, we could just as easily take a chief series $G = N_0^* \triangleright N_1^* \triangleright \dots \triangleright N_k^* = 1$, and label via

$$\lambda_*(E \triangleright F) = (i, \lambda_*^{i, \mathcal{N}_i}(\rho_i^*(E) \triangleright \rho_i^*(F))),$$

where N_i^*/N_{i+1}^* weakly separates $E \triangleright F$ and ρ_i^* is the projection to $[N_{i+1}^*, N_i^*]$. Depending on taste, the resulting EL -labeling of the dual lattice may even seem more natural.

4.1. Descending chains. If $E \subseteq F$ satisfies $E \cap N_{i+1} = 1$ and $EN_{i+1} = G$ while $F \cap N_i = 1$ and $FN_i = G$, then $EN_{i+1} = FN_{i+1} = G$ and $E \cap N_i = F \cap N_i = 1$. Thus, $E \subseteq F$ is separated by i , and thus a chain of complements is a descending chain, labeled $k-1, \dots, 1, 0$. By Thevnaz's theorem (Theorem 1.4), and since an EL -shellable lattice has the homotopy type of a bouquet of spheres in correspondence to the descending chains, the chains of complements are exactly the descending chains. (This is also straightforward to verify by induction.)

Similarly for the dual labeling λ_* . To summarize:

Proposition 4.3. *The descending chains of both λ and λ_* are exactly the chains of complements of the chief series used.*

Thus, the labelings we have constructed are the ones conjectured by Shareshian.

ACKNOWLEDGEMENTS

Thanks to Bruce Sagan, John Shareshian, Hugh Thomas, and the anonymous referee for their comments and suggestions.

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